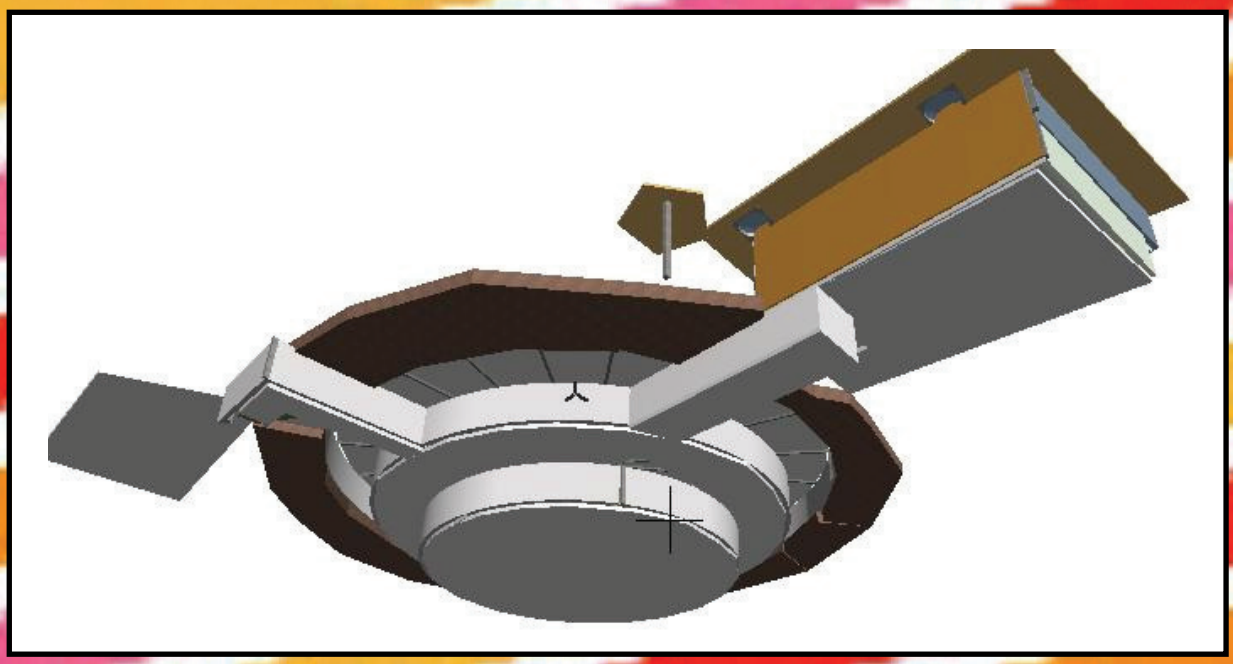
MULTISPACE & MULTISTRUCTURE. NEUTROSOPHIC TRANSDISCIPLINARITY

(100 COLLECTED PAPERS OF SCIENCES)

VOL. IV



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Neutrosophic Transdisciplinarity

(100 Collected Papers of Sciences)

Vol. IV

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Collected Eclectic Ideas

Preface

The fourth volume, in my book series of "Collected Papers", includes 100 published and unpublished articles, notes, (preliminary) drafts containing just ideas to be further investigated, scientific souvenirs, scientific blogs, project proposals, small experiments, solved and unsolved problems and conjectures, updated or alternative versions of previous papers, short or long humanistic essays, letters to the editors - all collected in the previous three decades (1980-2010) – but most of them are from the last decade (2000-2010), some of them being lost and found, yet others are extended, diversified, improved versions.

This is an eclectic tome of 800 pages with papers in various fields of sciences, alphabetically listed, such as: astronomy, biology, calculus, chemistry, computer programming codification, economics and business and politics, education and administration, game theory, geometry, graph theory, information fusion, neutrosophic logic and set, non-Euclidean geometry, number theory, paradoxes, philosophy of science, psychology, quantum physics, scientific research methods, and statistics.

It was my preoccupation and collaboration as author, co-author, translator, or co-translator, and editor with many scientists from around the world for long time. Many topics from this book are incipient and need to be expanded in future explorations.

I am very grateful to all my collaborators, translators, editors and publishers, advisers and friends, not only those cited in this book but all researchers, professors, students that exchanged messages, articles, books with me during these years and carefully listened to my ideas and helped me improve them, and I listened to theirs too.

I dreamt with the engineer and friend Vic Christianto to build a Lunar Space Base and travel from there inside the Solar System and outside in order to discover new planetoids and respectively exoplanets and to quantize the Universe. Or use the multispace and multistructure together with the physicist and editor-in-chief Dmitri Rabounski to re-interpret and extend scientific theories and even to induce New Physics if possible. Generalize the qu-bit to a *mu-bit* (multi-bit in a multi-space) for a (multi-)parallel computing (*mu-computing* with a *mu-supercomputer*), and search for an *SC* Potential.

Go to the outer-limits of science (to the Classes of Neutrosophic Paradoxes), not in a fiction but in a realistic way, and apply a neutrosophic interdisciplinary method of study and research, not being ashamed to ask and seek even elementary or impossible questions.

Thus, Neutrosophic Transdisciplinarity was born, which means to find common features to uncommon entities, i.e. for vague, imprecise, not-clear-boundary entity $<A>$ one has:

$<A> \cap <nonA> \neq \emptyset$ (empty set), or even more $<A> \cap <antiA> \neq \emptyset$,
where <*nonA*> means what is not <*A*>, and <*antiA*> means the opposite of <*A*>.
There exists a Principle of Attraction not only between the opposites <*A*> and <*antiA*> (as in dialectics), but also between them and their neutralities <*neutA*> related to them, since <*neutA*> contributes to the Completeness of Knowledge.
<*neutA*> means neither <*A*> nor <*antiA*>, but in between;
<*neutA*> is neutrosophically included in <*nonA*>.
But, we may also have $<A> \neq <A>$, since <*A*> could be endowed with different structures simultaneously, or <*A*> at some time could be different from <*A*> at another time.

This volume includes, amongst others, copies of some of my manuscripts confiscated by the Secret Police [*Securitate*], but other manuscripts were never returned although I asked for them back after the 1989 Revolution. There are four folders, summing a total of about 880 pages, monitoring me, consisting of reports and photos about my activities written and respectively made by secret police agents.

Also:

- Methods of doing research, of improving and generalizing known results.
- Reflections on the philosophy of science: where will science go?
- An algebraic generalization of Venn diagram for programming codification.
- Generalization and alternatives of Kaprekar routine and of SUDOKU.
- Improvement of statistics estimators.
- Alternatives to Pearson's and Spearman's Correlation Coefficients.
- How to construct a quantum topology?
- In quantum physics about: Brightsen Model, Klein-Gordon Equation, Ginzburg-Landau-Schrödinger type equations, LENR, PT-symmetry and Iso-PT symmetry, etc.
- Learn how to partially negate axioms, lemmas, theorems, notions, properties, and theories – degrees of negation.
- Classes of neutrosophic paradoxes and neutrosophic degree of paradoxicity; neutrosophic diagram; new neutrosophic operators (such as *neuterization* and *antonymization*).
- Applications of neutrosophic logic and set to the semantic web services, information fusion, image segmentation and thresholding.
- Sequences and metasequences.
- Introduction of the operators of *validation* and *invalidation* of a proposition, and extension of the operator of *S-denying* a proposition, or an axiomatic system, from geometry to respectively any theory in a given domain of knowledge. By S-denying a <notion> one can get a <pseudo-notion> (for example: S-denying the *norm* one gets a *pseudo-norm*).
- Tunnels of triangles, polygons and of *n-D* solids.
- Unification of Fusion Theories and Rules.
- Degree of Uncertainty of a Set and of a Basic Believe Assignment (Mass).
- *α*-Discounting Method for Multi-Criteria Decision Making (α-D MCDM).
- New fusion rules and conditioning fusion rules.
- Neutrosophic Logic as a Theory of Everything in Logics. N-norm and N-conorm.

- Neutrosophic philosophy in applied sciences. Neutrosophy is a MetaPhilosophy.

And so on.

We introduce the *non-standard quaternion space* and *non-standard biquaternion space* [and even a generalization of them to a general non-standard vector space of any dimension] as possible working spaces for connecting the micro- and macro-levels in physics.

Also, we consider that our multispace (with its multistructure of course) unifies many science fields. We write about parallel quantum computing and mu-bit, about multi-entangled states or particles and up to multi-entangles objects, about multispace and multivalued logics, about possible connection between unmatter with dark matter (what about investigating a possible existence of dark antimatter and dark unmatter?), about parallel time lines and multi-curve time, projects about writing SF at the quantum level as for example "The adventures of the particle-man" or "Star Shrek" – a satire to Star Trek (just for fun), about parallel universes as particular case of the multispace, and we advance the hypothesis that more models of the atom are correct not only the standard model of the atom, etc.

I coined the name *unmatter* as a combination of matter and antimatter – and a possible third form of matter - since 2004, in a paper uploaded in the CERN website, and I published papers about "unmatter" that is now the predecessor of unparticles, which are a type of unmatter (mixtures of particles and antiparticles).

These fragments of ideas and believes have to be further investigated, developed, and check experimentally if possible. {Actually, no knowledge is definitive in a field!}

The "multispace" with its "multistructure" is a *Theory of Everything* in any domain. It can be for example used in physics for the **Unified Field Theory** that tries to unite the gravitational, electromagnetic, weak, and strong interactions.

The author hopes that certain articles or ideas from this tome will inspire the reader in his/her further research and creation.

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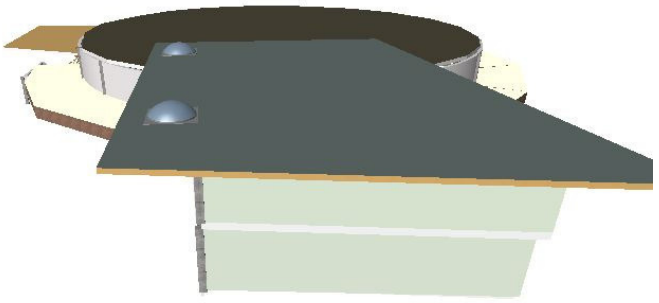

FEBRUARY 2008

FIRST LUNAR SPACE BASE 2009

PROPOSAL ENTRY FOR HOLCIM AWARDS 2008

By V. Christianto & F. Smarandache

INTRODUCTION / MOTIVATION

We interpret 'Sustainable construction' theme in its widest possible meaning, i.e. the preservation of sustainability of environment to support mankind. In this regard, it is realized that this Earth is likely to continue to deteriorate and therefore its capability to sustain mankind is diminishing.

Therefore some institutions have begun to study possibility to send mankind to space for long-time period. A year long experiment of mankind capability to survive in space has been conducted by International Space Station.

Then the next logical step would be how to find good location of international space base, possibly in the Moon surface. Therefore we design an imaginative concept for the First Lunar Space Base 2009.

We acknowledge that according to the competition rule, a design shall have 'high probability' to construct. But considering this program is likely to yield great interests not only for government and private sectors, therefore it is possible to conduct auction to put this design into reality. Alternative method to finance is to use some 'roof space' of this proposed space-base for advertising space. We expect that plenty of corporations would like to get their ads printed on the First Lunar Space Base, just like corporations put ads in the body of Russian space rocket.

PROPOSAL SUMMARY

Entry name : First Lunar Space Base 2009

Authors : V. Christianto, F Smarandache

Purpose : to put forward a realistic humanoid lunar-base within 2-3 years

Possible project financing: if no government support is available, financing may be obtained by selling wall space to become advertising display for supporting corporate.

CONCEPTUAL DESIGN

While it seems everybody knows that lunar base is the most logical first destination in new space exploration programs [1], apparently not so many people realize that it is dangerous place for some obvious and not so-obvious reasons, i.e. no water, no oxygen, no atmosphere, risk of asteroid shower, risk of lunar earthquake, risk of extreme exposure of sunlight including ultraviolet ray etc.

Therefore in this proposal we consider these risks in the design.

The conceptual idea came from some places, but one of the most intriguing source is what people call as 'Crop Circle', see Figure 1. [2] It's full of mysteries, yet it seems to offer a kind of 'hidden' geometry.

Then we modify this crop circle idea (see right image), and combine it with the 'Enterprise' starship in Star Trek movie, then we get the view as follows (see Figure 2,3 & 4)

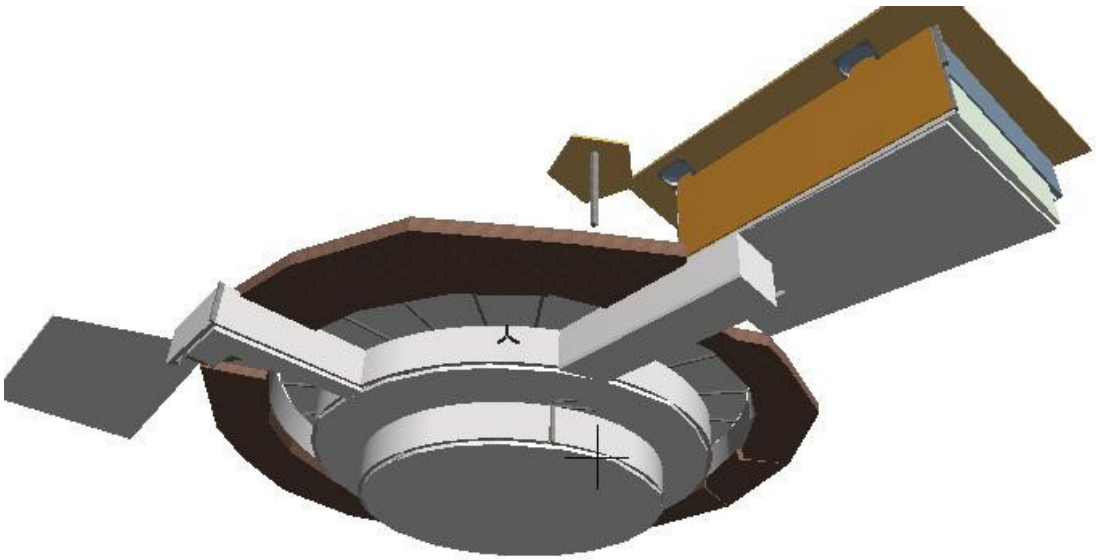

Figure 2: The proposed First Lunar Space base

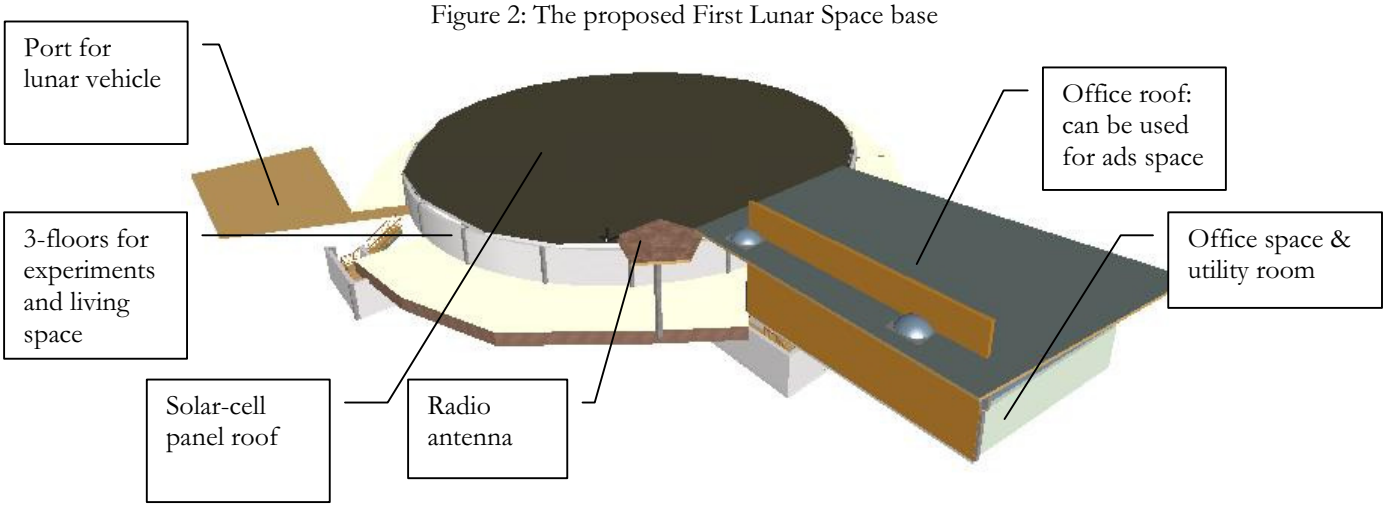
Port for lunar vehicle
3-floors for experiments and living space
Solar-cell panel roof
Radio antenna
Office roof: can be used for ads space
Office space & utility room

Figure 3. Top view of the First Lunar Space base

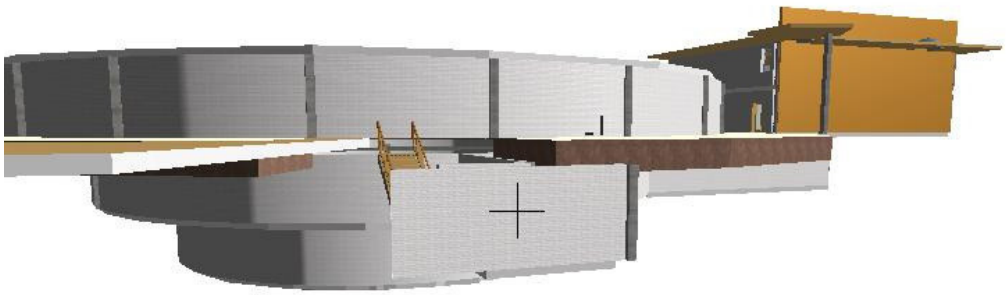

Figure 4. Side view of the First Lunar Space base

RATIONALE BASED ON DESIGN CRITERIA

As we write above, we interpret 'Sustainable construction' theme in its widest possible meaning, i.e. the preservation of sustainability of environment to support mankind. Nonetheless we describe here other considerations of design criteria:

1. Quantum change and Transferability

    1.1. We propose to use Lunar local material to as maximum as possible, therefore minimizing transportation needs to carry the raw material from Earth [1]. Even for concrete, we propose to use 'lunar concrete'. Of course, a particular construction obstacle would be how to guarantee that the process of concrete drying would be the same with process in Earth.

    1.2. Unlike 'standard' conventional design for Lunar space base in literature, in this design we propose to use 'regolith' to cover the roof. [1][3]

    1.3. The large flat-dome will be used for 'solar-cell panel' hence maximizing self-sufficient energy supply. Solar-cell panel can be applied using 'coating' (nanosolar-cell) method rather than having to carry and install complete solar-cell panels from Earth.

2. Ethical standards and social equity

    2.1. Detailed design and construction will maintain the highest ethical standard, for example it is not allowed to use processes that break human right. Also the use of local material is preferred in order to avoid prohibitive high-cost of material transportation from Earth to Moon.

3. Ecological quality and energy conservation

    3.1. By using the roof-space to be 'solar-cell panel' will ensure that energy usage is sustainable.

    3.2. Other technology which can be used is if possible to use 'hydrophonic' plant, therefore reduce using food coming from Earth.

4. Economic performance and compatibility

    4.1. Considering that it is likely that the cost to build this permanent base will be dominated by 'transportation cost', to bring the raw material to Earth, therefore economic efficiency /performance is directly proportional to percentage of local material that can be used instead of having to bring it from Earth.

5. Contextual and aesthetic impact

    5.1. With regards to the environment and physical context of Lunar surface (i.e. without atmosphere and risks of asteroid shower), then we put this safety and protection first over aesthetic consideration alone.

5.2. We also preserve the inherent quality of the landscape by using regolith and other local material as finishing, therefore aesthetically the proposed Lunar base will look 'compatible' with the surrounding environment.

5.3. Nonetheless, as a whole, we put aesthetic consideration from the viewpoint of basic concept (from Crop Circle and StarTrek) which can be noticed by anyone observing from distance. After all, who knows that someday there will be external cultures from outer space who may wish to stop by and look around?

In this regard, it is realized that this considering some obvious construction limitation, not all of the above critieria can be applied to a Lunar space base design. But we have done as far as possible to make this design a working example how a Lunar base can be designed with environment-sustainability in mind. In our opinion, the construction process itself is a kind of 'experimental research,' i.e. how to make sure that using present technologies we can build permanent Lunar base with minimum imported material as far as possible.

It is hoped that once this design can be put into reality, it can draw attention to sustainability construction.

SOME CONSTRUCTION OBSTACLES

There are obvious obstacles which may be considered. We list here only a few [1]:

(a) asteroid shower: may induce impact to the construction workers + machines. Perhaps need to build temporary (polycarbonate) shelter;

(b) no oxygen would mean that concrete drying process may not complete. Perhaps needs to bring oxygen chamber, and mixing concrete will be done inside the chamber;

(c) oxygen supply for the workers;

(d) design load shall take into consideration this asteroid shower and other possible impact load.

POSSIBLE RAW MATERIALS

MATERIALS THAT CAN BE USED IN THIS PROPOSED LUNAR BASE [1].

1. Steel: for columns and beams, are recommended if no concreting is allowed.

2. Aluminum. Very versatile and light material, suitable for inner utilities).

3. Fabric, can be used in various places, for instance: room separation, cladding (protection against extreme UV), and other function. Membrane can also be used for roof instead of rigid roof, but with greater risk of 'puncture' because of impact loads.

4. Compacted regolith, can be used to cover the roof.

5. Lunar concrete: can be used for slabs and beams, if concreting is allowed. Otherwise, slabs may use aluminum.

6. Glass: shall be used in minimum because of possible scratch or puncture under impacts.

7. Inflatable material can also be used, but in our opinion it is also suitable for temporary building, not a permanent space base, which is expected to be more durable and high-load resistance.

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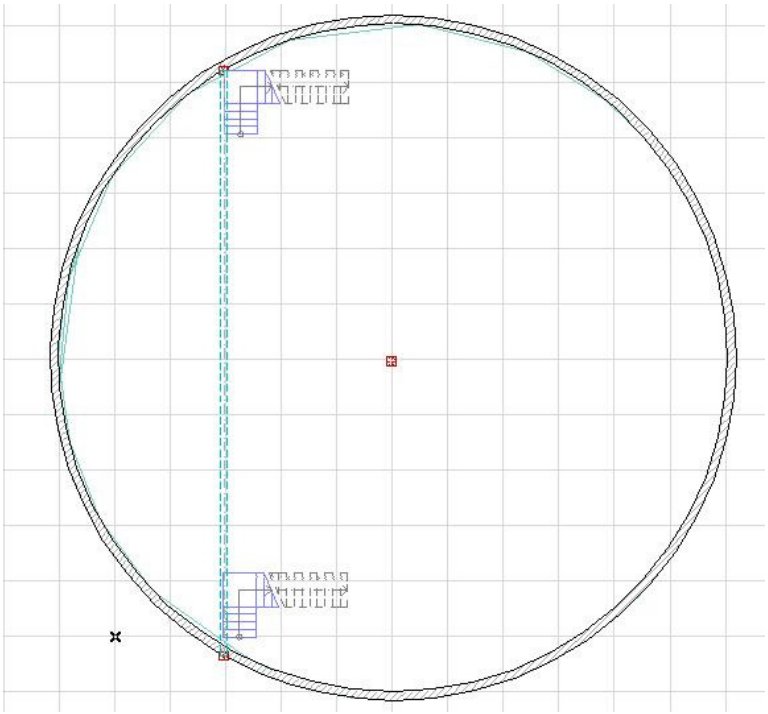
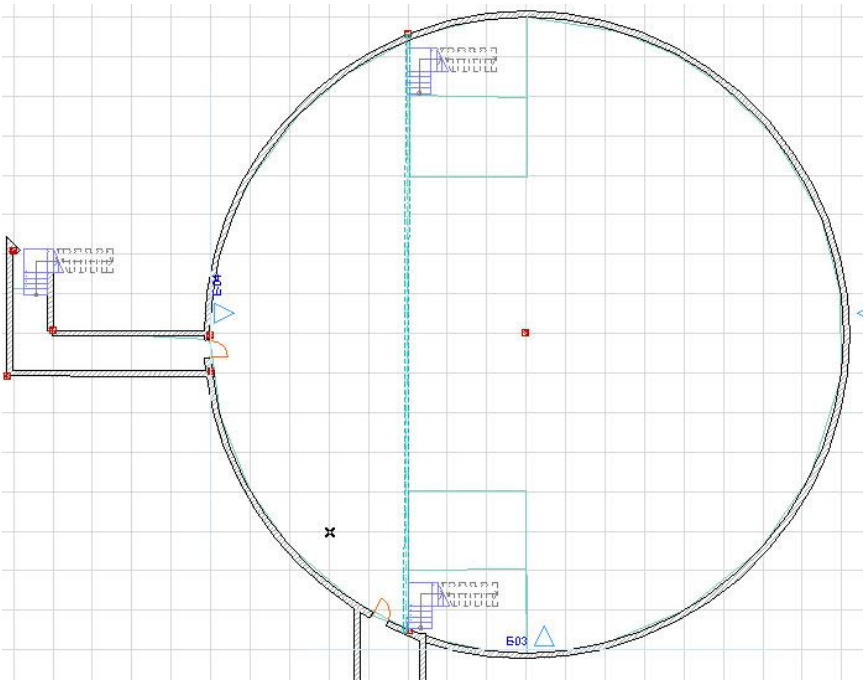
Created 1st: 28feb 2008. VC & FS, vxianto@yahoo.com & fsmarandache@yahoo.com

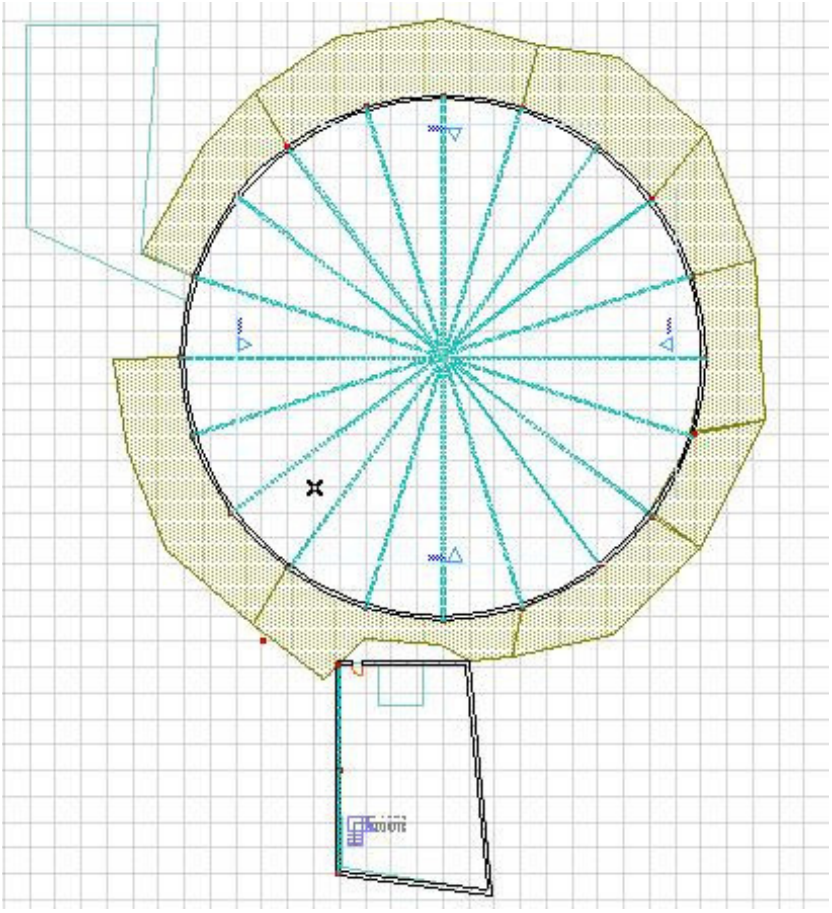
(a) Basement 2 level:



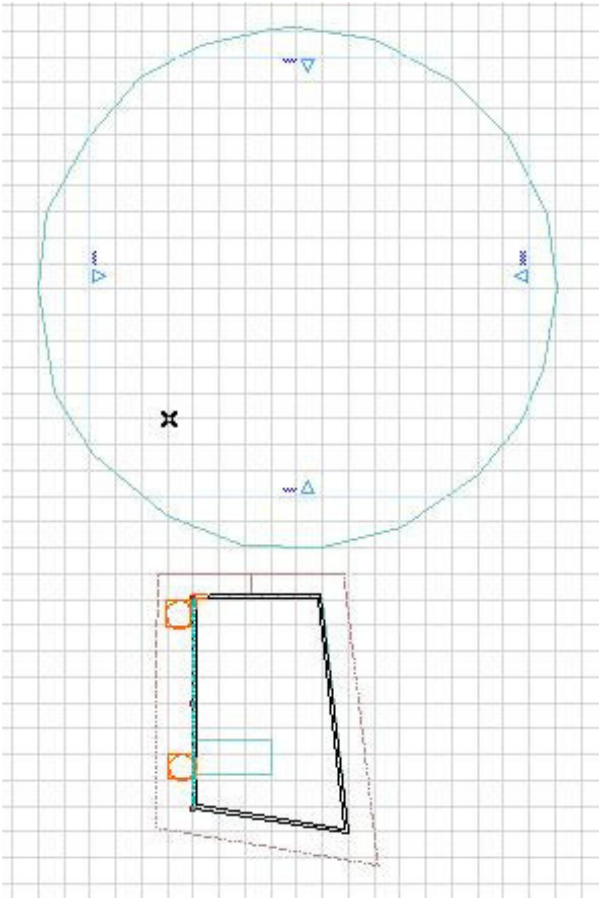
(b) Basement 1 level:



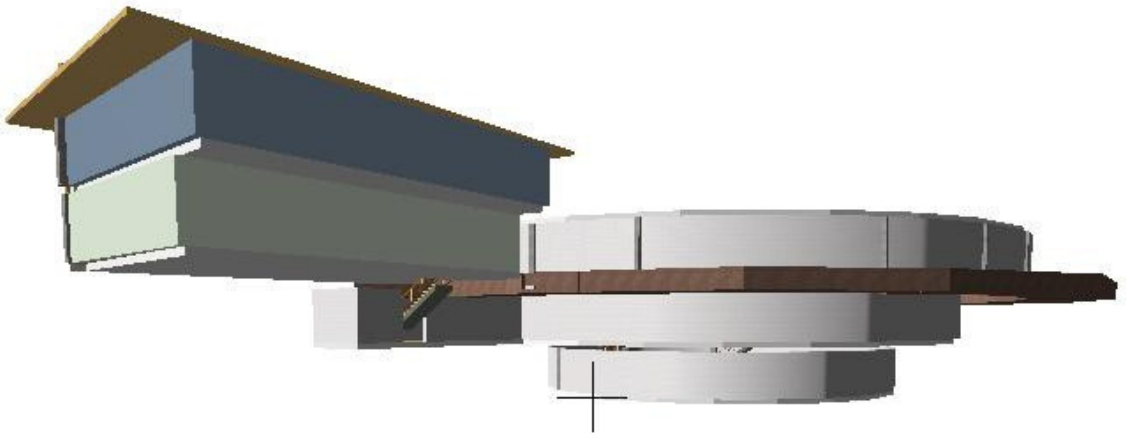
(c) Groundfloor level:



(d) Level 1:

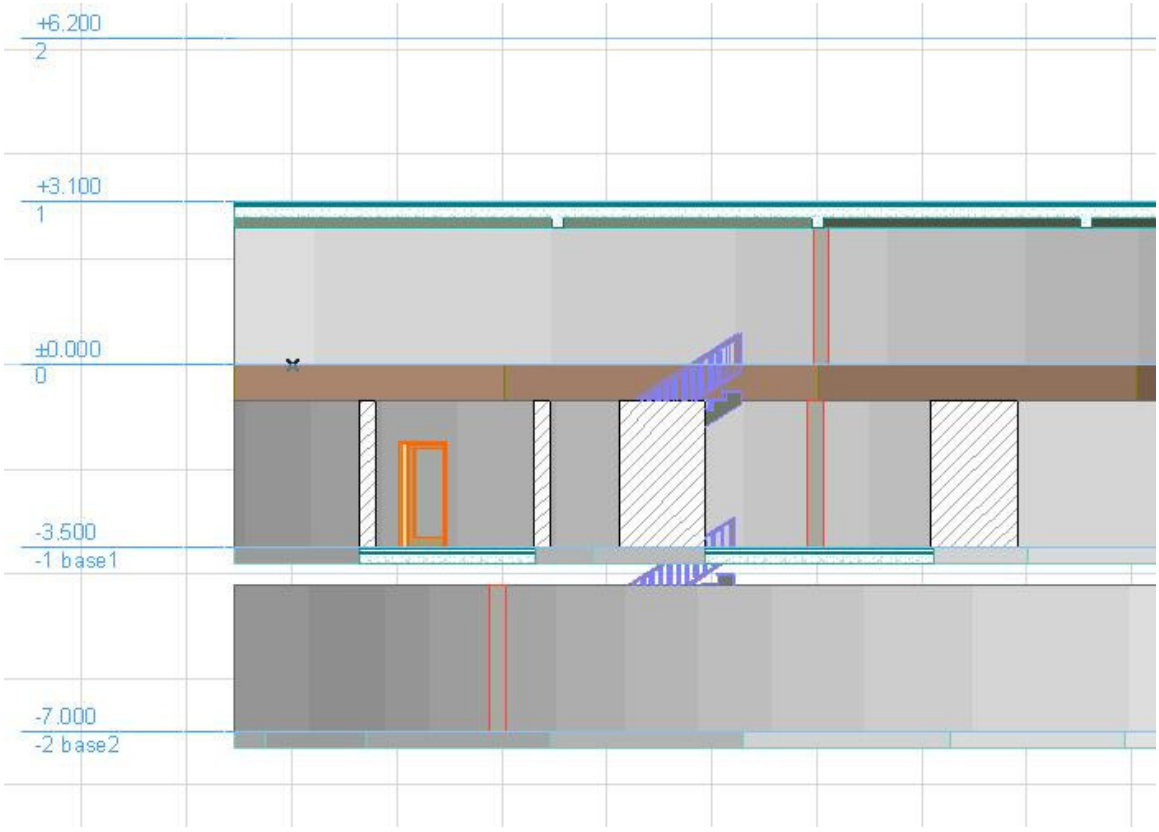


(e) Sideview

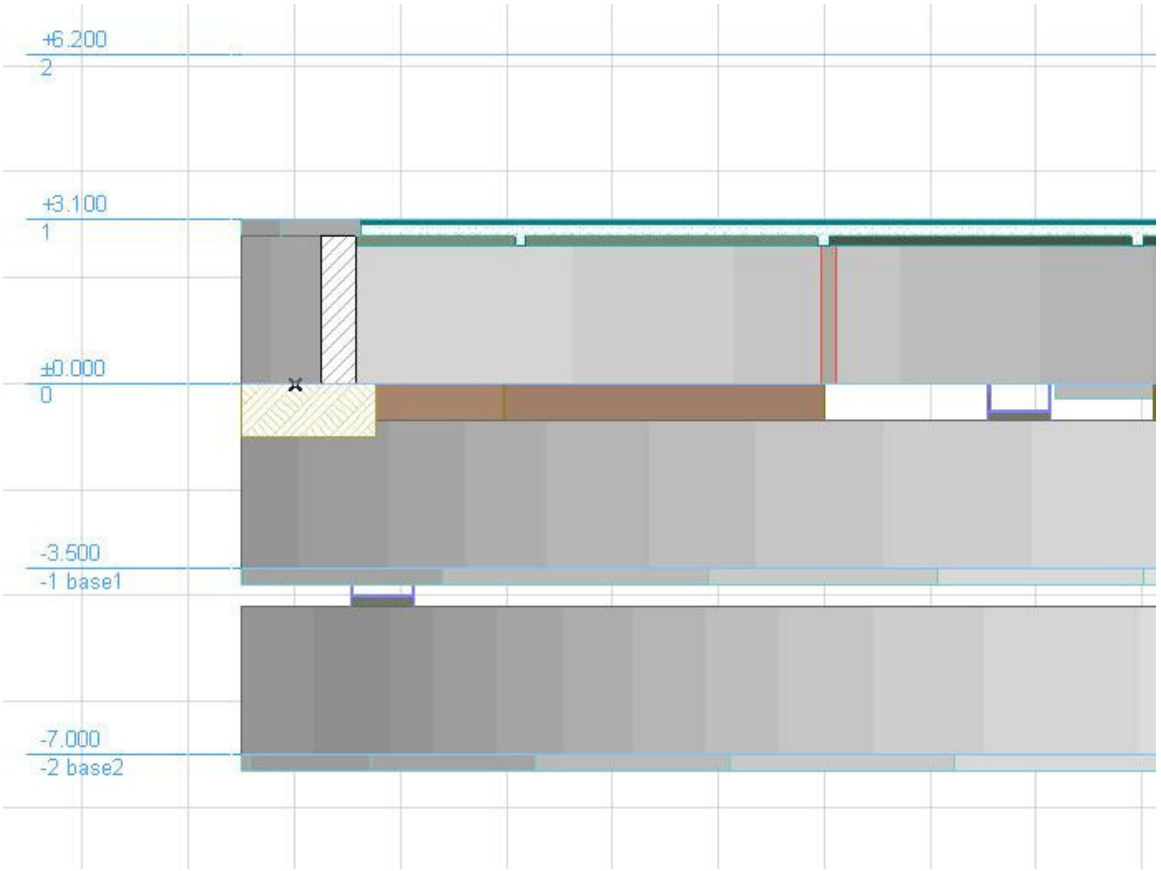


(f) North section

(g) South section



(h) East section



First Lunar Space Base 2009

[Project title]

HA08_PEUKC

[Office ID]

Project information

Project data

City
Moon

Country
United States

Type
Architecture (research/devel)

Status of planning
Final design stage

Status of formal permission
Approval/license not required

Estimated start of construction
Dec '09

Last modfied
Feb 29, 2008

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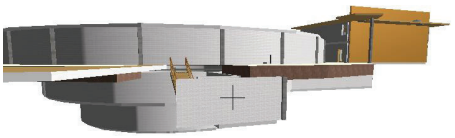

Right front-side view

Further author(s) & legal guardian(s)

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Prize money distribution statement
60% for the first author 40% for the second author

Project description

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Therefore some institutions have begun to study possibility to send mankind to space for long-time period. A year long experiment of mankind capability to survive in space has been conducted by International Space Station (NASA-ISS).

Then the next logical step would be how to find good location of international space base, possibly in the Moon surface. Therefore we design an imaginative (preliminary) design for the First Lunar Space Base 2009.

We acknowledge that according to the competition rule, a design shall have high probability to construct. But considering this program is likely to yield great interests not only for government and private sectors, therefore it is possible to conduct auction to put this design into reality. Alternative method to finance is to use some roof space of this proposed space-base for advertising space. We expect that plenty of corporations would like to get their ads printed on the First Lunar Space Base, just like corporations put ads in the body of Russian space rocket.

Therefore we expect that probably this design can be used in the next space station conducted by NASA-ISS [1][2]

This proposed design is using 'avant-garde' art design, i.e. non-art is also part of Art (Smarandache's Paradoxist theme). In this context, beautiful design is merely beyond just 'aesthetical accomplishment', but consistency with the purported theme.

29 February 2008
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On recent discovery of new planetoids in the solar system and quantization of celestial system

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The present note revised an old article discussing new discovery of a new planetoid in the solar system. Some recent discoveries have been included, and their implications in the context of quantization of celestial system are discussed, in particular from the viewpoint of superfluid dynamics. In effect, it seems that there are reasons to argue in favor of gravitation-related phenomena from boson condensation.

Keywords: quantization, planetary orbit, quantized superfluid, boson condensation, gravitation

Discovery of new planetoids

Discovery of new objects in the solar system is always interesting for astronomers and astrophysicists alike, not only because such discovery is very rare, but because it also presents new observation data which enables astronomers to verify what has been known concerning how our solar system is functioning.

In recent years a number of new planetoids have been reported, in particular by M. Brown and his team [1][2][3][4]. While new planet discoveries have been reported from time to time, known as *exoplanets* [9][10], nonetheless discovery of new planetoids in the solar system are very interesting, because they are found after a long period of silence after Pluto finding, around seventy years ago. Therefore, it seems interesting to find out implications of this discovery to our knowledge of solar system, in particular in the context of quantization of celestial system.

As we discussed in on old article, there are some known methods in the literature to predict planetary orbits using quantum wave-like approach, instead of classical dynamics approach. These new approaches have similarity, i.e. they extend the Bohr-Sommerfeld's quantization of angular momentum to large-scale celestial systems. This application of wave mechanics to large-scale structures [6] has led to several impressive results in particular to predict orbits of exoplanets [8][9][10]. However, in the present note we will not discuss again the physical meaning of wave mechanics of such large-scale structures, but instead to focus on discovery of new planetoids in solar system in the context of quantization of celestial system.

As contrary as it may seem to present belief that it is unlikely to find new planets beyond Pluto, Brown *et al*. have reported not less than four new planetoids in the outer side of Pluto orbit, including 2003EL61 (at 52AU), 2005FY9 (at 52AU), 2003VB12 (at 76AU, dubbed as *Sedna; i*t is somewhat different to our preceding article suggesting orbit distance = 86AU in accordance with ref. [14]). And recently Brown and his team reported new planetoid finding, dubbed as 2003UB31 (97AU). This is not to include *Quaoar* (42AU), which has orbit distance more or less near Pluto (39.5AU), therefore this object is excluded from our discussion. Before discovery of 2003UB31

(Brown himself prefers to call it '*Lila*'), Sedna has been reported as the most distant object found in the solar system, but its mass is less than Pluto, therefore one could argue whether it could be considered as a 'new planet'. But 2003UB31 is reported to have mass definitely greater than Pluto, therefore Brown argues that it is definitely worth to be considered as a 'new planet'. (See Table 1.)

Table 1. Comparison of prediction and observed orbit distance of planets in the Solar system (in 0.1AU unit)

Object	No.	Titius	Nottale	CSV	Observed	Δ (%)
	1		0.4	0.428		
	2		1.7	1.71		
Mercury	3	4	3.9	3.85	3.87	0.52
Venus	4	7	6.8	6.84	7.32	6.50
Earth	5	10	10.7	10.70	10.00	-6.95
Mars	6	16	15.4	15.4	15.24	-1.05
Hungarias	7		21.0	20.96	20.99	0.14
Asteroid	8		27.4	27.38	27.0	1.40
Camilla	9		34.7	34.6	31.5	-10.00
Jupiter	2	52		45.52	52.03	12.51
Saturn	3	100		102.4	95.39	-7.38
Uranus	4	196		182.1	191.9	5.11
Neptune	5			284.5	301	5.48
Pluto	6	388		409.7	395	-3.72
2003EL61	7			557.7	520	-7.24
Sedna	8	722		728.4	760	4.16
2003UB31	9			921.8	970	4.96
Unobserved	10			1138.1		
Unobserved	11			1377.1		

Moreover, from the viewpoint of quantization of celestial systems, these findings provide us with a set of unique data to be compared with our prediction based on CSV hypothesis [5]. It is therefore interesting to remark here that all of those new 'planetoids' are within 8% bound compared to our prediction (Table 1). While this result does not yield high-precision accuracy, one could argue that this 8% bound limit corresponds to the remaining planets, including inner planets. Therefore this 8% uncertainty could be attributed to macroquantum uncertainty and other local factors.

What's more interesting here is perhaps that some authors have argued using gravitational Schrödinger equation [12], that it is *unlikely to find new planets beyond Pluto* because density distribution becomes near zero according to the solution of Schrödinger equation [7][8][11]. From this viewpoint, one could argue concerning to how extent applicability of gravitational Schrödinger equation to predict quantization of celestial systems, despite its remarkable usefulness to predict exoplanets [9][10].

Therefore in the subsequent section, we argue that using Ginzburg-Landau equation, which is more consistent with superfluid dynamics, one could derive similar result with known gravitational Bohr-Sommerfeld quantization [13][15]:

$$a_n = GMn^2 / v_o^2 \tag{1}$$

where a_n,G,M,n,v_o each represents orbit radius for given n, Newton gravitation constant, mass of the Sun, quantum number, and specific velocity (v_o=144 km/sec for Solar system and also exoplanet systems), respectively [7][8].

Interpretation

In principle the Cantorian superfluid vortex (CSV) hypothesis [5] suggests that the quantization of celestial systems corresponds to superfluid quantized vortices, where it is known that such vortices are subject to quantization condition of integer multiples of 2π, or $\oint v_s.dl = 2\pi.n\hbar / m_4$ [5]. For a *planar cylindrical case* of solar system, this hypothesis leads to Bohr-Sommerfeld-type quantization of planetary orbits. It is also worthnoting here, while likelihood to find planetoid at around 90AU has been predicted by some astronomers, our prediction of new planets corresponding to n=7 (55.8AU) and n=8 (72.8AU) were purely derived from Bohr-Sommerfeld quantization [5].

The CSV hypothesis starts with observation that in quantum fluid systems like superfluidity, quantized vortices are distributed in equal distance, which phenomenon is known as vorticity. In a large superfluid system, we usually use Landau two-fluid model, with normal and superfluid component. Therefore, in the present note we will not discuss again celestial quantization using Bohr-Sommerfeld quantization, but instead will derive equation (1) from Ginzburg-Landau equation, which is known to be more consistent with superfluid dynamics. To our knowledge, deriving equation (1) from Ginzburg-Landau equation has *never* been made before elsewhere.

According to Gross, Pitaevskii, Ginzburg, wavefunction of N bosons of a reduced mass m* can be described as [17]:

$$-(\hbar^2 / 2m^*).\nabla^2\psi + \kappa|\psi|^2\psi = i\hbar.\partial\psi/\partial t \qquad (2)$$

For some conditions, it is possible to substitute the potential energy term ($\kappa|\psi|^2$) in (2) by Hulthen potential, which yields:

$$-(\hbar^2 / 2m^*).\nabla^2\psi + V_{Hulthen}.\psi = i\hbar.\partial\psi/\partial t \qquad (3)$$

where Hulthen potential could be written in the form:

$$V_{Hulthen} = -Ze^2.\delta.e^{-\delta r}/(1-e^{-\delta r}) \quad (4)$$

It could be shown that for small values of screening parameter δ, the Hulthen potential (4) approximates the effective Coulomb potential:

$$V_{Coulomb}^{eff} = -e^2/r + \ell(\ell+1).\hbar^2/(2mr^2) \quad (5)$$

Therefore equation (3) could be rewritten as:

$$-\hbar^2\nabla^2\psi/2m^* + \left[-e^2/r + \ell(\ell+1).\hbar^2/(2mr^2)\right]\psi = i\hbar.\partial\psi/\partial t \quad (6)$$

Interestingly, this equation takes the form of time-dependent Schrödinger equation. In the limit of time-independent case, equation (6) becomes similar with Nottale's time-independent gravitational Schrödinger equation from Scale relativistic hypothesis with Kepler potential [7][8][9]:

$$2D^2\Delta\Psi + (E/m + GM/r).\Psi = 0 \quad (7)$$

Solving this equation with Hulthen effect (4) will make difference, but for gravitational case it will yield different result only at the order of 10^{-39} m compared to prediction using equation (7), which is of course negligible. Therefore, we conclude that for most celestial quantization problems the result of TDGL-Hulthen (3) is *essentially* the same with the result derived from equation (7).

Furthermore, the extra potential to Keplerian potential in equation (5) is also negligible, in accordance with Pitkanen's remarks: "centrifugal potential $l(l+1)/r^2$ in the Schrödinger equation is negligible as compared to the potential term at large distances so that one expects that degeneracies of orbits with small values of l do not depend on the radius." [18]

It seems also worth noting here that planetoids 2003EL61 and 2005FY9 correspond to orbit distance of 52AU. This pair of planetoids could also be associated with Pluto-Charon pair. In the context of macroquantum phenomena of condensed matter physics,

one could argue whether these pairs indeed correspond to macroobject counterpart of Cooper pairs [16]. While this conjecture remains open for discussion, we predict that more paired-objects similar to these planetoids will be found beyond Kuiper belt. This will be interesting for future observation.

Furthermore, while our previous prediction only limits new planetoids finding until n=9 of Jovian planets (outer solar system), it seems that there are more than sufficient reasons to expect that more planetoids are to be found in the near future. Therefore it is recommended to extend further the same quantization method to larger n values. For prediction purpose, we have included in Table 1 new expected orbits based on the same celestial quantization as described above. For Jovian planets corresponding to n=10 and n=11, our prediction yields likelihood to find orbits around 113.81 AU and 137.71 AU, respectively. It is recommended therefore, to find new objects around these predicted orbits.

In this note, we revised our preceding article suggesting that Sedna corresponds to orbit distance 86AU, and included recently found planetoids in the outer solar system as reported by Brown *et al*. While our previous prediction only limits new planet finding until n=9 corresponding to outer solar system, it seems that there are reasons to expect that more planetoids are to be found. While in the present note, we argue in favor of superfluid-quantized vortices, it does not mean to be the only plausible approach. Instead, we consider this discovery as a new milestone to lead us to find better cosmological theories, in particular taking into consideration some recent remarkable observation of exoplanets as predicted by wave mechanics approach.

Acknowledgment

Special thanks go to Profs. C. Castro, M. Pitkanen, R.M. Kiehn and A Rubcic for their insightful suggestion and remarks.

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OPEN AND SOLVED ELEMENTARY QUESTIONS IN ASTRONOMY

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Student:
1) Let's consider a tunnel getting from a side to the other side of the Earth, and passing through the center of the Earth.

a) If one drops an object in the tunnel, will the object stop at the center of the Earth, or will oscillate like a pendulum about the center, up and down, and after a while will stop?
Will then the object float in the center?

Instructor:
Yes, we solved this problem in school using methods of Classical Mechanics.

Student:
b) If an elevator is freely left down in the tunnel, how much force would be necessary to push it up (especially from the Earth center) to the second side of the Earth?
Isn't any inertial force, from the falling force that might push the elevator beyond the Earth center towards the other side?

Instructor:
No. It is school problem too. Inertial forces will act on a body if only this body will be linked to this inertial field. Not this case.

Student:
c) Suppose the second side of the tunnel gets in the bottom of an ocean. Will the water flow down into the tunnel only up to the center of the Earth, or even lower near to the first side (to compensate/equilibrate somehow, about the Earth center, the water masses from both sides of the Earth center), or even will flood out the first side?

Instructor:
The water will oscillate like a pendulum in the first question.
Then the water will fill only ½ of the tunnel, so that part which is between the Earth centre and the ocean. If ocean is located at both sides of the tunnel, the tunnel will be highly full.

Student:
d) The above three questions for the case when the tunnel gets from a side to another side of the Earth, but doesn't pass through the Earth center. Would the midpoint of the tunnel play a similar role as the Earth center in the above three questions?

Instructor:
Yes, of course.

Student:
e) How will Coriolis force influence this?

Instructor:
Very little. The force of gravity is greater.

Student:
2) Is it possible to accelerate a photon (or another particle traveling at, say, 0.99c) and thus to get speed greater than c?

Instructor:
This is "double-question", linked to "no-speed-barrier" thesis. I mean it follows. General Relativity is the theory on **observable quantities**. **Absolute quantities** are also presented there - **absolute rotation** and **the deformation of the space**. This is well-known fact.
So, I think, you can accelerate a particle at 0.99c (where c = speed of light) and then more and more, but its observable motion
will asymptotically close to the light velocity anyway. The below is citation from Rabounski-Borissova's book "Particles here and beyond the mirror", for understanding this problem. They called it the **Blind Pilot Principle**:

"…We can outline a few types of frames of reference which may exist in General Relativity space-time. Particles (including the observer themselves), which travel at sub-light speed ("inside" the light cone), bear real relativistic mass. In other words, the particles, the body of reference and the observer are in the state of matter commonly referred to as "substance". Therefore any observer whose frame of reference is described by such monad will be referred to as *sub-light speed* (substantional) *observer*."

Particles and the observer that travel at the speed of light (i.e. over the surface of light hyper-cone) bear $m_0=0$ but their relativistic mass (mass of motion) $m\neq0$. They are in light-like state of matter. Hence we will call an observer whose frame of reference is characterized by such monad a *light-like observer*.

Accordingly, we will call particles and the observer that travel at super-light speed *super-light particles and observer*. They are in the state of matter for which $m_0\neq0$ but their relativistic mass is imaginary.

It is intuitively clear who a sub-light speed observer is, the term requires no further explanations. Same more or less applies to light-like observer.

From the point of view of light-like observer, the world around looks like colorful system of light waves. But who is a super-light observer? To understand this let us give an example.

Imagine a new supersonic jet plane to be commissioned into operation. All members of the commission are inborn blind. And so is the pilot. Thus we may assume that all information about the surrounding world the pilot and the members of the commission gain from sound that is from transversal waves in air. It is sound waves that build a picture that those people will perceive as their “real world".

Now the plane took off and began to accelerate. As long as its speed is less than the speed of sound, the blind members of the commission will match its ``heard" position in the sky to the one we can see. But once the sound barrier is overcome, everything changes. Blind members of the commission will still perceive the speed of the plane equal to the speed of sound regardless to its real speed. For the speed of propagation of sound waves in the air will be the *maximum speed of propagation of information* while the real supersonic jet plane will be beyond their ``real world" in the world of ``imaginary objects" and all its properties will be imaginary too.
The blind pilot will hear nothing as well. Not a single sound will reach him from the past reality and only local sounds from the cockpit (which also travels at the supersonic speed) will break the silence. Once the speed of sound is overcome, the blind pilot leaves the subsonic world for a new supersonic one. From his new viewpoint (supersonic frame of reference) the old subsonic fixed world tat contains the airport and the members of the commission will simply disappear to become an area of “imaginary values".

What is light? Transversal waves that run across a certain medium at a constant speed. We perceive the world around through sight, receiving light waves from other objects. It is waves of light that build our picture of the “true real world".

Now imagine a spaceship that accelerates faster and faster to eventually overcome the light barrier at still growing speed. From pure mathematical viewpoint this is quite possible in the space-time of General Relativity.
For us the speed of the spaceship will be still equal to the speed of light whatever its real speed is. For us the speed of light will be the maximum speed of propagation of information and the real spaceship for us will stay in another “unreal" world of super-light speeds where all properties are imaginary. The same is true for the spaceship's pilot. From his viewpoint having the light barrier overcome brings him into a new super-light world that becomes his “true reality". And the old world of sub-light speeds is gone to the area of “imaginary reality".

Student:
3) Would ever be possible to construct a flexible bridge between two planets and thus have "terrestrial" traffic between them? I mean what about gravity field of each planet (how to smoothly escape from one field and smoothly enter into the other filed)?
Another difficulty would be that planets are moving…

Instructor:
Yes, of course. Similar projects were developed from 1960’s. I mean Space Bridge or Space Lift, which will link the Earth surface and a satellite in a geostationary orbit (which is located over the

same point of the terrestrial equator) A cabin in such lift, moving upstairs, will be partially moved by inertial force, partially by a machine. The problem was a cable.
Geostationary satellites can be located in stable state in very high orbits, about 20,000 km minimum. Steel cable, linking the satellite and the terrestrial surface must be about 3metre in diameter. Then, in 60's, such projects had been stopped. Now it started again, because the recent developments of carbon cables give possibility to make such cable of necessary properties only centimeter width. I heard numerous commercial corporations started such projects the last year. They think to begin use the 1st Space Bridge in 2015.

Student:
4) Suppose we are able to dig around and cut our planet into two separated parts. In the first case, suppose the two parts are equal, while in the second case one part is much bigger than the second one. Will these parts mutually attract back to re-form a single planet, or will they separate each other?

Instructor:
It depends on those conditions in which they will be after the divorce. If they will be in relative rotation at a velocity, necessary for that they would be in stable condition - inertial forces will put the gravity force in equilibrium, then they will not form a unitary plane.
If they will not rotate, then they will join into a single planet.

Student:
5) Why from the Earth the Moon is seeing up, and from the Moon the Earth is seeing up too? (Let's consider a fixed point on the Earth; we are able to see the Moon from that point only when it is above the Earth, because when the Moon is diametrically opposed it cannot be seeing from that Earth point.
{Similarly when we consider a fixed point on the Moon where the Earth is visible from.}

Instructor:
I don't understand exactly…

BIOLOGY

Statistical modelling of primary Ewing tumours of the bone

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Abstract

This short technical paper advocates a bootstrapping algorithm from which we can form a statistically reliable opinion based on limited clinically observed data, regarding whether an osteo-hyperplasia could actually be a case of Ewing's osteosarcoma. The basic premise underlying our methodology is that a primary bone tumour, if it is indeed Ewing's osteosarcoma, cannot increase in volume beyond some critical limit without showing metastasis. We propose a statistical method to extrapolate such critical limit to primary tumour volume. Our model does not involve any physiological variables but rather is entirely based on time series observations of increase in primary tumour volume from the point of initial detection to the actual detection of metastases.

Key words
Ewing's bone tumour, multi-cellular spheroids, linear difference equations

I. Introduction

To date, oncogenetic studies of *EWS/FLI-1*1 induced malignant transformation have largely relied upon experimental manipulation of Ewing's bone tumour cell lines and fibroblasts that have been induced to express the oncogene. It has been shown that the biology of Ewing's tumour cells *in vitro* is dramatically different between cells grown as mono-layers and cells grown as anchorage-independent, multi-cellular spheroids (MCS). The latter is more representative of primary Ewing's tumour *in vivo* (Lawlor et. al, 2002).

MCS are clusters of cancer cells, used in the laboratory to study the early stages of avascular tumour growth. Mature MCS possess a well-defined structure, comprising a central core of necrotic i.e. dead cells, surrounded by a layer of non-proliferating, quiescent cells, with proliferating cells restricted to the outer, nutrient-rich layer of the tumour. As such, they are often used to assess the efficacy of new anti-cancer drugs and treatment therapies. The majority of mathematical models focus on the growth of MCS or avascular tumour growth. Most recent works have focused on the evolution of MCS growing in response to a single, externally-supplied nutrient, such as oxygen or glucose, and usually two growth inhibitors.

Mathematical models of MCS growth typically consist of an *ordinary differential equation* (ODE) coupled to one or more *reaction-diffusion equations* (RDEs). The ODE is derived from mass conservation and describes the evolution of the outer tumour boundary, whereas the RDEs describe the distribution within the tumour of vital nutrients

such as oxygen and glucose and growth inhibitors (Dorman and Deutsch, 2002). However studies of this type, no matter how mathematically refined, often fall short of direct clinical applicability because of rather rigorous restrictions imposed on the boundary conditions. Moreover, these models focus more on the structural evolution of a tumour that is already positively classified as cancerous rather than on the clinically pertinent question of whether an initially benign growth can at a subsequent stage become invasive and show metastases (De Vita et. al., 2001).

What we therefore aim to devise in our present paper is a bootstrapping algorithm from which we can form an educated opinion based on clinically observed data, regarding whether a bone growth initially diagnosed as benign can subsequently prove to be malignant (i.e. specifically, a case of Ewing's osteosarcoma) . The strength of our proposed algorithm lies mainly in its computational simplicity – our model does not involve any physiological variables but is entirely based on time series observations of progression in tumour volume from the first observation point till detection of metastases.

II. Literature support

In a clinical study conducted by Hense et. al. (1999), restricted to patients with suspected Ewing's sarcoma, tumour volumes of more than 100 ml and the presence of primary metastases were identified as determinants of poor prognosis in patients with such tumours. Diagnoses of primary tumours were ascertained exclusively by biopsies. The diagnosis of primary metastases was based on *thoracic computed tomography* or on

whole body bone scans. It was observed that of 559 of the patients (approx. 68% in a total sample size of 821) had a volume above 100 ml with smaller tumours being more common in childhood than in late adolescence and early adulthood. Extensive volumes were observed in almost 90% of the tumours located in femur and pelvis while they were less common in other sites ($p < 0.001$). On average, 26% of all patients were detected with clinically apparent primary metastases.

The detection rate of metastases was markedly higher in patients diagnosed after 1991 ($p < 0.001$). Primary metastases were also significantly more common for tumours originating in the pelvis and for other tumours in the Ewing's family of tumours (EFT); mainly the peripheral neuro-ectodermal tumours (PNET); ($p < 0.01$). Tumours greater than 100 ml were positively associated with metastatic disease ($p < 0.001$). Multivariate analyses, which included simultaneously all univariate predictors in a *logistic regression model*, indicated the observed associations were mostly unconfounded.

Further it has been found that the metastatic potential of human tumours is encoded in the bulk of a primary tumour, thus challenging the notion that metastases arise from sparse cells within a primary tumour that have the ability to metastasize (Sridhar Ramaswamy et. al., 2003). These studies lend credence to our fundamental premise about a critical primary tumour volume being used as a classification factor to distinguish between benign and potentially malignant bone growth.

III. Statistical modelling methodology

Assuming that the temporal drift process governing the progression in size of a primary Ewing tumour of the bone to be linear (the computationally simplest process), we suggest a straightforward computational technique to generate a large family of possible tumour propagation paths based on clinically observed growth patterns under laboratory conditions. In case the governing process is decidedly non-linear, then our proposed scheme would not be applicable and in such a case one will have to rely on a completely non-parametric classification technique like e.g. an Artificial Neural Network (ANN).

Our proposed approach is a bootstrapping one, whereby a linear autoregression model is fitted through the origin to the observation data in the first stage. If one or more beta coefficients are found to be significant at least at a 95% level for the fitted model then, in the second stage, the autoregression equation is formulated and solved as a *linear difference equation* to extract the governing equation.

In the final stage, the governing equation obtained as above is plotted, for different values of the constant coefficients, as a family of possible temporal progression curves generated to explain the propagation property of that particular strain of tumour. The critical volume of the primary growth can thereafter be visually extrapolated from the observed cluster of points where the generated family of primary tumour progression curves show a *definite uptrend* vis-a-vis the actual progression curve.

If no beta coefficient is found to be significant in the first stage, a non-linear temporal progression process is strongly suspected and the algorithm terminates without proceeding onto the subsequent stages, thereby implicitly recommending the problem to a non-parametric classification model.

The mathematical structure of our proposed model may be given as follows:

Progression in primary Ewing tumour size over time expressed as an n-step general autoregressive process through the origin:

$$S_t = \sum_{j=1}^{n} \beta_j S_{t-j} + \varepsilon \qquad \text{(I)}$$

Formulated as a linear, difference equation we can write:

$$-S_t + \beta_1 S_{t-1} + \beta_2 S_{t-2} + \ldots + \beta_n S_{t-n} = -\varepsilon \qquad \text{(II)}$$

Taking S_t common and applying the *negative shift operator* throughout, we get:

$$[-1 + \beta_1 E^{-1} + \beta_2 E^{-2} + \ldots + \beta_n E^{-n}] S_t = -\varepsilon \qquad \text{(III)}$$

Now applying the *positive shift operator* throughout we get:

$$[-E^n + \beta_1 E^{n-1} + \beta_2 E^{n-2} + \ldots + \beta_n] S_t = -\varepsilon \qquad \text{(IV)}$$

The *characteristic equation* of the above form is then obtained as follows:

$$-r^n + \beta_1 r^{n-1} + \beta_2 r^{n-2} + \ldots + (\beta_n + \varepsilon) = 0 \qquad \text{(V)}$$

Here r is the root of the characteristic equation. After solving for r, the governing equation can be derived in accordance with the well-known analytical solution techniques for ordinary linear difference equations (Kelly and Peterson, 2000).

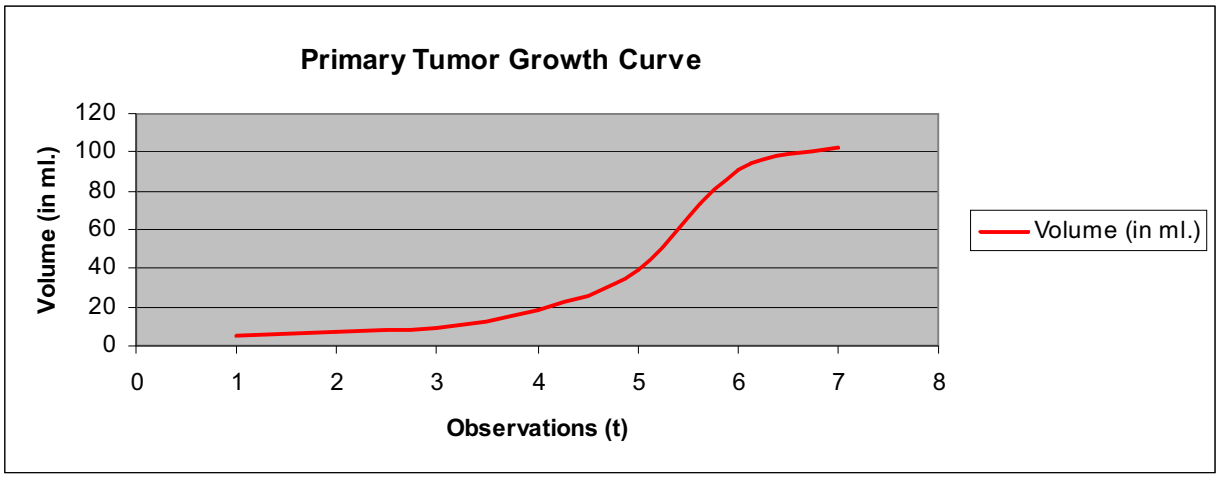
IV. Simulated clinical study

We set up a simulated clinical study applying our modelling methodology with the following hypothetical primary Ewing tumour progression data adapted from the clinical study of Hense et. al. (1999) as given in Table I below:

Table I

Observation (t)	Primary Ewing tumour volume (in ml.)
(At point of first detection) 1	5
2	7
3	9
4	19
5	39
6	91
7 (At the point of detection of metastasis)	102

Figure I



The temporal progression path of the primary growth from the point of first detection to the onset of metastasis is plotted above in Figure I.

We have fitted an AR (2) model to the primary tumour growth data as follows:

$$E(S_t) = -1.01081081S_{t-1} + 5.32365561S_{t-2} \quad \text{(VI)}$$

The R^2 of the fitted model is approximately 0.8311 and the *F*-statistic is 9.83832 with an associated p-value of approximately 0.04812. Therefore the fitted model definitely has an overall predictive utility at the 5% level of significance.

The residuals of the above AR (2) fitted model are given in Table II as follows:

Table II

Observation	***Predicted S_t***	***Residuals***
1	-5.05405405	12.05405405
2	19.5426024	-10.5426024
3	28.168292	-9.168292003
4	28.7074951	10.29250488
5	61.7278351	29.27216495
6	115.638785	-13.63878518

The average of the residuals comes to 3.044841. Therefore the linear difference equation to be solved in this case is as follows:

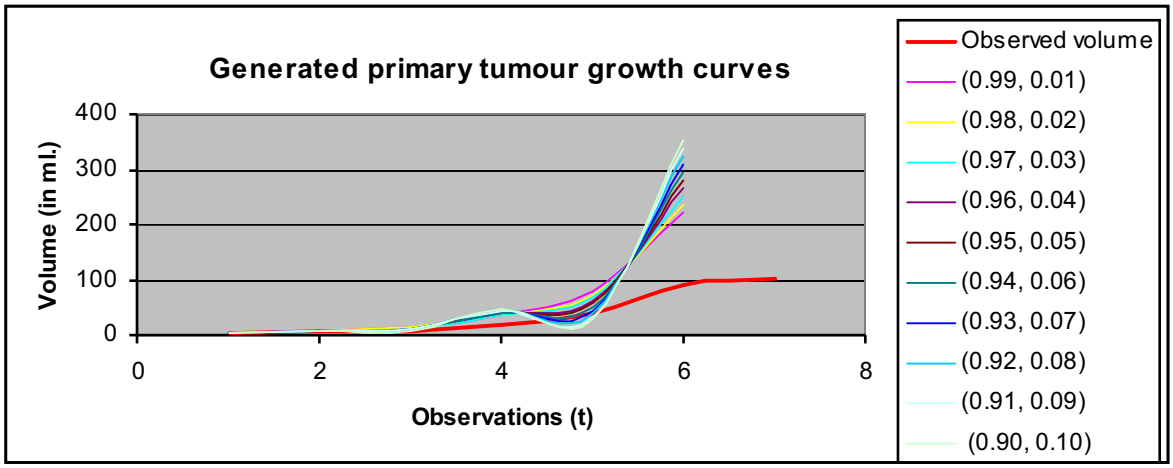
$$X_t = -1.01081081X_{t-1} + 5.32365561X_{t-2} + 3.044841 \quad \text{(VII)}$$

Applying usual solution techniques, the general solution to equation (VII) is obtained as follows:

$$X_t = c_1 (2.43124756)^t + c_2 (-3.44205837)^t \qquad \text{(VIII)}$$

Here c_1 and c_2 are the constant coefficients which may now be suitably varied to generate a family of possible primary tumour progression curves as in Figure II below:

Figure II



In the above plot, we have varied c_2 in the range 0.01 to 0.10 and imposed the condition $c_1 = 1 - c_2$. The other obvious condition is that choice of c_1 and c_2 would be such as to rule out any absurd case of negative volume. Of course the choice of the governing equation parameters would also depend on specific clinical considerations (King, 2000).

V. Conclusion

From Figure II, it becomes visually apparent that continuing increase in the observed size of the primary growth beyond approximately 52 ml. in volume would be potentially malignant as this would imply that the tumour would possibly keep exhibiting uncontrolled progression till it shows metastasis. This could also be obtained arithmetically as the average volume for t = 5. Therefore the critical volume could be fixed around 52 ml. as per the computational results obtained in our illustrative example.

Though our computational study is intended to be purely illustrative as we have worked with hypothetical figures and hence cannot yield any clinical conclusion, we believe we have hereby aptly demonstrated the essential algorithm of our statistical approach and justified its practical usability under laboratory settings. We have used a difference equation model rather than a differential equation one because under practical laboratory settings, observations cannot be made continuously but only at discrete time intervals. There is immediate scope of taking our line of research further forward by actually implementing an autoregressive process to model *in vitro* growth of MCS with real data.

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CALCULUS

A Triple Inequality with Series and Improper Integrals

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Abstract.

As a consequence of the Integral Test we find a triple inequality which bounds up and down both a series with respect to its corresponding improper integral, and reciprocally an improper integral with respect to its corresponding series.

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1. Introduction.

Before going in details to this triple inequality, we recall the well-known
Integral Test that applies to positive term series:
For all $x \geq 1$ let $f(x)$ be a positive continuous and decreasing function such that $f(n) = a_n$ for $n \geq 1$. Then:

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx \tag{1}$$

either both converge or both diverge.

Following the proof of the Integral Test one easily deduces our inequality.

2. Triple Inequality with Series and Improper Integrals.

Let's first make the below notations:

$$S = \sum_{n=1}^{\infty} a_n \ , \tag{2}$$

$$I = \int_1^{\infty} f(x)dx \ . \tag{3}$$

We have the following
Theorem (Triple Inequality with Series and Improper Integrals):

For all $x \geq 1$ let f(x) be a positive continuous and decreasing function such that $f(n) = a_n$ for $n \geq 1$. Then:

$$S - f(1) \leq I \leq S \leq I + f(1) \tag{4}$$

Proof.

We consider the closed interval [1, n] the function f is defined on split into n-1 unit subintervals [1, 2], [2, 3], …, [n-1, n], and afterwards the total area of the rectangles of width 1 and length f(k), for $2 \leq k \leq n$, inscribed into the surface generated by the function f and limited by the x-axis and the vertical lines x = 1 and x = n:

$$S_{inf} = \sum_{k=2}^{n} f(k) = f(2) + f(3) + \ldots + f(n) \quad \text{[inferior sum]} \tag{5}$$

and respectively the total area of the rectangles of width 1 and length f(k), for $1 \leq k \leq n-1$, inscribed into the surface generated by the function f and limited by the x-axis and the vertical lines x = 1 and x = n:

$$S_{sup} = \sum_{k=1}^{n-1} f(k) = f(1) + f(3) + \ldots + f(n-1) \quad \text{[superior sum]} \tag{6}$$

But the value of the improper integral $\int_1^{\infty} f(x)dx$ is in between these two summations:

$$S_n - f(1) = S_{inf} \leq \int_1^n f(x)dx \leq S_{sup} = S_{n-1} \tag{7}$$

where

$$S_n = \sum_{k=1}^{n} f(k). \tag{8}$$

Now in (7) computing the limit when $n \mapsto \infty$ one gets a double inequality which bounds up and down an improper integral with respect to its corresponding series:

$$S - f(1) \leq I \leq S \tag{9}$$

And from this one has

$$S \leq I + f(1) \tag{10}$$

Therefore, combining (9) and (10) we obtain our triple inequality:

$$S - f(1) \le I \le S \le I + f(1)$$

As a consequence of this, one has a double inequality which bounds up and down a series with respect to its corresponding improper integral, similarly to (9):

$$I \le S \le I + f(1) \tag{11}$$

Another approximation will be:

$$S_n \le S \le S_n + I_n \tag{12}$$

where

$$I_n = \int_n^{\infty} f(x)dx \quad \text{for } n \ge 1 \tag{13}$$

and $I_1 = I$, $S_1 = a_1 = f(1)$.
The bigger is n the more accurate bounding for S.

These inequalities hold even if both the series S and improper integral I are divergent (their values are infinite). According to the Integral Test if one is infinite the other one is also infinite.

3. An Application.

Apply the Triple Inequality to bound up and down the series:

$$S = \sum_{k=1}^{\infty} \frac{1}{k^2+4} \tag{14}$$

The function $f(x) = \frac{1}{x^2 + 4}$ is positive continuous and decreasing for $x \ge 1$. Its corresponding improper integral is:

$$I = \int_1^{\infty} \frac{1}{x^2 + 4} dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2 + 4} dx = \lim_{b\to\infty} \left[\frac{1}{2} \arctan \frac{x}{2}\right]_1^b$$

$$= \frac{1}{2} \lim_{b\to\infty} (\arctan \frac{b}{2} - \arctan \frac{1}{2}) = \frac{1}{2} (\frac{\pi}{2} - \arctan 0.5) \approx 0.553574.$$

Hence:

$$0.553574 = I \le S \le I + f(1) = 0.553574 + 1/(1^2 + 4) = 0.753574$$

or

$$0.553574 \leq S \leq 0.753574.$$

With a TI-92 calculator we approximate series (14) summing its first 1,000 terms and we get:

$$S_{1000} = \sum(1/(x^2+4),x,1,1000) = 0.659404.$$

Sure the more terms we take the better approach for the series we obtain.

4. Conclusion.

We found a triple inequality which helps approximates a series and in a similar way one can bound up and down an improper integral with respect to its corresponding series.

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IMMEDIATE CALCULATION OF SOME POISSON TYPE INTEGRALS USING SUPERMATHEMATICS CIRCULAR EX-CENTRIC FUNCTIONS

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0. ABSTRACT

This article presents two methods, in parallel, of solving more complex integrals, among which is the Poisson's integral, in order to emphasize the obvious advantages of a new method of integration, which uses the supermathematics circular ex-centric functions.

We will specially analyze the possibilities of easy passing/changing of the supermathematics circular ex-centric functions of a centric variable α to the same functions of ex-centric variable θ. The angle α is the angle at the center point O(0,0), which represents the centric variable and θ is the angle at the ex-center E(k,ε), representing the ex-centric variable. These are the angles from which the points W_1 and W_2 are visible on the unity circle – resulted from the intersection of the unity/trigonometric circle with the revolving straight line d around the ex-centric E(k,ε) – from O and from E, respectively.

KEYWORDS AND ABBREVIATIONS

C - Centric, Circular,
CC - Circular Centric,
E - Ex-centric,
EC - Ex-centric Circular,
F - Function,
H - Hyperbolic,
PI - Poisson Integral,
M - Mathematics,
CM - Centric Mathematics,
EM - Ex-centric Mathematics,
SM – Supermathematics,
FSM - **F** & **SM, FSM_EC- FSM** & **EC, FSM_EH-FSM** & **EH.**

1. INTRODUCTION

The discovery of the ex-centric mathematics (**EM**), as a vast extension of the centric common/mathematics (**CM**), which together form the SuperMathematics (**SM**), allows new simpler approaches, for resolving more complex integrals, among which we present (11) the Poisson integral (**PI**) [1]. To emphasize the new integration method, we will present, in parallel, the classic method of solving, only for **PI**, presented in [1] and the new method which utilizes **SM**'s ex-centric circular functions (**EC**) [2], [3], [4] .

The **SM-EC** functions, which will be in the center of our attention, are the radial ex-centric functions rex θ and Rex α and the derivatives ex-centric dex θ and Dex α, functions which are independent of the reference system selected.

The functions rex θ, of ex-centric variable θ, of the principal determination 1 and secondary 2, defined on the whole real axis for numeric ex-centricity $k^2 < 1$, and for $k^2 > 1$ exist only in the interval $\Im \in (\theta_i, \theta_f)$, in which $\theta_{f,i} = \pi + \varepsilon \pm \arcsin(1/k)$, $\alpha_{f,i} = \theta_{f,i} + \beta_{f,i}$ are

(1) $\text{rex}_{1,2}\,\theta = \text{rex}_{1,2}\,(\theta, E(k,\varepsilon)) = -k.\cos(\theta-\varepsilon) \pm \sqrt{1-k^2\sin^2(\theta-\eta)}$,

where **E(k, ε)** is a pole, called **ex-center**, which divides the straight line d ($d = d^+ \cup d^-$), revolving around this point, in the positive semi straight line d^+, on which is situated the first principal determination $\text{rex}_1\,\theta$, as function of ex-centric variable θ and, respectively, Rex α_1 , of centric variable α of the function and in the negative semi straight line d^-, on which is situated, along it, the second determination, secondary, of the function $\text{rex}_2\,\theta$ and Rex α_2. The expressions of the same entities (1), as functions of centric variable α, which exist on the whole real axis, no matter which is the numeric ex-centricity k, are

(2) $$\text{Rex}\,\alpha_{1,2} = \pm\sqrt{1+k^2-2k.\cos(\alpha_{1,2}-\varepsilon)}$$

These functions represent, as Prof. Dr. Math. Octav Gheorghiu observed, the **distance** in plane, as oriented segments, in polar coordinates, between two points: the ex-center E(k, ε) and the intersection points $W_{1,2}$ (1, $\alpha_{1,2}$) – between the straight line d and the unity circle CT[1,O(0,0)] with the center in the origin O of the system of coordinates axis, right Cartesian or polar reference point.

For an E which is interior to the unity disc, the segment EW_1 is situated on the positive direction of the semi straight line d^+, being, in this case, positive, that is Rex $\alpha_1 = \text{rex}_1\theta > 0$, while the oriented segment EW_2, positive oriented on the negative semi straight line is negative, that is Rex $\alpha_2 = \text{rex}_2\,\theta < 0$, as it can be seen in the Figure 1.

For $k = \pm 1$ at $\alpha_1 \in (0, 2\pi) \Rightarrow \theta \in (0, \pi)$ and at $\alpha_2 \in (0, 2\pi) \Rightarrow \theta \in (\pi, 2\pi)$. In other words, if, the straight line d rotates around $E(k,\varepsilon) \subset C(1,O)$ with an angular speed Ω ($\theta = \Omega t$), the points $W_{1,2}$ rotate on the unity circle *C*(1,O) with a double angular speed ($\alpha_{1,2} = 2\Omega t$) in half of the period and is stationary in the second half of the period ($\alpha_{1,2} = 0$) taking turns in E (k, ε).

If E is exterior to the unity disk, that is $|k| > 1$, then both determinations will be on the same semi straight line, being, successive, both positive and then, after the rotation of d of π, both negative, therefore are of the same sign, and this will make their product, in this case, positive, and, while in the precedent case, the product of the two determinations of the function was always negative (see Fig.1).

We must observe also that at $k > 1$ and for $\alpha_{1,2} \in (0, 2\pi) \Rightarrow \theta \in (\theta_i, \theta_f)$; the ex-centric variable θ diminishes the interval of existence of FSM-CE, between an initial value θ_i and a final one θ_f, with as much as the numeric growth of ex-centricity k. For $k \to \infty$ the interval is reduced to a single point on the real axis ***R***, for each determination.

The results presented so far, will also be obtained from the relations that will follow.

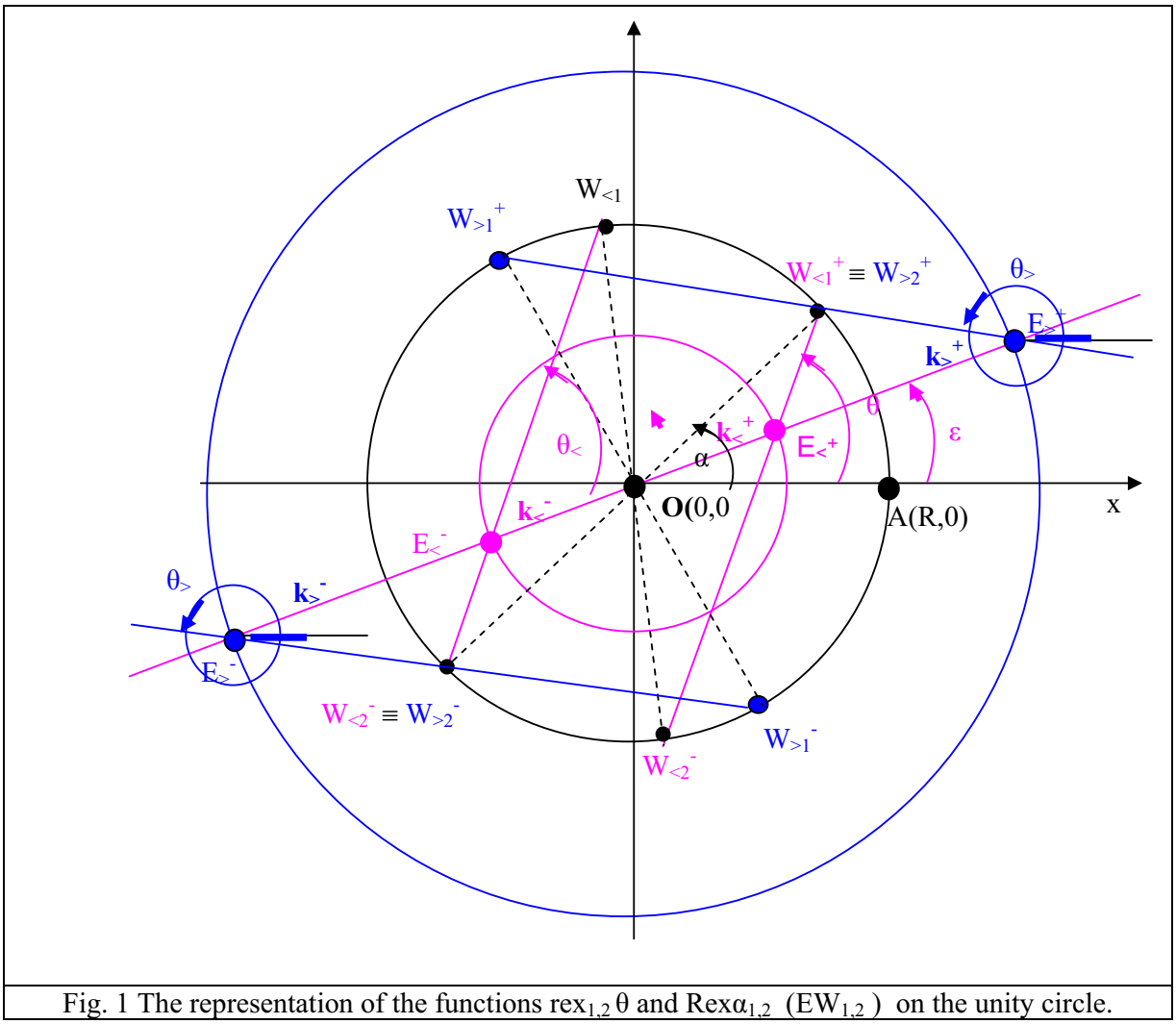

Fig. 1 The representation of the functions $rex_{1,2}\,\theta$ and $Rex\alpha_{1,2}$ ($EW_{1,2}$) on the unity circle.

The dependency between the two variables is:

(3) $\alpha_{1,2}(\theta) = \theta - \beta_{1,2}(\theta) = \theta \mp \arcsin[e.\sin(\theta - \varepsilon)]$

and, respectively

(4) $\theta(\alpha_{1,2}) = \alpha_{1,2} + \beta(\alpha_{1,2}) = \alpha_{1,2} + \arcsin\left(\frac{k.\sin(\alpha_{1,2}-\varepsilon)}{\pm\sqrt{1+k^2-2k.\cos(\alpha_{1,2}-\varepsilon)}}\right) =$

$= \alpha_{1,2} + \arcsin\left(\frac{k.\sin(\alpha_{1,2}-\varepsilon)}{Re\,x\alpha_{1,2}}\right)$

or

(4') $\theta(\alpha_{1,2}) = \alpha_{1,2} + \arctan\left(\frac{k.\sin(\alpha_{1,2}-\varepsilon)}{1-k.\cos(\alpha_{1,2}-\varepsilon)}\right) = \alpha_{1,2} + \arctan\left(\frac{\sin\beta(\alpha_{1,2})}{\cos\beta(\alpha_{1,2})}\right)$,

where

(5) $\cos\beta(\alpha_{1,2}) = \frac{1-k.\cos(\alpha_{1,2}-\varepsilon)}{Re\,x\alpha_{1,2}}$

and

(6) $\sin\beta(\alpha_{1,2}) = \dfrac{k.\sin(\alpha_{1,2}-\varepsilon)}{\mathrm{Re}\,x\alpha_{1,2}}$,

and the derivative of $d[\beta(\alpha)]/d\alpha$ is

(7) $$\frac{d\beta(\alpha)}{d\alpha} = \frac{k[\cos(\alpha-\varepsilon)-k]}{1+k^2-2.k.\cos(\alpha-\varepsilon)} = \frac{k[\cos(\alpha-\varepsilon)-k]}{\mathrm{Re}\,x^2\alpha_{1,2}}$$

From (1), it results, without difficulty, that the sum, the difference, the product, and the ratio of the two determinations of the functions rex are:

(8) $$\begin{cases} \sum\!\!=^{+} rex_1\theta + rex_2\theta = -2k.\cos(\theta-\varepsilon) \\ \sum\!\!=^{-} rex_1\theta - rex_2\theta = 2\sqrt{1-k^{2.}\sin^2(\theta-\varepsilon)} \\ \prod = \mathrm{rex}_1\theta.rex_2\theta = \begin{cases} k^2-1<0, \to k<1 \\ 0, \Rightarrow k=1 \\ \pm(1-k^2), \to k>1 \end{cases} \\ \bigcup = \dfrac{|rex_2\theta|}{rex_1\theta} = \dfrac{d\alpha_2}{d\alpha_1} = \dfrac{|1-k^2|}{\mathrm{Re}\,x^2\alpha_1} = \dfrac{d(rex_2\theta)}{d(rex_1\theta)} \end{cases}$$

A function similarly useful, in this article, is the ex-centric derivative function of a centric variable α, for which the form of expression is invariable at the position of the ex-center E is:

(9) $$\mathrm{Dex}\,\alpha_{1,2} = \frac{1-k.\cos(\alpha_{1,2}-\varepsilon)}{1+k^2-2k.\cos(\alpha_{1,2}-\varepsilon)} =: \frac{d\theta}{d\alpha_{1,2}} = \frac{d(\alpha_{1,2}+\beta_{1,2})}{d\alpha} = \frac{1}{dex_{1,2}\theta},$$

and

(10) $$\mathrm{dex}_{1,2} = 1 - \frac{k.\cos(\theta-\varepsilon)}{\sqrt{1-k^2\sin^2(\theta-\varepsilon)}}$$

and the nucleus of Poisson integral

(11) $$\mathrm{Nip}\,\alpha_{1,2} = \frac{d\gamma}{d\alpha_{1,2}} = \frac{d(\theta+\beta)}{d\alpha_{1,2}} = 1 + 2\,\frac{d\beta}{d\alpha_{1,2}} = \frac{1-k^2}{1+k^2-2k(\alpha_{1,2}-\varepsilon)} = \frac{1-k^2}{\mathrm{Re}\,x^2\alpha_{1,2}} = -\frac{\mathrm{Re}\,x\alpha_2}{\mathrm{Re}\,x\alpha_1}$$

2. THE INTEGRATION USING THE CLASSIC METHOD [1]

The Poisson's integral, with modified notations, in accordance with the supermathematics ex-centric circular functions (SM - EC), is

(12) $$\mathbf{PI}(k,\varepsilon) = \int_{-\pi}^{\pi} \frac{d\alpha}{1+k^2-2.k.\cos(\alpha-\varepsilon)},$$

in which $k \in \Re$ and $\varepsilon \in [-\pi,\pi]$ are the parameters and, in the same time, the polar coordinates of the ex-center E. This is resolved in [1] as a simple integral which is dependent of a real parameter $\lambda \equiv k$, which will be further denoted as k, and representing, in EM, numeric ex-centricity k = e/R, the ratio between the real ex-centricity e and the circle radius R on which are placed the intersection points W_1 and W_2. The integral is simple, but the integration is quite laborious, as we will see later, and it will become indeed simple, only when passed from **CM** to **EM** with the utilization of the new supermathematics functions.

Classical Solution: The periodic real function

(13) $f(\alpha) = \dfrac{1}{1+k^2-2k\cos(\alpha-\varepsilon)}$

is, as it is easily observed, the square of the radial ex-centric function of α

(14) $f(\alpha) = 1 / (Rex^2\ \alpha)$,

defined for any $k \in \Re - \{\pm 1\}$ and $\varphi \in [-\pi, \pi]$.

Remark: Only one from the two determinations of the function Rex $\alpha_{1,2}$ is null (!) when E belongs to unity circle, that is /k/ = 1; the second determination having the expression which will be presented bellow. Based on the new knowledge from **EM**, now we can assert that the radial ex-centric function is defined also for $k = \pm 1$.

If k = +1, then

(15) $rex_{1,2}\,\theta = -\cos(\theta-\varepsilon) \pm \sqrt{1-\sin^2(\theta-\varepsilon)} \rightarrow rex_1\,\theta = Rex\ \alpha_1 = 0$

and $rex_2\theta = Rex\ \alpha_2 = -2\cos(\theta-\varepsilon)$ and, for k = –1, it results

(16) $rex_{1,2}\,\theta = \cos(\theta-\varepsilon) \pm \sqrt{1-\sin^2(\theta-\varepsilon)}$

such that, now, $rex_1\theta = Rex\ \alpha_1 = 2\cos(\theta-\varepsilon)$ and $rex_2\theta = Rex\ \alpha_2 = 0$, which it results and it can be seen, equally easy, also from the graphic.

Because

(17) $Rex^2\ \alpha = [k - e^{i(\alpha-\varepsilon)}].[k - e^{-i.(\alpha-\varepsilon)}] = [k - rad(\alpha-\varepsilon)].[k - rad\text{-}(\alpha-\varepsilon)]$,

in which, the radial centric functions [5], or in short, radial (denoted **rad**), equivalent to the exponential functions are unitary vectors, of symmetric directions, in relation to the straight line which contains the points O and E, therefore:

(18) $rad\ (\alpha-\varepsilon) - rad[-(\alpha-\varepsilon)] = 2\cos(\alpha-\varepsilon)$

and

(19) $rad(\alpha-\varepsilon)\ .\ rad\text{-}(\alpha-\varepsilon) = \dfrac{rad(\alpha-\varepsilon)}{rad(\alpha-\varepsilon)} = 1$,

in which

(20) $rad\ \alpha = e^{i\alpha}$

is equivalent, <u>in centric</u> (for k = 0, when $\alpha_1 = \theta$ and $\alpha_2 = \theta + \pi$) of functions rex θ and Rexα [5].

The function $Rex^2\alpha$ (16) has the roots:

(21) $e^{\pm(\alpha-\varepsilon)} = rad[\pm(\alpha-\varepsilon)]$

which, for $\alpha = \varepsilon$ and also for $\alpha = \varepsilon - \pi$, din (14) it results

(22) $k = \pm 1$.

By introducing in **PI** the variable $\alpha' = \alpha + \pi$, the change will lead to the integral:

(23) $$\textbf{PI}\ (-k) = \int_0^{2\pi} \frac{d\alpha'}{1 + k^2 + 2k.\cos(\alpha' + \varepsilon)},$$

in which the numeric ex-centricity changes the sign, that is $k \to -k$, which is equivalent to the rotation of the ex-center E(k, ε) around the origin O (0,0) with π, on the circle with the radius k, that is $\varepsilon \to \pm$ ($\varepsilon \pm \pi$), or, yet, because of the inter-conversion properties of α with ε in the function cosine from (12), $\alpha \to \pm$ ($\alpha \pm \pi$).

Suppose that $k \neq \pm 1$, the change of the variable $\alpha' = \alpha + \pi$

(24) $$z = e^{i(\alpha' + \varepsilon)},$$

for which

(25) $$dz / z = \frac{der(\alpha' + \varepsilon)d\alpha'}{rad(\alpha' + \varepsilon)} = \frac{i.rad(\alpha' + \varepsilon)d\alpha'}{rad(\alpha' + \varepsilon)} = i\, d\alpha'$$

it will transform the segment $[-\pi, +\pi]$ in the unity circumference, going in trigonometric positive sense (sinistrorum / levogin). Then:

(26) $$\textbf{PI}(k, \alpha) = i \int_C \frac{dz}{(1 + k^2)z + (1 + z^2)k} = \frac{i}{k} \int_C \frac{dz}{z^2 + mz + 1},$$

in which m = k +1/k.

The poly-functions f(z) from under the sign of $\int_C$ are z' = – k and z" = – 1 / k with the residues $a'_{-1} = Rez[f(z), -k] = k / (k^2 - 1)$ and $a''_{-1} = Rez[f(z), -1 / k] = k / (1 - k^2)$, such that $a'_{-1} + a''_{-1} = 0$.

By applying the residues and semi-residues theorems, it results that for any angle $\varepsilon \in [-\pi, +\pi]$

(27) $$\textbf{IP}(k, \varepsilon) = \begin{cases} \dfrac{2\pi}{1 - k^2}, for\ |k| < 1; \\ 0,\ for\ k = \pm 1 \\ \dfrac{2\pi}{k^2 - 1}, for\ |k| > 1 \end{cases}$$

The zero value for $\lim_{k \to \pm 1} PI(k, \varepsilon)$ can be found choosing the contour Γ made of the circumferences C and γ (Fig. 2) , the last having the center in z" = 1 / k and the radius r < 1, from which we suppressed the interior portions of the reunion of the two circles. In this conditions, the integral $\int_\Gamma \frac{dz}{z^2 - kz + 1}$ is null even when $k \to 1$ (or –1), appearing as a principal value in the Cauchy sense. We can then write:

(28) $$\mathbf{PI}(k,\varepsilon) = \begin{cases} \dfrac{2\pi}{|1-k^2|}, for \Rightarrow k = \Re - \{\pm 1\} \\ 0, for \Rightarrow k = \pm 1 \end{cases}$$

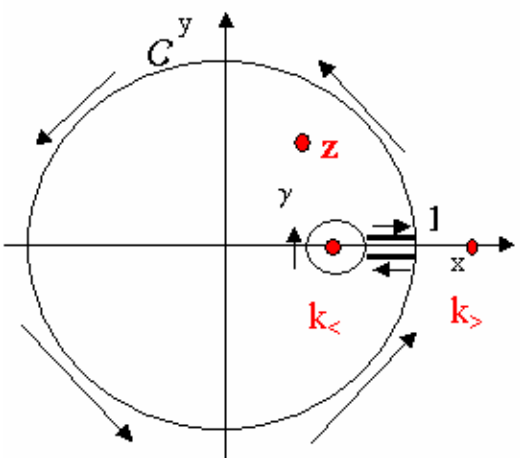

Fig. 2 Explanatory Drawing

Fig. 3 The functions $Rex_{1,2}$ and $rex_{1,2}$ of $k = \pm 1$ and $\varepsilon = 0$

The result (28), presented in [1], can also be established directly, knowing that from (14), for $k = 1$, $rex_1\theta = 0$, and for $k = -1$, $rex_2\,\theta = 2\cos(\alpha-\varepsilon)$ such that:

(28) $$\int_{-\pi}^{\pi}\frac{d\alpha}{\mathrm{Re}\,x^2\alpha_2} = \int_{-\pi}^{\pi}\frac{d\alpha}{4\cos^2(\alpha-\varepsilon)} = \frac{1}{4}\left|\tan(\alpha-\varepsilon)\right|_{-\pi}^{\pi} = \frac{1}{4}[\tan(\pi-\varepsilon)-\tan(-\pi-\varepsilon)] = 0$$

For $k \neq \pm 1$, will present the integral below.

3. THE INTEGRATION WITH THE HELP OF CIRCULAR EX-CENTRIC SUPERMATHEMATICS FUNCTIONS.

Multiplying **PI**(k, ε) with $(1-k^2) / (1-k^2)$ it results

(29) $$\mathbf{PI}\,(k,\varepsilon) = \frac{1}{1-k^2}\int_{-\pi}^{\pi}\frac{1-k^2}{\mathrm{Re}\,x^2\alpha_{1,2}}d\alpha = \frac{1}{1-k^2}\int_{-\pi}^{\pi}-\frac{\mathrm{Re}\,x_2\alpha}{\mathrm{Re}\,x_1\alpha}d\alpha =$$

$$= \frac{1}{1-k^2}\int_{-\pi}^{\pi}\frac{d(\theta+\beta)}{d\alpha}d\alpha = \frac{1}{1-k^2}\int_{-\pi}^{\pi}d(\theta+\beta) = \frac{1}{1-k^2}\left|\theta+\beta\right|_{-\pi}^{\pi} = \frac{2\pi}{1-k^2},$$

for $k < 1$ and

(30) $$\mathbf{PI}\,(k,\varepsilon) = \frac{1}{1-k^2}\int_{-\pi}^{\pi}\frac{\mathrm{Re}\,x\alpha_2}{\mathrm{Re}\,x\alpha_1} = \frac{-2\pi}{1-k^2},$$

for k > 1, in which we took in consideration the relation (9) and the sign of the functions Rex $\alpha_{1,2,}$ for k < 1 and for k > 1, that is an ex-center interior or exterior to unity disk and of the relation, for k < 1. The relations between the integration limits, taking in account of the dependencies [2]

(31) $$\begin{cases} \alpha_1 = \theta - \beta \\ \alpha_2 = \theta + \beta + \pi \end{cases}$$

knowing that $\beta_1 + \beta_2 = \pi$ [3] are :

(32) If $\alpha_1 \in [-\pi, \pi] \Rightarrow$ then $\theta \in [\pi - \beta_1, \pi + \beta_1]$

and their difference is $+2\pi$, and if

(33) $\alpha_2 \in [-\pi, \pi]$, then $\Rightarrow \theta \in [-2\pi - \beta, -\beta]$,

as it can be seen also in Fig. 1, and their difference is -2π.

(34) $$1+2\frac{d\beta}{d\alpha} = 1+2\frac{e[(\cos(\alpha-\varepsilon)-e]}{\mathrm{Re}\, x^2\alpha_{1,2}} = \frac{1-k^2}{\mathrm{Re}\, x^2\alpha_{1,2}} = \frac{d\alpha}{d\alpha} + \frac{d(2\beta)}{d\alpha} = \frac{d(\alpha+\beta+\beta)}{d\alpha} = \frac{d(\theta+\beta)}{d\alpha}$$

because $\theta = \alpha + \beta$, and for

(35) $$\alpha = \begin{cases} \gamma_1 = -\pi \rightarrow \theta = -\pi + 2\beta_{1,2} \\ \gamma_2 = \pi \rightarrow \theta = \pi + 2\beta_{1,2} \end{cases},$$

as it results also from the figure, therefore

(36) $\gamma_2 - \gamma_1 = 2\pi$

CONCLUSIONS

Because of the labor volume in the two variants, the conclusion is, evidently, in the favor of the new method of integration, taking in account, firstly, the degree of complexity of the integration.

By utilizing the existing relations in EM, as, for example the relation (28), which can be written by denoting $\gamma = \theta + \beta$, from which $d\gamma = d(\theta + \beta)$, but $\alpha = (\theta - \beta)$ and $d\alpha = d(\gamma - 2\beta)$ or $d\alpha = d(\theta - \beta)$, such that

$$d\gamma / d\alpha = 1 + 2.d\beta / d\alpha = -\frac{k.\cos(\theta-\varepsilon)}{rex_1\theta} = -\frac{\mathrm{Re}\, x\alpha_2}{\mathrm{Re}\, x\alpha_1} = \frac{1-k^2}{\mathrm{Re}\, x^2\alpha_{1,2}}$$

and **PI** is an immediate integral, k being a constant parameter, as we saw before. Furthermore, from the relation (29) it results the Poisson's integral value undefined as being:

(37) $$\mathbf{IP_N} = \int \frac{d\alpha}{\mathrm{Re}\, x^2\alpha} = \frac{1}{|1-k^2|}[\theta(\alpha)+\beta(\alpha)] = \frac{1}{|1-k^2|}[\alpha + 2\beta(\alpha)] = \frac{\alpha + 2\arcsin\frac{k\sin(\alpha\varepsilon)}{\mathrm{Re}\, x\alpha}}{|1-k^2|}$$

The integrals calculated in [1] with the help of the residues theorem

(38) $\mathbf{I_1} = \int_0^{2\pi} \frac{R - r\cos(\alpha - \varepsilon)}{R^2 + r^2 - 2r\cos(\alpha - \varepsilon)}$,

in which with r = k.R we denoted the real ex-centricity and with R the radius of a certain circle and

(39) $\mathbf{I_2} = \int_0^{2\pi} \frac{r\sin(\alpha - \varepsilon)}{R^2 + r^2 - 2r\cos(\alpha - \varepsilon)}$

which, by the classical method presented in [1, pp. 186-187] are equally laborious and, unfortunately, wrong; by the new method, from EM, these integrals are immediate. Reducing R from (38) and (39) it results the functions to be integrated:

(40) $F_1 = \frac{R - r\cos(\alpha - \varepsilon)}{R^2 + r^2 - 2r\cos(\alpha - \varepsilon)} = \frac{1 - k\cos(\alpha - \varepsilon)}{\mathrm{Re}\, x^2\alpha}$ = Dex $\alpha_{1,2} = \frac{d\theta}{d\alpha}$

such that the undefined integral is

(41) $\mathbf{I_{1N}} = \int \frac{d\theta}{d\alpha} d\alpha = \int d\theta = \theta(\alpha_{1,2}) = \alpha + \arcsin\left[\frac{k\sin(\alpha_{1,2} - \varepsilon)}{\pm\sqrt{1 + k^2 - 2k\cos(\alpha_{1,2} - \varepsilon)}}\right]$,

such that, the defined integral (38) will be:

- For k = +1 $\Rightarrow$ $\mathbf{I_1} = \pi$, because in the first determination 1 (principal) $\theta(\alpha = 0) = \pi/2$ and $\theta(\alpha = 2\pi) = 3\pi/2$ and the difference is π. If k = – 1 for the first determination $\theta(\alpha = 0) = \pi$ and $\theta(\alpha = 2\pi) = 2\pi$, such that the difference is the same π. It results that for |k| = 1 $\Rightarrow$ **$\mathbf{I_1} = \pi$.**
- For k > 1, the integral value I_1 is 0.
- For k < 1, the integral value is 2π.

The undefined integral $\mathbf{I_{2N}}$ is:

(42) $\mathbf{I_{2N}} = \int \frac{k.\sin(\alpha - \varepsilon)}{1 + k^2 - 2k\cos(\alpha - \varepsilon)} d\alpha = \int \frac{k.\sin(\alpha - \varepsilon)}{\mathrm{Re}\, x^2\alpha} d\alpha = \int \frac{1}{\mathrm{Re}\, x\alpha} \frac{d(\mathrm{Re}\, x\alpha)}{d\alpha} d\alpha$ = ln|Rexα

Therefore, the defined integral $\mathbf{I_2}$ is:

(43) $\mathbf{I_2} = \left|\ln\left|\mathrm{Re}\, x\alpha\right|\right|_0^{2\pi} = 0$,

for any k and ε, knowing that Rex0 = rex0 = Rex2π = rex2π.

More integrals can be resolved immediately in this way without difficulties, if one knows the expressions of some supermathematics functions.

More integrals are presented in [6].

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CHEMISTRY

POTENTIAL USE OF LIME AS NITRIC ACID SOURCE FOR ALTERNATIVE ELECTROLYTE FUEL-CELL METHOD

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1. INTRODUCTION

1.1. Background

1.1.1. Despite growing popularity for the use of biofuel and other similar methods to generate renewable energy sources from natural plantation in recent years, there is also growing concern over its disadvantage, i.e. that the energy use of edible plants may cause unwanted effects, because the plantation price tends to increase following the oil price. Therefore an alternative solution to this problem is to find 'natural plantation' which have no direct link to 'food chain' (for basic foods, such as palm oil etc).

1.1.2. Another choice is to use directly natural plants as substitute for components of Fuel Cell systems, such as Electrolyte Fuelcell Systems (EFS). Interestingly EFS methods have been investigated in recent years. This new use of natural plantation in EFS may be considered as potential applications of Green Chemistry [1].

1.1.3. In this regards, possible usage of nitric-acid in EFS (NA-EFS) has been identified and discussed in recent years by some authors. [2][3].

1.1.4. Nonetheless, this new NA-EFS have not been studied in Indonesia, despite plenty of tropical fruits can be found in this country which consists of nitric-acid compounds.

1.1.5. Therefore, in this proposal we will focus on possible use of nitric-acid from natural plants (in particular lime) in EFS. Furthermore, it can be expected that NA-EFS can be proved to be more efficient than other existing biofuel methods, thanks to the fact that NA-EFS does not need to grow plants which normally are parts of basic foods of common people.

1.2. Objectives

1.2.1. Objectives of this proposed research includes:

1.2.1.1. Studying chemical composition of some known source of nitric acid in nature, in order to find out which natural plantation is more suitable from the viewpoint of NA-EFS.

1.2.1.2. Studying (experimentally) which EFS method is the most suitable to be used in conjunction with nitric-acid extracted from Natural Plantation, in particular for tropical countries like Indonesia.

1.3. Expected Output

1.3.1. Result from experiments as well as theoretical studies on how nitric-acid shall be produced and used in EFS methods.

1.3.2. Prototype Design as guidance for practical use or further development.

2. STATE OF THE ART OF THE RESEARCH

2.1. Fuel cell definition: A fuel cell is an energy conversion device that consists essentially of two opposing electrodes, an anode and a cathode, ionically connected together via an interposing electrolyte. Unlike a battery, fuel cell reactants are supplied externally rather than internally. [2] Fuel cells operate by converting fuels, such as hydrogen or a hydrocarbon (e. g., methanol), to electrical power through an electrochemical process rather than by combustion. It does so by harnessing the electrons released from controlled oxidation-reduction reactions occurring on the surface of a catalyst. A fuel cell can produce

electricity continuously so long as proper reactants are supplied from an outside source. [2]

2.2. Some known types of ecological power sources [3]:

2.2.1. PEFC (polymer electrolyte membrane fuel cells): With ion-exchange membranes it is possible to create ‘green’ energy source, such as PEFC and RFB. [3]

2.2.2. HOFC (hydrogen-oxygen fuel-cell): In HOFC method, the hydrogen is oxidized at the anode and the protons migrate through a cation-exchange membrane to the anode where they react with oxygen, forming water. [3]

2.2.3. Redox flow battery (RFB): RFB method can be very efficient for large-scale energy storage. In this method, Cr3+/Cr2+ and Fe3+/Fe2+ are circulating through a cell divided by anion-exchange membrane. [3]

2.3. Existing fuel cell systems are typically classified based on one or more criteria: [2]

2.3.1. the type of fuel and/or oxidant used by the system;

2.3.2. the type of electrolyte used in the electrode stack assembly;

2.3.3. the steady-state operating temperature of the electrode stack assembly;

2.3.4. whether the fuel is processed outside (external reforming) or inside (internal reforming) the electrode stack assembly. In general, however, it is perhaps most customary to classify existing fuel cell systems by the type of electrolyte (i. e. , ion conducting media) employed within the electrode stack assembly. Accordingly, most state-of-the-art fuel cell systems have been classified into one of the following known groups: 1. Phosphoric acid fuel cells (e. g., phosphoric acid electrolyte); 2. Alkaline fuel cells (e. g., KOH electrolyte) ; 3. Molten carbonate fuel cells (e. g., Li2CO3/K2CO3 electrolyte); 4. Solid oxide fuel cells (e. g. yttria-stabilized zirconia electrolyte) ; 5. Proton exchange membrane fuel cells (e. g., NAFION electrolyte).

3. METHODS

3.1. The method of research can be described in 'phases'. Phase A consists doing experimental study on the chemical composition of some of known sources of nitric acid in nature. This phase A also includes studying the nature of 'nitric acid' of these tropical fruits, including: lime, lemon, mango, etc. Each of these fruits will be discussed and analyzed, and the results will be summarized.

3.2. Phase B consists of studying and experiments best method to implement EFS in conjunction with nitric-acid extracted from these tropical fruits. Each of these fruits will be tested using particular EFS (not yet determined at present), discussed and analyzed, and the results will be summarized.

3.3. Phase C consists comparing theoretical knowledge in the existing body of knowledge concerning EFS [4], with the findings obtained from Phase B. This will result in new/improved design for better NA_EFS for tropical countries.

4. TIME SCHEDULE

Phase	Feb08	Mar08	Apr08	May08	Jun08	Jul08	Aug08	Sep08	Oct08
A. Chem.	====								
B. Test EFS		====	====	=====	====				
C. Design					====	====	====		
D. Summary							====	====	
E. Final Report								====	====

5. PROPOSED BUDGET

6. PERSONAL INVESTIGATOR AND OTHER RESEARCHER

6.1. Team Leader: V. Christianto

6.2. Research team:

6.2.1. Prof. F. Smarandache (UNM)

6.3. Assistant:

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www.biotrop.org

Observation of Anomalous Potential Electric Energy in Distilled Water Under Solar Heating

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Abstract

In this paper, we describe a very simple experiment with distilled water which could exhibit anomalous potential electrical energy with *very minimum preparation energy*. While this observed excess energy here is less impressive than [1][2] and the material used is also far less exotic than common LENR-CANR experiments, from the viewpoint of minimum preparation requirement –and therefore less barrier for rapid implementation--, it seems that further experiments could be recommended in order to verify and also to explore various implications of this new proposition.

Introduction

There has been a somewhat regained awareness in recent years for the alternative energy technologies based on *low-energy reaction and also chemical-aided reaction* [1]. This process includes various different methods ranging from the well-known gas discharge process until the exotic processes such as microwave-induced reaction.[2][3] Some theoretical explanation has also been proposed in recent years. [4][5]

Nonetheless, from the viewpoint that our Earth is presently seeking a rapid change to alternative energy, one could imagine that it is required to find a 'less-exotic' energy source, which can be generated with minimum preparation. Therefore, the 'energy input' term should also include the energy amount needed to make preparation for the source and also for the equipment.

In this regard, we re-visit a well-known process of finding excess electrical energy out of 'distilled water.' It can be shown via simple experiment setting, that with very minimum preparation one can obtain anomalous excess electrical energy from distilled water, in particular under solar (photon) exposure. The result is summarized in Table 1.

In the last section we will discuss a few alternative approaches to explain this observed anomalous effect, for instance using the concept of 'zero point energy' of the phion-fluid condensate medium. [6]

Nonetheless, further experiment is recommended in order to verify or refute our proposition as described herein.

Experimental preparation and result

The basic idea of this experiment comes from reading various papers related to chemical aided reaction [1][2]. There is also an abstract requirement for minimum preparation energy, so that it would be easier for rapid implementation (if chance permits).

Therefore we come to analogue to dc battery: *a used battery will re-gain part of its electric energy once it is put under exposure to the Sun light for a few hours*. This analogy leads us to hypothesize that the Sun light emits photon flux with sufficient 'z*ero point energy'* which could trigger chemical reaction in the electrolyte. Then the re-gained electric energy of the used battery will last for a few more days.

Possible implication for this experiment could include usage of distilled water as an efficient method for *battery charger*, while possible future use in transportation etc. remains open. However, this simple experiment is merely at its very initial phase, so we haven't exercised thoroughly yet how it could be transformed into practical use. Our intention here is to explore another route which may have been forgotten in the plethora of CANR methods.

We also haven't made reasonable assumptions yet concerning the development of a commercial generator cell (for battery charger or other practical use), or what would be the expected electrical energy output per unit water volume per day.

Therefore, in this simple experiment we consider a few alternative scenarios, as follows:

(i) ordinary water without exposure to Sun light or to external dc potential (as control for this experiment);
(ii) ordinary water with exposure to Sun light;
(iii) distilled water without exposure to Sun light or to external dc potential;
(iv) distilled water with exposure to Sun light;
(v) distilled water with exposure to carbon alkali (chemical inside battery);
(vi) distilled water with exposure to external dc potential;
(vii) distilled water with exposure to Sun light and carbon alkali (chemical inside battery);
(viii) distilled water with exposure to carbon alkali and to external dc potential.

Distilled water is used in this experiment instead of heavy-water (deuterium) which is commonly used in various LENR-CANR experiments [1][2], with simple reason that it is easier to obtain almost anywhere. Therefore no excessive preparation for such water is needed. Of course, for better result it is recommended to repeat this experiment with heavy-water. (For instance, Belyaev *et al*. already reported various experiments with heavy-water.)

In the meantime, *carbon arc in water* experiments have been performed by a number of experimenters [2, p.1110], which may have similarity with type (viii) of our experiment.

The preparation for this experiment is described as follows. Distilled water which we use in this experiment was obtained from other sources (We don't distil water with our own process).

We use 20 mm-diameter aluminium tube and fill it with ordinary water for control, then we measure its electrical resistance and also its electrical voltage (Type iA experiment). Then we put this tube under the exposure of Solar daylight (high noon), and using a 60mm-diameter magnifying lens at its focal distance in order to focus the Solar's photon flux into our tube. Then we measure again the electrical resistance and also its electrical voltage. (Type iB experiment)

After around 1 hour, we use another 20 mm-diameter aluminium tube and fill it with distilled water, then we put these tubes under the exposure of Solar daylight (Type iiB).

Thereafter we repeat the procedure once again after introducing an external 1.5V DC potential into the electrolytes. Then we measure again the electrical resistance and also its electrical voltage. (Type iiC) After around 5-10 minutes, we release the external potential (1.5 DC volt) and put the tube again under solar light exposure. (Type iiD)

We repeat the procedure after filling the tube with carbon alkali from used-batteries 1.5V DC. Then we measure again the electrical resistance and also its electrical voltage. (Type iiiA) Thereafter we repeat the procedure once again after introducing an external 1.5V DC potential into the electrolytes. Then we measure again the electrical resistance and also its electrical voltage. (Type iiiC) After around 5-10 minutes, we release the external potential (1.5 DC volt) and put the tube again under solar light exposure. (Type iiiD).

The experimental configuration is shown in the following diagrams, both with and without external 1.5Volt DC potential.

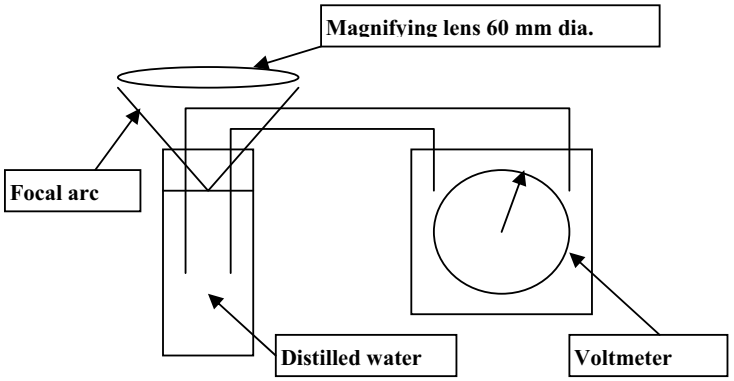

Diagram1. Experiment with distilled water and no external DC (Type iiA)

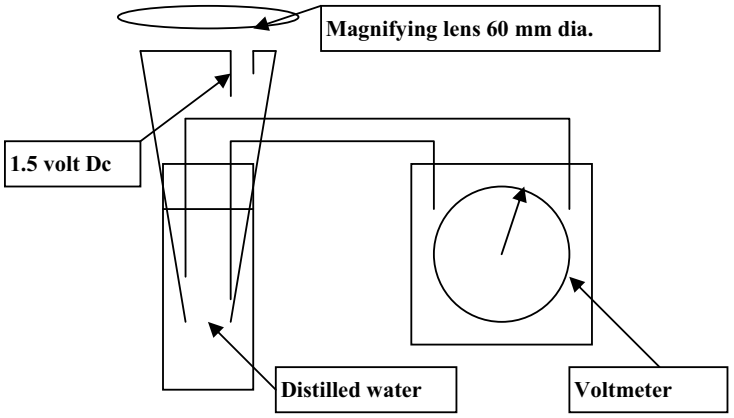

Diagram2. Experiment with distilled water and external 1.5V DC (iiC)

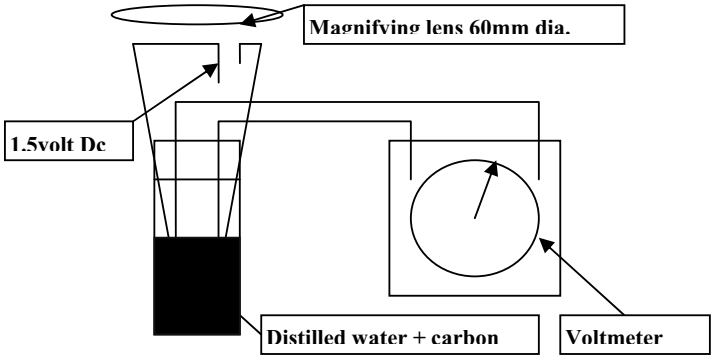

Diagram 3. Experiment with distilled water with carbon alkali and external 1.5Volt DC (Type iiiC + iiiD)

In simple words, in this experiment we want to know whether the effect of Solar heating (photon flux) is similar with introducing carbon alkali material or introducing 1.5V DC potential into the electrolytes. As shown in Table 1 below, it turns out that both photon flux and external 1.5V DC potential could induce significant impact to the observed anomalous potential, while carbon alkali almost has no further effect (at least to the experimental configuration as described herein).

In each experiment, we fill the 20mm-diameter tube with 100mm high of distiller water, meaning that we use more or less ~ 120cc of distilled water for each phase of experiment.

The experiment was conducted in the backyard, around 21st Aug. 2006.

Table 1. Observation result with distilled water

Description	Without solar exposure	With solar exposure (magnifying lens)	Before external 1.5V DC. Without solar exposure	After external 1.5V DC. With solar exposure (magnifying lens)
	A	**B**	**C**	**D**
Ordinary water [i]	V=0 Volt; R>>1000 Ω	V=0 Volt; R>>1000 Ω		
Distilled water [ii]	V=0 Volt; R>>1000 Ω	V=0.2 Volt; R=600Ω ~1000Ω	V=0.8-1.0 Volt; R=600Ω ~1000Ω	V=0.6-0.8 Volt; R=100Ω ~600Ω
Distilled water with carbon alkali material [iii]	V=0.2 Volt; R>>1000 Ω	V=0.6 Volt; R=600Ω ~1000Ω	V=0.6-0.8 Volt; R=600Ω ~1000Ω	V=0.6-0.8 Volt; R=100Ω ~600Ω

From Table 1 we can observe a few interesting results, as follows:

(i) That *within bounds of experimental precision limits* we observe that there is anomalous potential energy in distilled water as much as 0.6-0.8 Volt (DC) after sufficient exposure to solar light, and after a few minutes introducing external 1.5Volt (DC) potential into the electrolytes. (Type iiC)

(ii) Using carbon alkali material will add no further effect into this anomalous observed potential energy (Type iiiC). The exact source of this observed anomalous potential energy remains unknown.

(iii) Furthermore, it is also interesting to note here that after around two hours (the external 1.5Volt DC potential has been released), measurement reading for configuration [iiD] remains showing anomalous potential electric energy ~ 0.4-0.6 Volt and resistance R=~100Ω.

(iv) After around 24 hours (the next day), measurement reading for configuration [iiD] remains showing anomalous potential electric energy ~ 0.1-0.2 Volt and resistance R=~100Ω.

(v) Therefore we can conclude to summarize this experimentation, that a small DC potential and photon flux (Solar light) could play significant role in the LENR/CANR-type processes which so far this effect has been almost neglected in reported LENR/CANR experiments.[1][2]

For clarity, we draw diagram showing observed anomalous potential energy (the lower bound value) in experiment type iiA, iiB, iiC, iiD for the first 24 hours of this experiment (Table 2 and Diagram 4). It is clear here that the peak of anomalous potential energy was observed after introducing external 1.5Volt DC potential, and its impact not last yet after 24 hours.

Table 2. Observation result in each step of experiment Type ii

Step	Hours	Observed potential (volt)
Without solar light	0	0
After solar light	0.2	0.2
With external 1.5Volt	0.4	0.8
Without external 1.5Volt, after solar light	0.5	0.6
After 2 hours	2.5	0.4
After ~24 hours	24	0.1

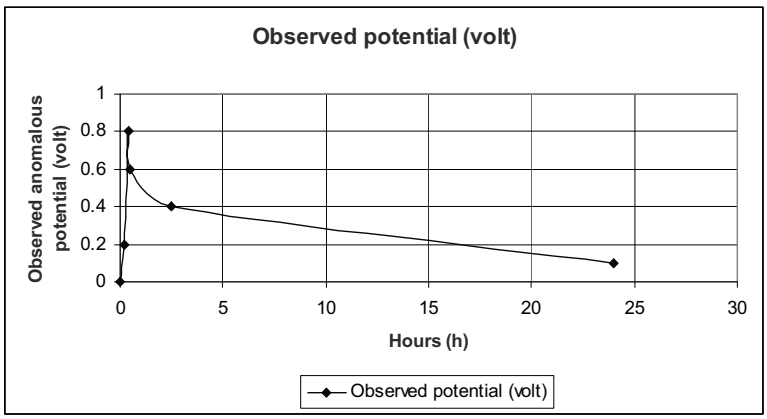

Diagram 4. Observation result in each step of experiment Type ii

In our opinion, it is very likely that this *photon flux* could trigger effect just like in '*photo-synthesis' process* which is known in various biological forms of life. However, this proposition requires further theoretical considerations.

If this proposition corresponds to the facts (concerning the role of photo-synthesis), then perhaps this experiment does not belong to typical LENR-CANR experiments [1][[2], instead it is perhaps more convenient to call it PSCR (PhotoSynthesis-catalyzed Chemical Reaction).

We also note here that the energy dissipated by an electric field flowing through the water resistance could waste low-grade heat. However, it shall also be noted that in our experiments as described above, the photosynthesis process seem to affects the distilled water resistance, down to as low as R=~100Ω after 24 hours. Therefore it is recommended to find out how likely is the chance to transmit electromagnetic field via distilled water in this low resistance condition.

Nonetheless, it should also be noted here that there is shortcoming of this experimentation, for instance we don't exactly measure how much carbon alkali material has been introduced into the electrolyte, nor how long the solar light exposure shall be maintained (it could take 5-10 minutes). It is because this experiment is merely to assess the viability of the idea, instead of becoming a rigorous experiment. Further experiments are of course suggested to verify this proposition with better precision.

Furthermore, as precaution, it is worth noting here that perhaps the tube material (aluminium, in this experiment) may have contributed significantly to the anomalous effects reported here. Repeating this experiment with different tube material may affect the result.

A few alternative interpretations of the above anomalous effect

In order to explain the above anomalous potential energy, we consider a few possible alternative interpretations, as follows:

- photon magnified energy;
- photon Hall effect;
- photon condensate's zero point energy;
- phion condensate's Gross-Pitaevskii energy.

The rationale for each of these alternatives is discussed as follows:

(a) Photon Magnified Energy. It can be shown by the use of special relativity that the energy momentum relation actually also depends on the 'scale' of the frame of reference. Therefore the use of magnifying lens that focuses photon energy in the electrolyte will be not the same again with E=p.c for the area of magnifying lens, but:

$$E_{focused} = n^2 . E_{photon-flux} \qquad (1)$$

Where n represents scaling factor, similar to refractive index.

(b) Photon Hall effect. It is known that photon takes the form of boson [10]. Now it is possible also to assume that the photon condensate will induce Hall effect [8][9],

therefore we could use total particle momentum expression instead of conventional momentum [9]:

$$p = mv + m\Omega \times r + qA \qquad (2)$$

Therefore the energy-momentum relation becomes:

$$E = pc = (mv + m\Omega \times r + qA).c \qquad (3)$$

If we neglect the first term (assuming photon is massless), then:

$$E = pc = (qA).c \qquad (4)$$

We shall note here that Vigier and others suggested photon has mass.

(c) Photon condensate's Zero Point Energy. Starting with the assumption that photon is Bosonic, then we could also use zero point energy of Bose condensate for photon [10]. It is also known that zero point energy could play significant role in LENR experiments [2]. The zpe for Bose condensate could be expressed as follows [10, p.13]:

$$\in = \frac{1}{v}\left\langle \hat{H}_{QFT} \right\rangle_{vac} \qquad (5)$$

Nonetheless it is not yet clear, how zpe could trigger anomalous effect. This zpe could have linkage with interpretation of Dirac's negative energy [5].

(d) Phion condensate's Gross-Pitaevskii energy. We could also start with assumption that there exists phion fluid medium which is unobserved [6][12]. Recent paper by Moffat [6a] has shown that phion condensate model is at good agreement with CMBR temperature and also with galaxies rotation curve data. It could also be shown that using Gross-Pitaevskii equation one could derive Schrödinger equation, also planetary quantization.[7] Using the mechanism of photon-photon interaction [6], the solar's photon flux interacts with the surrounding phion condensate medium. And therefore the energy collected by the magnifying lens is not only its own 'photon flux' energy but also includes the energy of the phion condensate medium. This energy then triggers chemical reaction in the electrolyte. It is known that Ginzburg-Landau (Gross-Pitaevskii) equations have free energy term due to its nonlinear effect [13], therefore it perhaps could explain why the effect on the electrolyte remains quite significant (more than 0.2volt) after a few hours.

Further experiments are of course recommended in order to verify or refute these alternative explanations.

Concluding remarks

We have described here an experiment which could exhibit anomalous electrical energy in distilled water with *very minimum preparation energy*. While this observed excess energy here is less impressive than [1][2] and the material used is also far less exotic than common LENR/CANR experiments, from the viewpoint of minimum preparation requirement –and therefore less barrier for rapid implementation--, it seems that further experiments could be recommended in order to verify and also to explore various implications of this new proposition.

Practical implication of this experiment could include possibility to use distilled water+carbon alkali for battery charger, as an alternative to polymer electrolyte charger (PEFC) method introduced by DoCoMo in July this year (2006). Nonetheless, this simple experiment is merely at its very initial phase, so we haven't exercise thoroughly yet how it could be transformed into practical use.

Furthermore, as precaution, it is worth noting here that perhaps the tube material (aluminium, in this experiment) may have contributed significantly to the anomalous effects reported here.

We shall note here that perhaps this experiment does not belong to 'standard' LENR-CANR experiments [1][[2], instead it is perhaps more convenient to call it PSCR (Photo-Synthesis-catalyzed Chemical Reaction). Nonetheless, the present simple experiment was reported merely to encourage further experiments along similar line of thought.

Acknowledgment

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Simplified Biographies

- F. Smarandache, a mathematician now teaches at Department of Mathematics & Science, University of New Mexico, Gallup, USA. His theoretical work includes a generalization of fuzzy logic, Neutrosophic Logic, as well as numerous papers and books in various areas of applied mathematics. He is also an associate editor of *Progress in Physics* journal, www.ptep-online.com.
- V. Christianto, a self-taught experimenter with recent interests on new alternative energy experiments, and also a reader in physics science. His experiments were performed mainly in backyard. With engineering background, but now he works in information technology field. He is also an administrator of independent preprint service, www.sciprint.org.

First version: 21st Aug 2006. 1st revision: 23rd Sept. 2006.

COMPUTER PROGRAMMING CODIFICATION

Algebraic Generalization[1] of Venn Diagram

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Abstract.

It is easy to deal with a Venn Diagram for $1 \leq n \leq 3$ sets. When n gets larger, the picture becomes more complicated, that's why we thought at the following codification. That's why we propose an easy and systematic algebraic way of dealing with the representation of intersections and unions of many sets.

Introduction.

Let's first consider $1 \leq n \leq 9$, and the sets $S_1, S_2, \ldots, S_n$.
Then one gets 2^n-1 disjoint parts resulted from the intersections of these n sets. Each part is encoded with decimal positive integers specifying only the sets it belongs to. Thus:
part 1 means the part that belongs to S_1 (set 1) only, part 2 means the part that belongs to S_2 only, ..., part n means the part that belongs to set S_n only.
Similarly, part 12 means that part which belongs to S_1 and S_2 only, i.e. to $S_1 \cap S_2$ only.
Also, for example part 1237 means the part that belongs to the sets S_1, S_2, S_3, and S_7 only, i.e. to the intersection $S_1 \cap S_2 \cap S_3 \cap S_7$ only. And so on. This will help to the construction of a base formed by all these disjoint parts, and implementation in a computer program of each set from the power set $\mathcal{P}(S_1 \cup S_2 \cup \ldots \cup S_n)$ using a binary number.
The sets $S_1, S_2, \ldots, S_n$, are intersected in all possible ways in a Venn diagram. Let $1 \leq k \leq n$ be an integer. Let's denote by: $i_1 i_2 \ldots i_k$ the Venn diagram region/part that belongs to the sets S_{i1} and S_{i2} and … and S_{ik} only, for all k and all n. The part which is outside of all sets (i.e. the complement of the union of all sets) is noted by 0 (zero). Each Venn diagram will have 2^n disjoint parts, and each such disjoint part (except the above part 0) will be formed by combinations of k numbers from the numbers: 1, 2, 3, …, n.

Example.

Let see an example for n = 3, and the sets S_1, S_2, and S_3.

[1] It has been called the **Smarandache's Codification** (see [4] and [3]).

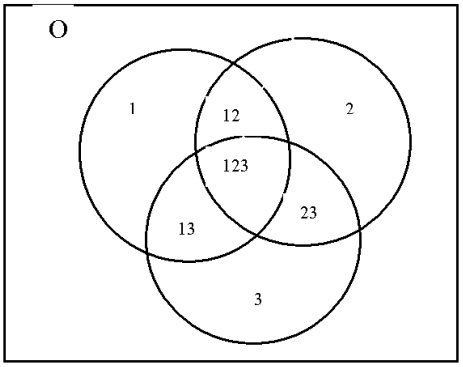

Fig. 1.

Unions and Intersections of Sets.

This codification is user friendly in algebraically doing unions and intersections in a simple way.
Union of sets S_a, S_b, …, S_v is formed by all disjoint parts that have in their index either the number a, or the number b, …, or the number v.
While intersection of S_a, S_b, …, S_v is formed by all disjoint parts that have in their index all numbers a, b, …, v.
For n = 3 and the above diagram:
$S_1 \cup S_{23}$ = {1, 12, 13, 23, 123}, i.e. all disjoint parts that include in their indexes either the digit 1, or the digits 23;
and $S_1 \cap S_2$ = {12, 123}, i.e. all disjoint parts that have in their index the digits 12.

Remarks.

When $n \geq 10$, one uses one space in between numbers: for example, if we want to represent the disjoint part which is the intersection of S_3, S_{10}, and S_{27} only, we use the notation [3 10 27], with blanks in between the set indexes.
Depending on preferences, one can use other character different from the blank in between numbers, or one can use the numeration system in base n+1, so each number/index will be represented by a unique character.

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ECONOMICS, BUSINESS, AND POLITICS

Introduction to Poly-Emporium Theory in Economics

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We propose the **poly-emporium theory**. A search done in Google on May 3rd, 2008, for the term "poly-emporium" returned no entry, so we introduce it for the first time.

Thus "poly-emporium" etymologically comes from *poly* = many, and *emporium* = trade center, store with a wide variety of selling things; therefore *poly-emporium* is the study of interactions among many (big and small) firms in the market.

Poly-emporium is different from *oligopoly* since poly-emporium takes into consideration the small firms too (not only the big firms that dominate the market as in oligopoly). Poly-emporium considers the *real situation* of the market, where big firms and small firms co-exist and interacting more or less.

First, let's present the *duopoly* theory, which is a theory of two firms that dominate and interact in the market, proposed by A. Cournot (1801-1877) in year 1838.

In Cournot's model, if one firm changes its output, the other will also change its output by the same quantity, and eventually both firms will converge towards equilibrium.

In 1883 Bertrand's duopoly model, devised by Joseph Bertrand (1822-1900), if one firm changes its price and the second firm follows, eventually both firms would reach a price (equilibrium) where they would stay.

Both models are similar to two mathematical sequences that little by little converge towards the same limit.

Bertrand's model is criticized because it ignores the *production cost* and market entry by *new firms*.

In oligopoly, which is an extension of duopoly, a small number of selling firms control the market. There is a big degree of interaction among these firms, which set the price, and the price is high and rigid. There is a perfect oligopoly, where all firms produce an identical product, and imperfect

oligopoly, where the firms' products are differentiated but in essence are similar.

Sir Thomas More (1478-1535) used this theory in his "Utopia" (1516) and then A. Cournot. Each firm can act as a leader on its market share, or they collude, or one firm sets the price and others follow.

An analogue of oligopoly is the *oligopsony*, where a few buying firms control the market. They set the price which is normally low and rigid.

The cartel (or trust) influences the price too by regulating the production and marketing, but its influence is of less degree than monopoly's or oligopoly's.

Inflexible price or administered pricing (1930s) is set in monopolies, oligopolies, government organizations, cartels.

Poly-Emporium Theory.

How would interact n firms, F_1, F_2, ..., F_n, for $n \geq 3$, producing a similar product in the same market? A firm can be a business, a corporation, a proprietorship, or a partnership.

There are three cases of the poly-emporium, which will be detailed below:

1) All firms are large and they dominate the market, so we have an oligopoly or oligopsony.

2) Some firms are large, and dominate a big share of the market, while others are small, and do not dominate.

   In this sub-case, either the small firms are grouped around some of the large firms (as satellites) just as in growth-pole theory, other small firms might exit the competition.

   This case also includes the possibility that new firms enter the market, so they commence by small investments and later can grow.

   The relationship between large firms in this case can lead either to oligopoly/oligopsony if they succeed to eliminate the small competitors, or to semi-oligopoly/

semi-oligopsony if they control a big part of the market, but not the whole market.

Small firms might collude and form larger firms.

3) All firms are small and they do not dominate the market.

As in mathematics, it is akin having n sequences, which interact, that we need to study their limit. Would they converge towards the same limit?

Surely, there would always be a *monopolistic competition* between them.

As in *monopoly*, each firm attempts to dominate the market, to prevent competition, in order to control the price. But monopoly is outlawed in most capitalistic countries. If one firm, let's say (without lost of generality) F_1, alters its output, the others F_2, ..., F_n, should also respond, otherwise they loose customers.

If it's an imperfect competition, i.e. a market with a large number of sellers and buyers but having differentiated products, the interaction between these firms is less than in a

perfect competition, and they all tend towards a so-called in our opinion **multi-equilibrium**, as in a weighting machine with many balances, or as in a mathematical weighted average.

Nevertheless, if these firms produce a homogeneous product for many buyers, as in perfect competition, their interdependence increases. Disequilibrium of one firm would affect others.

If superior technology commences to be introduced by some firms, the quality of their product will increase and the price decrease.

This may generate the theory of growth-pole, enunciated by Sir William Petty (1623-1687) and François Perroux (1903-1987), which refers to the fact that smaller firms are grouped around a central core of firms that become catalysts. Maximum growth and product excellence for these firms presumes optimal management.

In it's a monopsony, then a single buyer dominates the market forcing sellers to accept buyer's conditions. Therefore, in this case, the firms compete under buyer's conditions. For

example, this would be the case if the government controls the cultural economics, the government will then set the prices.

If some firms co-operate, as in collusion theory, entailing similar output levels and prices, then other firms should either join the collusion, making a block or monopoly that controls the market, but this is outlawed in capitalistic countries, or they can alter their output by lowering price or improving production for better output quality.

Another alternative would be for the non-collusion firms to form themselves a separate collusion that will counter-balance the first one, or also have some firms to merge. Some firms may exit the market, while new firms would enter the market.

If the government controls the cultural economics, then trade unions of cultural workers should be created for counter-balancing. Because this gives birth to a bilateral monopoly, which is a market with a single buyer and a single seller, mostly referring to the government dealing conditions and salaries with unions of workers.

The dynamicity of the market keeps the firms in a permanent competition, and competition means progress.

We extend Engel's law (1857), that the proportion of income spent on food falls as individual income increases, to a similar law related to cultural economics:

As individual income increases, the proportion of amount spent on cultural event decreases.

Thus, as individual income increases an acceleration of cultural economics occurs.

Moreover, adjusted from the absolute income hypothesis (1936, 1960s, and 1970s) by J. M. Keynes and later refined by James Tobin (b. 1918), we derive the **absolute income cultural hypothesis** applicable to the cultural economics: as income rises, cultural consumption rises but generally not at the same rate.

The 18th century absolute advantage theory, which states that people and nations trade since they have exceeding production in some particular field, does not apply in cultural economics. Nor comparative advantage approach that superseded absolute advantage theory works, because we can't really compare cultures.

Comparative cost, developed by Robert Torrens (1780-1864) and David Ricardo (1772-1823), which is a feature of comparative advantage, asserts that trade between countries is benefic even if one country is more efficient, because of the variety of products. Similarly, cultural economics benefits from its cultural difference. The more distinguishable is a culture, the better chance of increasing the cultural economics.

Economic culture is part of entertainment industry, and depends on taste, advertisement, curiosity, history, and the quality of being diverse, distinctive, with a large spectrum of varieties.

The most interesting case is the third one, where all n firms are small and they do not dominate the market. Let's see, for example, a network of independent restaurants in a city. They interact little with each other. The quality, taste, distance, and price of course make the difference between them.

They do not collude but in rare situations since each of them has its specific, its exotism, which they don't want to loose. They cannot make an oligopoly since new restaurants

may easily enter the market with its specific, and because the taste changes periodically. They remain into multi-equilibrium. Similarly for international cultural economics, where each culture has its specific, and that's what attracts visitors, tourists.

In general, the n firms eventually tend towards multi-equilibrium, where they stay for a while. In multi-equilibrium each firm tends towards its specific sub-equilibrium.

Periodically this multi-equilibrium is partially or totally disturbed, due to technology, government intervention, wars, crises, reorganization of the firms, change in customers' taste and preferences, but then again the firms return to stability. This period of multi-disequilibrium is a natural state, since economy is dynamic, and the disturbance is a launching pad to refreshment; in order to rebalance the market, these n firms must improve their technology, their structure, cut production cost, or else they exit the competition. "All the bad for the good", says a Romanian proverb, so disequilibrium brings later new blood into economy.

In conclusion:

The cycle of multi-equilibrium - multi-disequilibrium repeats continuously.

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Global Totalitarianism and the Crisis of Neo-Lib Movement [i]

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Abstract

During the past decades there are growing concerns on the escalation and massive deprivation of the public life's quality, which some researchers attribute these as effects of the so-called 'Globalization'. To cite a few books, Prof. J. Stiglitz's book "*Globalization and its Discontent*" has sparked debate all over the world. A somewhat less known book which is worth to mention here is N. Hertz's book "*The Silent Takeover*."

The present article may not offer something new compared to the aforementioned 'standard literature' in the critical analysis of globalization, but we discuss some hints on deep the root causes of the problems related to globalization, not only at phenomenological-social level but also at mathematical foundations of economics theory itself, namely the notion of 'utility'.

Introduction

During the past decades there are growing concerns on the escalation and massive deprivation of the public life's quality, which some researchers attribute these as effects of the so-called 'Globalization'. To cite a few books, Prof. J. Stiglitz's book "Globalization and its Discontent" has sparked debate all over the world. A somewhat less known book which is worth to mention here is N. Hertz's book "The Silent Takeover."

The present article may not offer something new compared to the aforementioned 'standard literature' in the critical analysis of globalization, but we discuss some hints on deep the root causes of the problems related to globalization, not only at phenomenological-social level but also at mathematical foundations of economics theory itself, namely the notion of 'utility'. Furthermore, at an 'empirical' level to supports our arguments, we cite some quotations from Perkins [5], and Prof. M. Chossudovsky, who has studied globalization extensively in the past decades. [6-8].

This so-called "globalization" is called on Internet in the following different ways and names, such as: global totalitarianism, neo-totalitarianism, new world order, global fascist order, neo-fascism, today's new fascism, semi-colonialism, neocolonialism, global cyber hegemony (global control and manipulation of the Internet), global dictatorship, etc. where a few elites from some power countries try to *take over* the whole globe, which would become a prison planet. These unscrupulous, immoral, corrupted, genocidal, power-hungry elites will exercise an apartheid policy against the whole world, controlling people's soul through their totalitarian regulation

coercions.

While the present article may not offer something new from analytical viewpoint, by relying on both phenomenology-sociological and mathematical results, we submit the viewpoint on the validity of the hypothesis presented herein. The purpose of this article is to call attention from national leaders on the extensiveness and criticalities of the issues discussed here, and suggest them to take actions together.

Our only hope for liberty would be… to move to another planet!

This global colonization, which we call in the present article Glob-Colonization, means that a few powerful circles of elites wish to transform the third world countries into politically, ideologically, militarily, economically, financially, spiritually, culturally dominated territories.

While at first glance this proposition sounds like a fantasy, comparable to the dark picture described in Orwell's book 1984, and some others would think that globalization is inevitable in order to reach global prosperity; in the following sections we will cite a number of phenomenology-social facts which supports our argument.

It is generally accepted among academicians (notably S. Huntington) that in a rapid movement towards globalization, societies tend to become unstable and therefore we observe disintegration almost everywhere in the World. His hypothesis is supported with plenty of statistical data in his famous book.

Nonetheless at this point we can also ask, what if instead of 'natural tendency' towards instability as he supposed, the reality is that those small countries are merely 'the puppets' under the strings played by the masters of Global Totalitarianism? In this alternative scenario, then the national leaders of small countries are nothing more than actors who want to maximize the 'utility' function of their role as national leaders, and actually don't care at all if by doing so they serve the 'grandmasters' who want to take advantage of the people in their countries?

This alternative hypothesis, while unknown so far to the majority of academicians who cling to the same belief of 'natural tendency' hypothesis of S. Huntington, are indeed supported by rigorous research by Prof. Chossudovsky [6-8] and also confessions book by Perkins [5].

To support this alternative hypothesis, let us cite a number of observations in the following section.

Phenomenology-Sociology observation

These powerful circles of elites do the followings in order to dominate the underdeveloped countries:

(a) Install puppet or at least semi-puppet governments and presidents in underdeveloped countries, easily manipulated and subordinated to them. Many times, they put in power

and support dictators, hated by the local population (see for example Pakistan, then some countries in Latin America, in some Arabic countries, etc.).

(b) Falsify and manipulate underdeveloped countries' local elections in order to bring to power marionettes subordinated to them. The secret services of these powers start by publicizing, before local elections, spurious "Gallup poll" or "opinion statistics" that show as favorable (of course!) their marionette politicians – in order to psychologically prepare the local population for accepting these marionettes.
When the falsification of the elections does not succeed, a flood of slandering, defamatory propaganda is lunched by the secret services (sheltered by these power countries' embassies/consulates/missions, etc.) against the democratically elected government.
If learning from history can be useful at all, let us cite that in the 17th-19th centuries, Romania and other countries under the Ottoman Empire had eastern leaders, called "Phanariotes" (i.e. Greeks nobles from the Phanar district of Istanbul), now local population jokes that their leaders are… *western-Phanariotes* [or *neo-Phanariotes*].

(c) Destroy the industry of underdeveloped countries, making the populace poorer, jobless, and thus obliged to emigrate to the west as cheap and discriminated labor. In this kind, they eliminate industrial competition. A country in general cannot be rich without industrialization.

(d) Dumping third world countries' agriculture system in order to destroy their peasants' small economies, and thus make the citizens of the third world countries dependent of the dominant powers.

(e) Break up underdeveloped countries into small parts, by pedaling on regional differences between various ethnic groups.

(f) Send so-called "peaceful traders" in underdeveloped countries, who in realities are spies who collect information and stir an ethnic group against another in these underdeveloped countries in order to provoke regional turbulence, encourage separatist groups, and try to destabilize these countries.
See for example Czechoslovakia, then Yugoslavia and afterwards Serbia [because they are Slavic countries], and who would be dismembered next?
Attempts were made against Romania too.
Also, attempts to break Brazil, since it is too big and becomes a dangerous competitor, into South Brazil (a rich part) and North Brazil (a poor part), or to remove Amazon's jungle from Brazil because, because as they say: Amazon's jungle belongs to the planet not to Brazil.
Maybe, Indonesia will follow next (?) (East Timor was already cut off from it.) There are some scenarios showing that this process may already be apparent.
As in paradoxism, the Balkanization of the world is "performed" by non-Balkan powers, the vile actors on the world scene.
But the same powerful circles of elites do not want to hear about, for example, dividing Canada into two parts, the French part, Québec – that many times asked for

independence, and the Anglo part; or splitting Belgium into two parts, French part (Wallonie) and Flemish part; or letting Ireland unite with Northern Ireland… .

So, what kind of globalization is that in which, instead of unifying, it divides? These powerful circles of elites do whatever they can for dominance by force and by deceiving. To point out the basic scheme here; this can be observed quite easily:

(i) Destabilize local-national governments by supporting the two opposite sides simultaneously [5];

(ii) Replace the national leaders which apparently sound too strong or too 'vocal' against globalization;

(iii) Destruction of national economies and therefore create the necessity to break up into smaller regions;

(iv) Totally dominate the smaller regions by means of contracts with MNCs (see Noreena Hertz, The Silent Takeover);

(v) Deprivation of public quality of life, and therefore create national economics dependence towards international bank resources (the so called 'bail out' game);

(vi) Create more pressures to the public thereafter.

(g) Entangle, by any mean, countries of same language or culture to unite (for example Arabic countries, or Hispanic countries, or all Islamic countries), so they do not become powers.

(h) Create international organizations that pretend serving the whole globe but, in reality, they only serve the interests of a few powers against independent non-obedient states.

What kinds of democracy promoted by these international organizations when some countries are allowed to have sophisticated arms and others are not? Clearly, they are biased. In our opinion, all countries should disarm - but this is a utopia today.

Also, why some countries have the "right of veto"? That's not fair.

These international organizations look for pretexts (saying that they bring "international aid") to intervene in the affairs of underdeveloped countries.

(i) Create an International Court of Justice where these powerful circles of elites punish those who do not obey to them, by biased and set up trials. The whole world is judged upon the laws and interests imposed by these power country elites.

(j) Erase the collective memory of other nations by defaming, slandering, ridiculing, detracting nations' history, language, personalities, traditions, culture – in order to destroy them and to impose a cultural dominance over the whole world. This is a cynic strategy for abolishing other nations. Instigate the young generations from these power elites countries - through their infiltrated secret service agents - to ignore, boycott, insult, invent lies, discredit underdeveloped countries' scientists, writers, artists and their creations. Teach and instigate the young generations from these power elites countries to hate other cultures, other traditions, and third-world countries' scientists, writers, artists and their creations. Launch an international campaign of denigration and lies against those who succeed to promote their research without approvals from these power countries' elites, and against those who dare to defend the researchers insulted by these power countries' elites.

(k) Humiliate a whole underdeveloped nation through a propaganda that throws the particular to the general, i.e. blameworthy facts of a few individuals from an underdeveloped country are generalized to the whole nation they belong to; that's the intentional way of how mass-media of these powerful circles of elites transmit lies to the whole world.

These powerful circles of elites mutually promote at an international level the racial idea of "superior nations" [which is a kind of *neo-Aryanism*] by humiliating other nations (using lies, speculations, slandering, boycotting, ridiculing realizations and people of third world countries). It is indeed a *Global Aryanism*.
This means an attempt to culturally, spiritually, intellectually, etc. exterminating other nations.
These powerful circles of elites try to intimidate by inspiring international fear and slavishness.
They publish and promote all kinds of reports, various encyclopedias, handbooks, movies, documentaries, propagandistic news, web sites, etc. in order to indoctrinate the whole world that they detain the hegemony in every field.

(m) Indoctrination of third world countries with these few powerful circles of elites' ideology, culture, religion, politics, propaganda, while suppressing local values.

(n) Calumniation of underdeveloped nations' traditions, customs. Powerful countries' secret agents pay dishonest local journalists to write and speak against their own countries' culture, history, traditions, but of course praising the dominants.

(o) Ignore, ridicule, detract and boycott underdeveloped countries' realizations, personalities, men and women of arts and letters, scientific research. Falsify the local history.
This is part of denationalization and brain washing!
From the national poet Eminescu, to high historical leaders as Ştefan cel Mare (Stefan the Great), Mihai Viteazu (Michael the Brave), and to the Romanian folklore characters Făt-Frumos and Ileana Cosânzeana, everything is under a flood of organized denigrations [9], while those who dare to defend them are blacklisted and constantly insulted. It is a way to erase the collective memory of third world country nations.

(p) Weaken the national education system in underdeveloped countries and intoxicate it with these power countries' propaganda, ideology, identity.

(q) International Banks lend money to underdeveloped countries with the pretext of "helping" them, but under cover these banks interfere with underdeveloped countries' political, ideological, economical affairs undermining them, imposing regulations in the interest of a few power countries these banks belong to, and transforming the underdeveloped countries in semi-colonies.

(r) An international swindle done by these few powerful circles of elites is the so-called "convertibility" of only their currencies (or only their currencies to be considered "hard money") in foreign exchange, and not of other countries' currencies. This international

financial cunning gave these powerful circles of elites a huge advantage over the world, since it was extremely cheap for them (i.e. only the cost of ink and paper) to print *colored papers* [= their currencies] and pay in the whole world for all kind of goods and services with their 'colored papers': from oil and agriculture products to secret agents acting in third world countries to destabilizing them.

Third world countries should not recognize these colored papers, and ask in the international trade to get in exchange: gold, silver, diamond, or other concrete goods and services, but not colored papers.

(s) Those who dare to think otherwise, or countries that do not obey these powers are labeled "undemocratic", "politically incorrect", and accused of not respecting the "human rights". These powerful circles of elites pretend to promote democracy, but actually they only adhere to a phony democracy, i.e. "democracy of men with money", since democracy in the classical Greek sense means "power of the people" [in Greek *demokratia* = *demos* (people) + *kratos* (strength), therefore: strength/power of the people], not power of a governmental junta. When, according to pool investigation, majority of people are against a war, and millions demonstrate against the war, but the governmental junta still goes to war, is that a manifestation of the power of the people? Of course not!

There is no much difference between the Stalinist dictatorship and today's so-called "democracy": in the Stalinist dictatorship the citizen were not allowed to say anything; in today's self-called "democracy" you are allowed to speak up, but the effect is the same as in the dictatorship (I mean: there is no effect!... because today's totalitarian governmental junta do whatever it pleases). People can say whatever they want, but it has no consequence! Allowing people to say everything is a psychological tactic from the part of the governmental junta, since people release their anger, and doing that many times without any consequence they would eventually stop… There is a total ignorance from the part of the powerful Klan with respect to the people. Those who dare to criticize these phony democracies are called "unpatriotic" … .

Today's world meaning of "democracy" is subordination to these powerful circles of elites, so unfortunately "democracy" became a propaganda and a pretext of the powerful circles of elites to interfere in the third world countries' affairs!

In addition, countries having a bi-partite political system are less democratic that those having a pluri-partite political system since the last ones offer more alternatives of policies and governance.

Another example of lack of democracy and dominance of some elites over the normal citizen is the *lobby* in the American Congress; this lobby is unfortunately on *official corruption* where firms with money bribe senators to vote for firms' interest laws which are in citizens' disadvantage (a such example is the law that obliges each driver to have car insurance, money which in most cases the citizens pay for nothing… they pay like a tax for wind and for illusions!).

Further, this "politically incorrect" syntagme is a contemporary form of censorship and denial of freedom of speech (you're not allowed to criticize the dominance... the dominance pretends detaining the global "absolute" truth in any field.).

While by respecting the "human rights" they maybe mean: these powers' "human rights" of dominating other nations! The secret services of these powers and their

paid influence agents provoke disarray, disorder, and systematic psychological harassment against the governments of disobedient countries.

(t) Countries that oppose the dominancy of those powerful circles of elites are destroyed with bombs, while those countries that yield to the dominants are destroyed with the pen, as Prof. Michel Chossudovsky plastically wrote [6-8], in the sense that local deregulations took place and external regulations from dominant powers were implemented.

In the last category, Eastern Europe countries, such as Romania, Bulgaria for example, had their industries destroyed, their citizen required to pay high taxes to the government, and each whole country required to pay millions of euros for various European Union projects in Western countries, while Eastern European countries receive very little in exchange and their own projects are systematically rejected. As a result, a small percentage of Eastern Europeans became very rich and the majority very poor, while the degree of population's dissatisfaction – most people were plunged into misery - is very high. The majority's disgust and discomfort is reflected today by young generation's movement in poetry and writing called "grievism" [coming from 'grieve'] that it is often seeing on its Internet creations.

Alas, the majority of people in USA feel the same too, that they are merely 'boiled frog' in their own country, because of these practices by powerful circles of elites.

European Union (EU), as part of the global totalitarianism (i.e. globalization), exercises - besides an internal neocolonialism of Western European countries against Eastern European countries which transformed eastern countries into the wasted garbage of the west - also an external neocolonialism of European firms against African, parts of Asian, and Latin American countries, forcing these underdeveloped countries to open their markets to EU firms whose products surpass the local products, bringing to ruins the local economies.

These few powerful circles of elites use bombs, tortures (defying Geneva convention), invasions, genocides, deceptions, lies against third world countries - pretending they "fight for democracy"… (actually, it is the democracy of the most powerful elites that suck the natural and human resources of the neo-colonies).

Eastern European analysts consider that their countries are today under double occupation… .

(u) There are national and international deceptive agencies of so-called "human rights" movement created by these powerful circles of elites, such as Division of Human Rights, Amnesty International, Equal Employment Opportunity Committee (EEOC), etc. that pretend defending the human rights of citizens in the world, but in reality they go after so-called by them "rogue countries" [i.e. countries that do not subordinate to them] and they look for pretexts to interfere in these countries' affairs. These deceptive agencies don't even protect the ordinary citizens in these power countries from the abuse of the elites, not even do much for their discriminated minorities. These so-called agencies of human rights have a propagandistic role. [5]

(v) Encourage local population NOT to learn its country's history, culture, traditions, etc. transforming them in just "speaking servants" (we adjust the Latin "speaking tools" at

today's reality), or worse "global working animals" for these powers. This is part of the robotization of the people. This population is thus embezzled from its identity…
The more somebody knows, the more he or she demands from the society - which is inconvenient for these powers. That's why the dominance tries to turn the thinking populace into an amorphous ignorant and less educated crowd, and in consequence this populace will simply be pushed into obsequiousness.

(w) Encourage the local creators to imitate and follow these power countries' ideas in arts, letters, science, etc. while discouraging them from having original ideas and creations; these power countries' elites pretend they detain the monopoly of creation;

(x) These power countries try to control and manipulate the information at the global level, as part of globalization, by controlling the national and transnational mass-media, the Internet, and by defaming people who are independent and thus not obeying to them.

(y) Award pompous international awards [with exaggerated epithets such as: "the best in the world", "the genial creator", "the genial theory" (for useless theories that many people contest), etc.] in science, arts, and letters to these powerful circles of elites' servants, since these powerful circles of elites manipulate the awards as well…
Transform the third world countries in cheap leisure places for the vacation of the powerful circles of elites.
The middle class is thinning in all countries, and so is the real democracy.
Even in U. S. there is no universal medical system as in other developed and even underdeveloped countries, the medical assistance cost is sky rocketing, the medical insurance agencies are simply business companies not medical helpers; the social system is bankrupting, and the retirement system in bad shape menacing contemporary working class to remaining without pensions… .The rich become richer and the middle class poorer.

Despite the initial good features of globalization (amongst them the free circulation of people and ideas across borders), it has drastic negative impacts. The uniformization imposed by global totalitarianists reduces or even annihilates the countries' national specific differences, which are the flavor of foreigners' attraction.

It is obvious to say that those powerful circles of elites who planned and set up the globalization did it in their own advantage/profit.

A global totalitarianism of a few elites is installing today against the whole world.

Equilibrium of powers at the planetary level is needed for a healthy global atmosphere. Today's unipolarity is abusive, aggressive, corrupted, yoking. Therefore, it is hoped that maybe China, India, Brazil and other modest countries will develop in order to counterbalance the arrogance of today's totalitarian power.

History teaches us that no empire lasts forever, consequently sooner or later this glob-colonialism/totalitarianism will fail.

At Mathematical Level

Besides the aforementioned phenomenological-social observation, we can also mention that at theoretical-mathematical level part of the problem comes from basic economics belief started by Adam Smith's 'invisible hand' [4][2]. It can be shown [4] that this belief than subsequently leads to an illusion of 'utility function' as an integrable function. Prices, dynamics, market equilibriums, are supposed to be 'derived' from utility. However, this assertion cannot be proved empirically, from Walras (who assert this function from 'auction' model) to Samuelson (his market-demand equilibrium is only a myth). [4]

In particular, economists assume that price is the gradient of utility in equilibrium, but it can be shown instead that price as the gradient of utility is an integrability condition for the dynamics of an optimization problem in economic control theory [4]. One consequence of this new proposition is that, in a nonintegrable dynamical system, price cannot be expressed as a function of demand and supply variables [4]. This can be observed most vividly in the very-high oil price last year (mid of 2008) which some analysts believed this effect was not supported by the reality of market demand-supply.

Therefore for evidence of stability of prices in free markets simply has not been found.[4] This new finding apparently can affect so much in the design of national economics policies, i.e. instead of pursuing equilibrium at all costs, efforts can be directed toward more 'active' measures to make the best out of the market dynamics of non-equilibrium itself. New types of economics theories can be expected therefore, with the most essential part shall be studying non-equilibrium theories, which are well-known in chemistry studies.

Concluding remarks

We have discussed a number of phenomenology-sociology observations which indicated that the global destabilization processes have taken place.

To point out the basic scheme here; this can be observed quite easily:

(a) Destabilize local-national governments by supporting the two opposite sides simultaneously [5];
(b) Replace the national leaders which apparently sound too strong or too 'vocal' against globalization;
(c) Destruction of national economies and therefore create the necessity to break up into smaller regions;
(d) Totally dominate the smaller regions by means of contracts with MNCs (see Noreena Hertz, *The Silent Takeover*);
(e) Deprivation of public quality of life, and therefore create national economics dependence towards international bank resources (the so called 'bail out' game);
(f) Create more pressures to the public thereafter.

In other words, it is obvious to say that a global totalitarianism of a few elites is installing today against the whole world. Those powerful circles of elites who planned and set up the globalization did it in their own advantage/profits, and nothing they have in common with the public interests, both at developed countries and also at underdeveloping countries. Even in developed countries like USA, the majority of people feel that they are only 'boiled frog' whose life quality experiencing deprivation at massive scale.

Equilibrium of powers at the planetary level is needed for a healthy global atmosphere. Today's unipolarity is abusive, aggressive, corrupted, yoking. Therefore, it is hoped that maybe China, India, Brazil and other modest countries will develop in order to counterbalance the arrogance of today's totalitarian power.

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A Note on Exchange Rate Management and Gravity Equation: Developing Country's Viewpoint

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Abstract

In the last few decades, the flexible exchange rate has become the predominating policy implemented by most countries in the World, except only a few countries who can keep their exchange rates fixed. The predominating position can be attributed mainly to Milton Friedman's strong support. What is less known, nonetheless, is a hidden premise that the flexible exchange rate policy will help fiscal policy in the sense that only governments who do not manage the fiscal policy properly will get 'punished' by exchange rate decreases. But as the US-Japan experience showed [1], this widely-held assumption is often not realistic. The same experience has been observed in other countries too, i.e. that financial liberalization including flexible exchange rate often became precursor of financial instability [2].

While in the past year, in Indonesia for particular, the exchange rate remains stable, it does not mean that it would be free from troubles in the future. Therefore it is worthwhile to explore some other choices for better exchange rate policy.

In the present paper will discuss, albeit in somewhat 'crude' manner, some long-term approaches which have been discussed in the literature, and also not-so conventional approaches which may be suitable for short term purposes.

The basic proposition in the present paper is that we argue in favor of returning to fixed (or pegging) exchange rate, but of course it is not realistic to promote this policy for the short-term future. Therefore we explore some unconventional alternatives for short-term. It is our hope that the proposition would be useful to explore further by the economics policy makers.

Introduction

In the last few decades, the flexible exchange rate has become the predominating policy implemented by most countries in the World, except only a few countries who can keep their exchange rates fixed. The predominating position can be attributed mainly to Milton Friedman's strong support. What is less known, nonetheless, is a hidden premise that the flexible exchange rate policy will help fiscal policy in the sense that only governments who do not manage the fiscal policy properly will get 'punished' by exchange rate decreases. But as the US-Japan experience showed [1], this widely-held assumption is often not realistic. The same experience has been observed in other countries too, i.e. that financial liberalization including flexible exchange rate often became precursor of financial instability [2].

For instance, a number of economists have revealed a caveat of financial liberalization which issue has often been discussed in monetary policy sessions, i.e. studies revealed that liberalization is neatly linked and often precedes financial instability. In other words, the magic word has now become the peril for the financial-liberalization supporters: [2]

> "Following liberalization, many developing countries found themselves involved in a condition of high instability and increasing fragility of their financial systems. Therefore, the question arises as to why countries should enact policies that move their financial systems from a situation of relative stability to one of potential instability."

In a somewhat similar tone, Krugman [1] has summarized the US experience with flexible exchange rate during 1980s:

(a) Exchange rate instability has resulted from reasonable market responses to changes in policies –but also from failures in the international financial market (p.77);

(b) Traditional fear that floating exchange rates will be subject to destabilizing speculation is unfortunately supported by the evidence of the 1980s (p.77);

(c) In his classic defense of floating rates, Milton Friedman argued that exchange markets would never be subject to destabilizing speculation per se.

(d) Nonetheless the fact during 1980s seems to support the argument of Ragnar Nurkse instead of M. Friedman.

These lines of arguments have led P Krugman to promote returning to fixed exchange rate (albeit not a radical one, but using a wide '*target zone'* approach). According to P Krugman [1] the basic yet not often understood premise behind flexible exchange rate is that the government will get punishment (in terms of fluctuating exchange rate) in precise proportion to the fallacies in their fiscal policies. The actual experience, nonetheless, has shown that the exchange rate can

fluctuate very large, without the fiscal policies big error. In other words, there are tendencies that exchange rate fluctuations are going on irrespective of fiscal policies.

With this new insight, we try to look again the conventional exchange rate policy in the literature.

A review of history of Flexible exchange rate, and Long-term policy

In accordance with Krugman, The Bretton Wood system was created after WWII to restore the international monetary system stability, by introducing fixed exchange rate. [1][5] Nonetheless, during the course of history, some major countries have found it very difficult to keep their fixed exchange rates, which then lead to the gradual acceptance of flexible exchange rate until these days.

In other words, while some economists would agree that fixed exchange rate is the best system, but the majority of them would also agree that it is not so realistic to keep the fixed exchange rate. Therefore the moderate choice can be a choice, either using 'pegging exchange rate' or 'target zone' [1]. These belong to long term policy, which often needs coordination among some countries in the same region in order to maintain stability.

The problems become more adverse in developing countries because various institutional or non-institutional factors, including [2]:

Table 1. Factors affecting exchange rate management in developing countries [2]

No	Problems	Plausible solution(s)
1	Following liberalization, many developing countries found themselves involved in a condition of high instability and increasing fragility of their financial systems. Therefore, the question arises as to why countries should enact policies that move their financial systems from a situation of relative stability to one of potential instability.	
2	Lack of independence on the part of regulators is a second source of uncertainty. Regulators often fail to perform because of the failure of institutions or because of political interference. Moreover, regulators often find themselves involved in a conflict situation between the preservation of the system and the interests of atomistic depositors.	

3	Contagion spreads within the domestic financial system, often originating from the less regulated and supervised sectors and affecting the core sectors of the system.	
4	Incompleteness of markets and institutions, undermining the efficiency of investment allocation	
5	Distortion, due to government intervention and the nature of regulation.	
6	Sequencing is important. The order in which to expect the components and its factors to develop is of crucial importance. Regulation and supervision should be strengthened before privatization of financial institutions takes place.	

What we can observe here is that the condition is far more complicated for developing countries, in particular because after financial liberalization and deployment of flexible exchange-rate, their economies are for more prone than before.

Not so often cited alternatives to Flexible exchange rate policy (Short term choices)

Other than the often cited methods (long-term policies), such as:

- Returning to fixed exchange rate regime (often it is not realistic approach);
- Or turning to pegging exchange rate;
- Or introducing target zone (unless this policy is adopted as regional policy, instead of implemented by individual country);

In the present paper we will explore some new unconventional approaches, including:

(a) News management for incomplete information;
(b) Modified gravity equations and re-assessment of national assets;
(c) Gradual changes toward target-zone (regional monitor);
(d) Considering (tax or fiscal) incentives (and disincentives) for conversion from exchange-rate market investment to industrial investment.

We will discuss first a method we call 'News management'. The idea is that from Stiglitz we learned that actually the information that forms decision in marketplace is incomplete. To summarize:

(*) Incomplete information --> Biased decision --> Biased price

Therefore the price that shows up at NYSE, for instance only reflects a partial amount of information that was received by traders (or investment managers).

Since Information was read from newspaper then the (*) can be written:

(**) Incomplete news --> Incomplete information --> Biased decision --> Biased price

Therefore one possible way to alleviate this problem (or at least reduce the imbalanced information) is via 'news management':

(***) More complete news --> More complete information --> Less Biased decision --> Less Biased price

In other words, in the context of this short-term policy, the government and other exchange-rate 'keepers' will be more proactively release news (of course, not crafting 'illusion' news which in tgurn can deepen the market distrust to the government capabilities). The news then can be broadcasted via numerous ways, often in real time, for instance using digg.com, blogging etc. At the end, the purpose is to reduce the 'imbalance' of information to the decision makers of investment companies.

Another approach that broadcasting-news, is targeted-news delivery. In this method, news is delivered to targeted people only, for instance to a few investment managers who take care a huge amount of foreign investment. By using Pareto theory, these people can often be identified by selecting the predominating investment companies.

The next possible method is gradual changes toward target zone. Since this method will require significant backup (just like to tackle DDOS attack in network management), therefore it can only be implemented by for instance finance ministers of South East Asia countries, not by individual country.

The next method is to implement (albeit in gradual way) the concept advocated by Tobin sometimes ago, but this time we propose a more positive way. Instead of punishing investors for investing in exchange market, we think it would be more convenient to offer tax/fiscal incentives if the investment companies would like to relocate their investment outside the exchange market. Hence the investment managers can calculate themselves based on opportunity costs (rational choice).

In the next section we will discuss a modification of gravity equation.

Concluding remarks

We have summarized some basic issues related to flexible exchange rates and long-term and short-term choices to manage their related volatility. Some unconventional approaches have been discussed too.

We acknowledge that this study is far from being complete, and therefore would like to invite others to contribute to its further development.

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Salesm@xx Outline

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Asymmetrical information is referring to the case when one side of the market (seller or buyer) has information about the product that other side doesn't. This information can be used by the knowledgeable side in its advantage.
The lack of information in the other side increases its inability of distinguishing the quality of the product.
In setting the price policies, firms normally use the average cost of production and then they add a profit margin, yet this profit should not be rigid but varying with the market demand.
We are talking about a flexible profit.

Salesm@xx is a new cutting edge concept of managing your sales activities using the below VEST formula, proven methods of sales process, along with Sales Management Solution technology.

We introduce a new word VEST, which is abbreviated of a quite simple formula:

$$\mathbf{V = E \times S \times T}$$

where the meaning shall be clearer if we use words instead of letters:

$$\mathbf{Volume(new\ sales) = Exposure \times SuccessRate \times TransactionRate}$$

The word 'VEST' itself has its own meaning, which is a bulletproof suite:

1. *vest n*. A sleeveless garment, often having buttons down the front, worn usually over a shirt or blouse and sometimes as part of a three-piece suit (cf. *www.answers.com/topic/**vest**);*
2. a waist-length garment worn for protective purposes: a bulletproof vest. (cf. *dictionary.infoplease.com/**vest***);
3. *bulletproof vest*. Body Armor Protective covering and other equipment designed to guard individuals in combat.
   (cf. *www.answers.com/topic/bulletproof-**vest**-2).*

Basic Formula:

SV(sales volume) = NS(new sales) + CSB(current sales base);
New sales (NS) = $\sum$ e × SR × TV.
Current Sales Base (CSB) = Customer life time value.

Salesm@xx formula : **SV = ($\sum$ e × SR × TV) + CSB**.

Sales Volume.
Sales volume is the end result. It is not happening overnight. It's like growing rice in the rice field which needs time and effort. Based on our experience SV is a sum of success from several activities (variable) which for easiest way to remember I try to simplify to a Salesmaxx formula which mentioned above. Let's take a look one by one.

New Sales = $\sum$ e × SR × TV.
New sales is the key driver to get the sales growth. Without the strong success of this activity the sales growth is going no where. In our perspective New Sales is a success from 3 group of activities (variable) :

$\sum$e = Sigma of exposure.

This is the effort of exposing the product and service to the market. It's requiring quality & quantity marketing activities approach. Quality mean talking about the right market segmentation & targeting that fit with your product or vice versa. If you hit wrong target segment, then your effort is useless and you just burning the sales & marketing expense with low ROI. If you hit the right target segment with the right product than the chance of success is huge now it is depend on the other variable.

SR = Success Rate.
This is talking about sales process. Like any other process, sales also have a process from identifying the right prospect, approaching, showing the value of the product & service, negotiating the deal and closing the sales. In short it takes 4 steps but it depend the nature of the business it could takes 4, 5, 6 and more steps to close the sales. Salesmaxx will explain in depth about this process.

The success of each step in the sales process requires a selling skill and experience to be more effective and efficient on moving from one step to another. There is no born sales person. I believe everyone born with a sales talent , as an example baby selling the idea to get milk by crying, we dress up nicely, act nicely and talk nicely to attract our opposite sex , isn't that similar on how to attract our potential customer? The different is a successful sales person practicing and sharpening their talent. They are reading selling book, going into sales training and having the right attitude and passion on doing their job. Using Stephen Covey words of Habit, Sales is a Habit. The successful sales person having 3 elements of habit: Knowledge, Skill and Attitude. They know what to do (knowledge) , how to do it (skill) and want to do it (Attitude).

TV – Transaction Value.
The last variable on getting high new sales is depending on the average transaction value you gain. If you highly success on the other variable but very low TV than your sales volume is also low. High & low TV also depend on the nature of the business, if you are selling a natural drinking water in small packaging than it's so obvious that you need very high success on the Sigma of Exposure to get the high sales volume whilst if you are selling a premium property such as house or apartment than you may just need to get one closing per month to have millions new sales volume.

Current Sales Base.
It is depending on the nature of the business, CSB is play very important role when you are in the portfolio business. If your business not a portfolio than you can put zero result on the formula or just simply pass this chapter.

Portfolio business is the accumulation of a routine transaction volume. It is like a consumable product such as : Food & Beverage product, cleaning chemical, banking, executive club, etc. You expect your loyal customer to keep buying your product or services in timely basis (daily, weekly, monthly, etc). It is a retain sales or maintenance sales or a sales foundation to gain more new sales.
The question is how to keep them loyal while your competitor is like a hungry lion want to grab your customer every time everyday. They are proposing better offer, better product with cheaper price or better benefit. Fail on this activity will result low to zero growth of your sales volume regardless how success your new sales activities. But if you have a highly success on this activities, then every new sales will just topping up the base and your sales volume growth just go higher and higher. So yes, in portfolio business the CSB play a great vital to grow your sales portfolio. Salesmaxx will explain more detail on this.

Salesm@xx. Why using @ ??
In to day world, the hyper growth of computing technology, internet and mobilization lifestyle have been changing the way people work and life. In short I call it CIP technology (Computer, Internet, Phone / PDA). This CIP now is no longer a luxurious gadget but it is becoming more and more an important tools to do the job / business. How can you do your job or business without computer? How will you communicate with customer, friend, colleague without email, sms, or phone? Sending a brochure to potential customer by mail is very outdated to day, if you still doing it than try to see if your competitor already sending their brochures & proposal by email. It goes faster and right to the person yet cheaper. Furthermore, maybe they already inviting the customer to just simply clicking their web site and get more information in very nice graphic, illustration, animation, photo, etc, etc. Can you compare the perception impact with your brochure which probably will arrive 1 week later and still in the office mailbox while your potential customer still traveling for another week?

Salesm@xx is introducing the new cutting edge concept of managing your sales activities using the CIP technology. The Salesm@xx selling skill and knowledge will simplify your learning process to be more effective on doing your sales activities while the CIP solution will leveraging your skill to maximizing the sales outcome. So in short Salesm@xx is a new mind set, a new skill set and a new tool set to have better manage and grow your sales volume. It is just a one shot comprehensive solution to be more effective on maximizing your sales outcome. Salesm@xx solution especially designed for a b2b type of business which requires one on one sales approach. If you are a salesman or managing b2b sales organization than you definitely need to consider using salesm@xx approach to maximizing your sales result.

Comment

If we talk of (Expected) sales volume rather than as an 'exact formula' then the SV term above shall be expressed as a probably outcome of a set of activities by sales forces along with tactical approach of advertising etc. It shall also include probable activities by the competitors, in other words, given the number of competitors raise, and then chance is the 'expected sales volume' will be decreasing.

Therefore, perhaps the actual Expected sales volume shall be written as follows:

Expected Sales Volume (ESV) : $\mathbf{ESV = (\sum e \times SR \times TV)/n + CSB}$

where n represents the number of competitors. This is why, as the number of consumer/retail products grow, the producers shall go advertising with massive ads in television, otherwise their marketing efforts will be nothing compared to the number of competitors.

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SalesMaxx:
Step by Step Proven Techniques to Boost Your Sales

D. Handriyanto, V. Christianto, F. Smarandache

Forward.

Have you ever met a salesman who is so confident and has excellent sales performance with apparent ease?

This type of salesmen belongs to 'natural born salespeople' [1], those who rely more on their intuition, gift, unconscious competent, or whatever you may call it. They have something hidden that makes him/her an excellent salesman. But as study shows this type is only approximately 20% percent of the sales talent pool, which can be called "*Eagles*." The remaining 80% are ready, willing, and capable in their sales duties, but they need training, methods, tools, and also proven techniques. These are majority of salespeople who will get benefits from sales training, pitch letter, etc.

This paper is written for those 80% salesmen out there who needs step-by-step proven and consistent methods to get their jobs done, and makes increasing sales performance. We will discuss how a step-by-step proven method –called '*sales process'*- will ensure your team's sales performance.

But this paper is also written for the marketing and sales manager level people, those who have to coordinate and supervise the others. As experience will show [1], an effective management is critical to sales process and sales performance. An effective sales manager will monitor, coach, manage, and supervise their sales force using a *consistent* sales process. Otherwise the sales force will find the guidelines inconsistent and this fact alone will reduce the morale of his/her sales team.

The basic idea of this paper is based on "*The New Solution Selling: The Revolutionary Sales Process that is Changing the Way People Sell"* book, by Keith M. Eades [1]. His concepts have been successfully implemented by computer industries giants like IBM, Microsoft etc. In this paper, we will adapt these concepts into different industries other than software / hi-tech markets. The principles are more or less the same, but the sales process and steps may be different here and there. We also include short discussion on new marketing techniques using internet. There are plethoras of those methods, so we will limit our discussion on the most common methods such as blogging etc. [2]

Moreover, this paper is written based on more than 14 years experience in corporate marketing, which is often called (in technology jargon) as 'B2B' market.

We've tried our best to write this article in as simple and accessible as possible, and removing the plethora or marketing jargons which can make you confusing. We also include some funny stories here and there as refreshment to readers who are new to marketing world [3]. For those experienced salespeople and marketing specialists, please skip the jokes and just cling with the ideas.

At the end of this paper we also include a review of sales process software which may be found useful as illustration on how the concepts described herein can be implemented consistently. We call this software 'SalesMaxx' and can be accessible for test purpose at www.salesmaxxonline.com.

We hope you to let the ideas in this paper to help you to build solid sales team based on solid sales process to reach solid sales performance.

Wishing you plenty of happy sales moments!

January 2009

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1. Introduction

When you hear the word '*sales*', what is your first reaction in your mind? Some of you will recall happy moments when you got the first clients, and some of you recall worst moments where your clients moved to another product. If you are in management level, most likely the same word will remind you to the nights that you have to go home late because you should attend sales meeting.

As in other specialized fields, the sales and marketing world have their own meaning for words, especially for hi-tech sales people. See for instance Table 1, just for hint.

Table 1. The Dictionary: what hi-tech salespeople say and what they mean

Word spoken by sales people	It's real meaning
New	Different color from previous design
All new	Parts not interchangeable with previous design
Unmatched	Almost as good as the competition
Designed simplicity	Manufacturer's cost cut to the bone
Foolproof operation	No provision for adjustments
Advanced design	The advertising agency doesn't understand it
Field-tested	Manufacturer lacks test equipment
High accuracy	Unit on which all parts fit
Direct sales only	Factory had big argument with distributor
Years of development	We finally got one that works
Breakthrough	We finally figured out a way to sell it
Futuristic	No other reason why it looks the way it does
Distinctive	A different shape and color than the others
Hand-crafted	Assembly machines operated without gloves on
Performance proven	Will operate through the warranty period
Meets all standards	Ours, not yours
Broadcast quality	Gives a picture and produces noise
High reliability	We made it work long enough to ship it.
Microprocessor controlled	Does things we can't explain

Source: http://jokes.maxabout.com/jid0017483/the_dictionary_what_hitech_sa.aspx.

Oh, yes, it aim's that bad at all. After all as salespeople you can boast yourself as the real people who do real jobs to get things done by meeting real customers who actually buy your products. In other words, you are the 'user-interface' -- in computer parlance -- that customers will touch, and speak to, and for sure –uh- express their feelings too, be it good or bad mood.

Therefore in this article we focus on real, daily sales issues, not the high-level strategic marketing issues, which belong to those in executive management level.

To simplify the present paper, we will begin with a summary of ideas to be discussed in the remaining sections of this paper.

1.1. **Summary of ideas**

To simplify our ideas, we introduce a new word VEST, which is abbreviated of a quite simple formula:

$$\mathbf{V = E \times S \times T}$$

Where the meaning shall be clearer if we use words instead of letters:

Volume(new sales) = Exposure × SuccessRate × TransactionRate

The word 'VEST' itself has its own meaning, which is a bulletproof suite:

1. *vest n*. A sleeveless garment, often having buttons down the front, worn usually over a shirt or blouse and sometimes as part of a three-piece suit. (cf. *www.answers.com/topic/**vest)***
2. a waist-length garment worn for protective purposes: a bulletproof vest. (cf. *dictionary.infoplease.com/**vest***)
3. *bulletproof vest*. Body Armor Protective covering and other equipment designed to guard individuals in combat.
   (cf. *www.answers.com/topic/bulletproof-**vest**-2)*

In other words, the word VEST would mean that using the step-by-step proven methods described in this article would enable you and your sales team to secure sales performance when the marketing competition seem increasingly difficult to get in. And more over, how to keep your customers despite fierce competition).

This is simply a set of proven methods to keep your business 'bulletproof'.

1.2. **Volume (New Sales)**

New sales are the key driver to get the sales growth. Without the strong success of this activity the sales growth is going no where. In my perspective New Sales is a success from 3 group of activities (variable), i.e. Exposure, SuccessRate, and TransactionVolume.

And the New Sales is related to the total Sales Volume by the virtue of this relation:

Sales Volume = New Sales + Present Sales Volume

In other words, without introducing New Sales, your Sales Volume is stagnant. And worse than that, it is likely that your Present Sales Volume is not a static number, because your products also have their own life-time expectation. In other words, what customers like ten years ago may be different with what they like today. In hi-tech industries this lifetime could be less than 3 years. Therefore you can also introduce 'depreciation rate' into your Present Sales Volume, if you wish to emphasize that what you got now is not for eternal.

1.3**. Exposure (E)**

This variable is expressed in number (or times of exposure).

This is the effort you shall make in order to expose the products and services to the marketplace. It requires quality and quantity marketing activities approach. Quality mean talking about the right market segmentation and targeting that fit with your product or vice versa.

For example, if you hit wrong target segment than your effort is useless and you are just burning the sales and marketing expenses with low ROI. If you hit the right target segment with the right product than the chance of success is huge now it is depend on the other variables.

1.4. **SuccessRate (S)**

This variable is expressed in percentage (%).

This is about sales process. Like any other processes, sales also have a set of steps from identifying the right prospect, approaching, showing the value of the product and service, negotiating the deal and closing the sales. In short it takes 4 steps but depending on the nature of the business it could takes 4, 5, 6 and more steps to close the sales. We will also discuss the **Salesm@xx** solution will explain in depth about this process.

The success of each step in the sales process requires a selling skill and experience to be more effective and efficient on moving from one step to another. There is no born sales person. I believe everyone born with a sales talent. The different is a successful sales person practicing and sharpening their talent. They are reading selling books, going into sales training and having the right attitude and passion on doing their job. Using Stephen Covey's words of Habit, Sales is a Habit. The successful sales person having 3 elements of habit: Knowledge, Skill and Attitude. They know what to do (knowledge) , how to do it (skill) and want to do it (Attitude).

1.5. **TransactionRate (T)**

This variable is also expressed in percentage (%).

The last variable on getting high new sales depends on the average transaction value you gain. If you are highly successful on the other variables but gets very low TransactionRate (%) than your sales volume is also low.

High and low TV also depends on the nature of the business. For example, if you are selling a natural drinking water in small packaging than it's so obvious that you need very high transaction rate in order to get a high sales volume, while if you are selling a premium property such as house or apartment than you may just need to get one closing per month to have a big new sales volume (in terms of currency).

The difference between TransactionRate and SuccessRate is between the industry-wide or nature of your business (be it helicopter, property, or toy), and the inside-nature of your company, i.e. how you follow methodically a step-by-step bulletproof methods which ensure that each prospect can be handled properly until the sales is achieved.

1.6. **Present Sales Volume (P)**

Present sales volume (P), is also called 'Current Sales Base'.

Depending on the nature of the business, Present sales volume (P) plays very important role when you are in the portfolio business. If your business not a portfolio than you can put zero result on the formula or just simply pass this section.

Portfolio business is the accumulation of a routine transaction volume. It is like a consumable product such as: Food & Beverage product, cleaning chemical, banking, executive club, etc. You expect your loyal customer to keep buying your product or services in timely basis (daily, weekly, monthly, etc.). It is retained sales or maintenance sales or a sales foundation to gain more new sales.
The question is how to keep them loyal while your competitor is like a hungry cheetah want to grab your customer every time everyday. They are proposing better offers, better products with cheaper price or better benefit. Failing to do these activities will result low to zero growth of your sales volume regardless how success your new sales activities.

But if you are highly successful on these activities than every new sales will just get topping up the base and your sales volume growth just go higher and higher. So yes, in portfolio business the CSB play a great vital to grow your sales portfolio.

1.7. **What is Salesm@xx**

Salesm@xx is a new cutting edge concept of managing your sales activities using the above VEST formula, proven methods of sales process, along with Sales Management Solution technology.

Acquiring the **Salesm@xx** selling knowledge will simplify your learning process to be more effective on doing your sales activities while the Sales Management Solution will enable you leveraging your skills to maximize the sales outcome.

In short **Salesm@xx** is a new mind set, a new skill set and a new tool set to have better sales management and grow your sales volume. It is just a one shot comprehensive solution to be more effective on maximizing your sales outcome. **Salesm@xx** solution is especially suitable for a B2B type of business which requires one-on-one sales approach. If you are a salesman or managing B2B sales organization than you definitely need to consider using **Salesm@xx** approach to maximizing your sales result.

Of course, if you are managing a B2C-type of business, you can also reap the benefits from **Salesm@xx** selling knowledge too, because it is very easy to adapt the concepts described herein.

The last section will discuss **Salesm@xx** solution will explain in depth about this process.

2. **Exposure**

The exposure concept is not so difficult to grasp, in particular if you already had your feet wet in sales and marketing world. How to get things done right, and in effective way, that is the real question.

The principle is also very simple, i.e. as an old saying*: "you will reap what you sow."* Of course before we sow, first of all we shall find out what kind of seed to sow, and whether it is appropriate to the fields. Furthermore, you shall count the quantity of seeds needed, and also the quality, otherwise they may be damaged during the season and therefore useless.

Now we turn our discussion to real marketing world.

2.1. **Segmentation**

Let's begin with a story, probably a real one (thought not exactly sure who did what when).

> A door-to-door vacuum cleaner salesman manages to bull his way into a woman's home in a rural area.
>
> "This machine is the best ever" he exclaims, whilst pouring a bag of dirt over the lounge floor.
>
> The woman says she's really worried it may not all come off, so the salesman says, "If this machine doesn't remove all the dust completely, I'll lick it off myself."
>
> "Do you want ketchup on it?" she says, "we're not connected for electricity yet!"[1]

So perhaps you know what is the message here, i.e. there is no point to sell vacuum cleaner if the house has no electricity in the first place.

Not only that, customers is becoming more diverse, according to studies by geodemographers, people who study the population characteristics. For instance, Clarita determined in the 1970s that 40 lifestyle segments were sufficient to define US populace. Today, the number has grown to 66, a 65% increase. [2]

Moreover, researchers begin to talk about the significance of ethnographic marketing, in particular if your region is ethnically diverse. In principle ethnographic marketing discuss how to use anthropologists' findings to understand how people use and relate to products and services.

To summary, goals of ethnographic marketing include:

- In-depth understanding of consumer;
- Seeing things from customer's viewpoint;
- Being open to different points of view;
- Exploring contexts and conditions;
- Emotions behind consumer behavior;
- Multi-vocal methodology.

As R.V. Kozinets (Kellog School of Mgmt) puts it:

> "Anthropology offers marketing ways to understand a variety of concerns important to marketers and marketing researchers, including language, taste consumption linkages, desires, motivations, and the decision making influences of particular consumers and consumer groups."

2.2. **Getting Prospects, the Offline Way**

2.2.a. **Advertising**

To boost exposure, of course the simplest way is to put some ads at places that your potential prospects will find them. These places could be newspapers, radios, television, movies, apartment walls, booklets, magazines, etc. One can also think of numerous unique ways to put these ads, for instance by installing a large balloon at the top of high-rise office building will increase chance of people to see, if nothing else but attractive color and fancy design.

2.2.b. **Events**

Events can be music performance, art performance, sport events, but they can also special gatherings like thanksgiving parties, high-school parents' meeting etc. The point here is not only the quantity of people who gather, but also how they could possibly relate to your products / services. It is not advisable to dumb all your expenses on a few great

performances only to get your banner displayed but no follow-up calls to potential customers.

If none of these events meet your requirement, you can invite local artist to do special dances or music performance but at unusual place near market or other public place.

2.2.c. **Publications**

Publications here don't only mean your CEO meets someone from outer space then your products get attention by media. You can get publication almost freely by doing something uniquely. Or at least ask a special team doing that for you, for instance roller-skating at the roads, wearing red-sunglasses etc.

Body Shop has its own unique way to get publication, i.e. by developing solid reputation for green products. That way people know where the products came from and they buy not only with their pocket, but also with their hearts.

In marketing parlance, this method is called doing marketing the spiritual level. It is called the best method a marketer can do.

2.2.d. **Telemarketing**

Telemarketing is also an effective method, in particular to get initial on your customer needs, We will learn on the next sections, that various methods depend on at what step is your sales process. Telemarketing is probably most suitable to get access and collect information from your customer (what they do, what they want, what are their problems, what are their pains).

In order to be more effective, it is also advisable to collect database first of your potential prospects, and select them to find out whether their business are suitable to your products and services.

2.2.e. **Other nonconventional methods: guerilla methods**

Some of those techniques described above may require large expenses, therefore may be suitable if you work with large budget or large companies (unless perhaps telemarketing).

As most companies are only followers, not leaders, then the methods of the leaders may not be suitable for them. Therefore in recent years, there is more attention on highly-efficient method called guerrilla marketing.

But beware, as telemarketing and other cost-efficient online marketing methods grow nowadays, some people say that guerilla marketing is dead. [5][6][7]

2.3 **Getting Prospects, the Online Way**

A tagline of Yahoo's search marketing says 'Customers don't go on tree, they grow online."[4] This is exactly the message of this section, in the high-paced business nowadays, you need to reach your prospects not only at work hours, or at workplace, or at their houses. You need to be able to reach them not only at any time, any place, 24x7 hours a week, but also at the time that they want to, and at the place they wish to hear your message. Too much often we hear the story of dropped-off salespeople who try to reach the wrong person, at the wrong time, and at the wrong mood!

Online presence is one possible way to resolve this issue, i.e. by leveraging your exposures effectively while keeping your marketing expenses effective. Here we discuss a number of possible ways you can do. Of course we discuss here a few simpler ways that you can do quickly; for more advanced discussion on this subject you can find good books on internet marketing, etc.

2.3.a. **Homepage**

Homepage doesn't mean only static text like boring books that you dumb quickly under your desk. It should also not be so flash-heavy animated contents that took weeks to prepare and some minutes to open.

But at least your homepage should be informative, and functionally effective. If you want to trigger sales from your homepage, then at least it should display your products/services in attractive way, and enable visitors to ask for more information if they need to. It shall also enable receiving-payment, either the simple way (direct postal), or using advance method such as paypal or clickbank.

2.3.b. **Newsletter, etc**.

Apart of coverage/publication, exposures also require reputation. And reputation is built based on relationship and trust. This alone cannot be achieved via massive advertising; one shall instead build trust via online methods, such as email-based newsletter. If you keep on your prospects and customers with newsletter, they will know that you care about them. And from this trust is built, at the end you will achieve customer loyalty.

Of course, there is trade-off between customer loyalty and marketing expenses, so that CEO in last years think at Customer Relationship Management as an expensive methods to keep customer loyal, when in fact customer loyalty is next to illusion, The key here is perhaps to use effective Customer Relationship methods which offer highest leverage while keeping costs at minimum. Newsletter is one of this method.

2.3.c. **Blog marketing**

Blog is like diary you write at the end of the day. But the difference is it's online and plenty of folks can read your diary. You can use free blogging method, and then post your content using RSS feeder directories or using digg.com.

The most important things to remember when you develop specific blog for your company's products/services are [3]:

- Offer your visitors advice, tips and other opportunities, like free reports or something else related to your business.
- Encourage your visitors to post comments and/or suggestions.
- Post as often as you can. It is recommended that you post daily.
- You can allow other people to repost your material on their websites as long as you let them know that the resource box must remain the same, this means that they cannot change your links in any way.
- Always keep good content on your site. Make sure you stay up-to-date with your information.
- Make sure you link to other sites from within your site. Add affiliate links through banners etc.

2.4. **The Role of Database.**

There are already some books written on the subject of *database marketing*. The idea is that you instruct your database to do more work using advanced techniques to detect patterns or habits of your customers.

For example, SuperStuff [2], a US retailer uses such an advanced technique called CHAID with their database to find out that 310 out of 508 of their department stores have EBIT (earnings before interest and taxes) less than 2.8%, while the remaining 198 outlets have EBIT more than 6.4%. But then they find out that those stores which have averagely less EBIT at 2.8% have their competitor's outlets nearby. Not only that, they can also find out that stores located in neighborhoods with higher-income household get larger EBIT rate.

And based on these data, they can remodel ideas, to find out that of 188 stores which are located in neighborhoods with high-income household and have the competitor's outlets nearby, those who have allocated more than 50% of their square footage to apparel have larger EBIT margins of 5.3%.

This information can be used by management to remodel or reallocate their stores and increase their EBIT margins significantly for other stores.

The crux of the idea is using advanced statistical methods (e.g. CHAID [2]) to identify which numbers are statistically meaningful. That is between two or more groups which have different EBIT we can find out whether the difference is more-or-less in the same pattern, or they indicate some reasons behind these differences.

And so on, you can find more ideas by digging on advanced machine learning or database marketing subject. We limit our discussion here because it will be more tedious.

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Cultural Advantage as an Alternative Framework: An Introduction

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Abstract

Despite the economics jargon on 'rational choice', nowadays the entire world has nothing else to choose except to succumb under the spell of magic words of modern economics, i.e. 'neoliberalism', 'financial liberalization', 'free market' (*laissez-faire*), and 'globalization'. All of these can be shown to be part of a preconception, i.e. far beyond the 'neutral' idea of natural sciences.

In Fritjof Capra's book 'Turning Point' (Bantam Books, 1982) these phenomena are summarized as follows: economics thinking have started by assuming that in economics sciences one can achieve the same generality and universality that physicists enjoy in doing Natural Sciences. In other words, economists try to become through their work 'hard science' rather than recognizing that in economics the subject of their study is human/people which is far from being predictable, either as individual or as society.

In our humble opinion, economics is a mixture of both, hard and soft sciences. In order to show this, we introduce a new study, called **Poly-Emporium Theory**, where we show that phenomena from hard science and soft science co-exist and interact in economics. Poly-Emporium Theory is the study of interactions among many (big and small) firms in the market, and it is different from *oligopoly* since poly-emporium takes into consideration the small firms too (not only the big firms that dominate the market as in oligopoly).

The above logic of thinking is the starting point to submit a new idea, under the heading of 'Cultural Advantage.' The first book in the series has title: Cultural Advantage for Cities: An alternative for Developing Countries. This presentation summarizes its basic ideas, with a hope that these ideas may be found interesting to develop further. For clarity the readers are referred to the book.

Introduction

In simple words, the entire history of economics as 'science' can be summarized as systematic methods to give reasonable explanation of human behavior in order to fulfill their needs. Furthermore, the progress was inspired by the remarkable success of Newtonian mechanics in describing the 'world' [2].

In the same way, economists since Adam Smith strived so hard to bring 'order' into the apparently chaotic phenomena with respect to human responses to various variables (government taxation rules, market competition, etc.). In conclusion, the strange history of Economics can be summarized as follows:

> "These days' people like to call neoclassical economics "mainstream economics" because most universities offer nothing else. The name also backhandedly stigmatizes as oddball, flaky, deviant, disreputable, perhaps un-American those economists who venture beyond the narrow confines of the neoclassical axioms. To understand the powerful attraction of those axioms one must know a little about their origins. They are not what an outsider might think. Although today neoclassical economics cavorts with neoliberalism, it began as an honest intellectual and would-be scientific endeavor. Its patron saint was neither an ideologue nor a political philosopher nor even an economist, but Sir Isaac Newton. The founding fathers of neoclassical economics hoped to achieve, and their descendents living today believe they had, for the economic universe what Newton had achieved for the physical universe." [2]

Despite the economics jargon itself on 'rational choice', nowadays the entire world has nothing else to choose except to succumb under the spell of magic words of modern economics, i.e. 'neoliberalism', 'financial liberalization', 'free market' (*laissez-faire*), and 'globalization'. All of these can be shown to be part of a preconception, i.e. far beyond the 'neutral' idea of natural sciences.

Another implication of this neoclassical economics can be summarized as follows:

> "Neoclassical economics is by its own axioms incapable of offering a coherent conceptualization of the **individual** or **economic agent**. From where do the preferences that supposedly dictate the individual's choice come from? Not from interpersonal relations, because if individual demands were interdependent, they would not be additive and thus the market demand function – neoclassicalism's key analytical tool – would be undefined. And not from society, because neoclassicalism's Newtonian atomism translates as methodological individualism, meaning that society is to be explained in terms of individuals and never the other way around." [2]

A caveat of financial liberalization has often been discussed in monetary policy sessions, i.e. studies revealed that liberalization is neatly linked and often precedes financial instability. In other words, the magic word has now become the curse and peril for the modern-economics believers [3]:

> "Following liberalization, many developing countries found themselves involved in a condition of high instability and increasing fragility of their financial systems. Therefore, the question arises as to why countries should enact policies that move their financial systems from a situation of relative stability to one of potential instability."

In Fritjof Capra's book 'Turning Point' (Bantam Books, 1982) these phenomena are summarized as follows: economics thinking have started by assuming that in economics sciences one can achieve the

same generality and universality that physicists enjoy in doing Natural Sciences. In other words, economists try to become through their work 'hard science' rather than recognizing that in economics the subject of their study is human/people which is far from being predictable, either as individual or as society.

> "As we know, natural sciences are normally considered as '*hard science'*, while social sciences are considered as '*soft science'*. This terminology can be traced back to Fritjof Capra, etc. In the meantime, some economists consider themselves as doing 'hard science' while other seem to be inclined to 'soft science'. Not surprising, therefore, that some economists seem very accustomed to prescribing solutions to economics problems, using hard technologies, hard methods, *vis a vis* humanistic considerations. See also E. F Schumacher's thinking on 'meta-economics'.
> Therefore, by considering Cultural advantage here, we are practically introducing more '*soft sciences'* into economic thinking. In other words, unlike modern economics that is more likely to be 'alienated' to the cultural context of the 'people' where they are implemented, here we propose to introduce more 'Cultural studies' before prescribing a new solution, especially for developing countries." [1]

With this new insight, we try to look again to human as human, not only as 'measure' of economics textbook, or just an object in the annual economic progress report.

With respect to development theory, the implications of those modern economics concepts can be summarized in terms of conventional belief that to become prosperous all countries should take the same industrialization path as other countries in the First World have taken. This is known as Rostow's development theory , which can be summarized as follows: [11]

> "The process of industrialization entails a transition from an agricultural to an industrial society, associated with a movement towards higher per capita income and productivity levels."

Despite all the jargons surrounding this development theory, it is recognized that the development via industrialization method has not been so useful so far, in other words most countries remain in the same problems as before: [13]

> "This development is, unfortunately, often more symbolic than real for many countries and actually helps these societies very little. Industrialization is not the solution for many countries seeking to improve conditions for their citizens." [13]

In other words the development theory is quite similar to an 'ideology' rather than a science [12]; it is full of premises based on perception or interpretation of history in the so-called First World countries [14].

In an attempt to make a connection between economics as hard science and economics as soft science we propose a new theory on Poly-Emporium, which will be described in the last section of this presentation. In our humble opinion, economics is a mixture of both, and soft sciences.

A modified gravity equation and some implications

The so-called gravity equation has been known by economists for more than 4 decades with various degree of acceptance. There are numerous studies that have estimated gravity equations to quantify impact of various trade costs on bilateral trade flows. [5, p.5]

While this model is widely-known for its simplicity, part of the critics addressed to this model is caused by its precision to the actual situation. It is also often cited that the gravity equations have no sufficient theoretical grounds [5, p.5]. We can call this issue as 'representation problem.'

In this section we discuss first a review of existing literature on this equation, and how it can be modified to represent better the actual condition.

(a) Existing models of gravity equations.

In accordance with Anderson and van Wilcoop, the gravity equation can be written as follows [5, p.6]:

$$x_{ij} = \frac{y_i y_j}{y^w}\left(\frac{t_{ij}}{P_i P_j}\right)^{1-\sigma} , \qquad (2)$$

Where x represents the nominal demands of country j from goods from country i, and y^w represents the World output, respectively. The other parameters are normally determined by curved fitting plot. [5, p. 7-8]. The Pi and Pj parameters are often cited as Dixit-Stiglitz consumer-based price levels [7, p.4].

Another expression of 'gravity equations' can be expressed as follows [6, p.48, eq. 4.12]:

$$\Pr X_{ij} = \frac{Y_i^{0.5} Y_j^{0.5}}{2} , \qquad (3)$$

Which is often cited to be not realistic and oversimplified. Other studies for exchange market problem have been reported, for instance see [8][9].

(b) A modified gravity equation.

What is clear from the above summary of gravity equation is that the role of geography (distance) between countries affects the trade between them [10]. Therefore it is worthwhile to take into consideration not only geographical distance, but also geographical assets and cultural assets into the 'gravity potential' of trade between two countries (sometimes it is related to potential FDI, see [7][8][9]).

In other words, the proposed modified gravity equation here can be expressed as follows:

$$\Pr X_{ij} = \frac{Y_i^{0.5} Y_j^{0.5}}{2} + \sum GCP_i + \sum GCP_j \ , \tag{4}$$

Where the GCP with index i represents the sum of geographical and cultural potential (assets) of the country i, and the GCP with index j represents the same potential for country j, respectively.

Rationale for this modification is because the role of location can be introduced into gravity equation to achieve better representation of the model.

This equation (4) can take into consideration the 'demand pull' of *eco-tourism* of a country, for instance. And the other pull factors can be introduced into the equation; this is why we introduce the 'sigma' symbols.

Therefore it can be expected that equation (4) can lead into more realistic economics model.

Interpretation of the equation:

(a) The gravity equation (2)-(3) represents bilateral trade magnitude between two countries given their distance and GDP. Of course, one can ask whether GDP alone can 'pull' the bilateral trade. For instance, small countries can have larger GDP than China, for instance Belgium, but we know that almost all Europe's large companies are heading toward investing or relocating into China, not Belgium. Therefore GDP alone is not triggering bilateral trade. What seems make more sense is that the size of the economies shall be taken into consideration too into (3). Therefore perhaps it would be more appropriate to replace the GDP with 'economy scale', which is GDP times population of the country in question.

(b) Another thing we can conclude from gravity equation (3) is that GDP will trigger FDI, not the other way around. This appears to be in contradiction with the common assumption in development theory, i.e. that FDI will improve GDP of the country in question. Sounds like circular logic?

(c) Furthermore, in equation (4) we introduce new terms in the right hand side of the equation. Given the electronic integration of the global marketplace, it would mean that there could be economics bilateral trade despite the distance of two or more countries. With comparison with instantaneous action at distance in Quantum Mechanics, then perhaps one can think of possible 'economics entanglement' between countries in distance.

(d) To include sigma symbols into the original gravity equation (3) will give no clue for the situation, except perhaps if we consider a 'group' (or cluster) of countries, for instance EURO to Latino America, etc.

Other plausible ideas:

- Is it possible to mixing gravity equations (1) and (2) and then adding the cultural stuff.
- How to write a gravity equation for a group of countries?

Poly-Emporium Theory (F. Smarandache)

We now propose the poly-emporium theory. A search done in Google on May 3rd, 2008, for the term "poly-emporium" returned no entry, so we introduce it for the first time.

Thus "poly-emporium" etymologically comes from *poly* = many, and *emporium* = trade center, store with a wide variety of selling things; therefore *poly-emporium* is the study of interactions among many (big and small) firms in the market.

Poly-emporium is different from *oligopoly* since poly-emporium takes into consideration the small firms too (not only the big firms that dominate the market as in oligopoly). Poly-emporium considers the *real situation* of the market, where big firms and small firms co-exist and interacting more or less.

First, let's present the *duopoly* theory, which is a theory of two firms that dominate and interact in the market, proposed by A. Cournot (1801-1877) in the year 1838. In Cournot's model, if one firm changes its output, the other will also change its output by the same quantity, and eventually both firms will converge towards equilibrium.

In 1883 Bertrand's duopoly model, devised by Joseph Bertrand (1822-1900), if one firm changes its price and the second firm follows, eventually both firms would reach a price (equilibrium) where they would stay.

Both models are similar to two mathematical sequences that little by little converge towards the same limit.

Bertrand's model is criticized because it ignores the *production cost* and market entry by *new firms*.

In oligopoly, which is an extension of duopoly, a small number of selling firms control the market. There is a big degree of interaction among these firms, which set the price, and the price is high and rigid. There is a perfect oligopoly, where all firms produce an identical product, and imperfect oligopoly, where the firms' products are differentiated but in essence are similar.

Sir Thomas More (1478-1535) used this theory in his "Utopia" (1516) and then A. Cournot did. Each firm can act as a leader on its market share, either they collude, or one firm sets the price and others follow.

An analogue of oligopoly is the *oligopsony*, where a few buying firms control the market. They set the price which is normally low and rigid.

The cartel (or trust) influences the price too by regulating the production and marketing, but its influence is of less degree than monopoly's or oligopoly's. Inflexible price or administered pricing (1930s) is set in monopolies, oligopolies, government organizations, cartels.

How would interact n firms, F_1, F_2, …, F_n, for $n \geq 3$, producing a similar product in the same market? A firm can be a business, a corporation, a proprietorship, or a partnership.

There are three cases of the *poly-emporium*, which will be detailed below:

1) All firms are large and they dominate the market, so we have an oligopoly or oligopsony.

2) Some firms are large, and dominate a big share of the market, while others are small, and do not dominate.

   In this sub-case, either the small firms are grouped around some of the large firms (as satellites) just as in growth-pole theory, other small firms might exit the competition.

   This case also includes the possibility that new firms enter the market, so they commence by small investments and later can grow.

   The relationship between large firms in this case can lead either to oligopoly/oligopsony if they succeed to eliminate the small competitors, or to semi-oligopoly/ semi-oligopsony if they control a big part of the market, but not the whole market.

   Small firms might collude and form larger firms.

3) All firms are small and they do not dominate the market.

As in mathematics, it is akin having n sequences, which interact, that we need to study their limit. Would they converge towards the same limit?

Surely, there would always be a *monopolistic competition* between them.

As in *monopoly*, each firm attempts to dominate the market, to prevent competition, in order to control the price. But monopoly is outlawed in most capitalistic countries. If one firm, let's say (without lost of generality) F_1, alters its output, the others F_2, …, F_n, should also respond, otherwise they loose customers.

If it's an imperfect competition, i.e. a market with a large number of sellers and buyers but having differentiated products, the interaction between these firms is less than in a perfect competition, and

they all tend towards a so-called in our opinion **multi-equilibrium**, as in a weighting machine with many balances, or as in a mathematical weighted average.

Nevertheless, if these firms produce a homogeneous product for many buyers, as in perfect competition, their interdependence increases. Disequilibrium of one firm would affect others.

If superior technology commences to be introduced by some firms, the quality of their product will increase and the price decrease.

This may generate the theory of growth-pole, enunciated by Sir William Petty (1623-1687) and François Perroux (1903-1987), which refers to the fact that smaller firms are grouped around a central core of firms that become catalysts. Maximum growth and product excellence for these firms presumes optimal management.

If the government controls the cultural economics, then trade unions of cultural workers should be created for counter-balancing. Because this gives birth to a bilateral monopoly, which is a market with a single buyer and a single seller, mostly referring to the government dealing conditions and salaries with unions of workers.

The dynamicity of the market keeps the firms in a permanent competition, and competition means progress.

We extend Engel's law (1857) that the proportion of income spent on food falls as individual income increases to a similar law related to cultural economics:

As individual income increases, the proportion of amount spent on cultural event decreases.

Thus, as individual income increases an acceleration of cultural economics occurs.

Moreover, adjusted from the absolute income hypothesis (1936, 1960s, and 1970s) by J. M. Keynes and later refined by James Tobin (b. 1918), we derive the **absolute income cultural hypothesis** applicable to the cultural economics: as income rises, cultural consumption rises but generally not at the same rate.

The 18th century absolute advantage theory, which states that people and nations trade since they have exceeding production in some particular field, does not apply in cultural economics. Nor comparative advantage approach that superseded absolute advantage theory works, because we can't really compare cultures.

Comparative cost, developed by Robert Torrens (1780-1864) and David Ricardo (1772-1823), which is a feature of comparative advantage, asserts that trade between countries is benefic even if one country is more efficient, because of the variety of products. Similarly, cultural economics benefits from its cultural difference. The more distinguishable is a culture, the better chance of increasing the cultural economics.

Economic culture can be viewed both as part of cultural economy, art (craftsmanship) economy, and also part of (music) entertainment industry, and depends on taste, advertisement, curiosity, history, and the quality of being diverse, distinctive, with a large spectrum of varieties.

The most interesting case is the third one, where all n firms are small and they do not dominate the market. Let's see, for example, a network of independent restaurants in a city. They interact little with each other. The quality, taste, distance, and price of course make the difference between them.

They do not collude but in rare situations since each of them has its specific, its exotism, which they don't want to loose. They cannot make an oligopoly since new restaurants may easily enter the market with its specific, and because the taste changes periodically. They remain into multi-equilibrium. Similarly for international cultural economics, where each culture has its specific, and that's what attracts visitors, tourists.

In general, the n firms eventually tend towards multi-equilibrium, where they stay for a while. In multi-equilibrium each firm tends towards its specific sub-equilibrium.

Periodically this multi-equilibrium is partially or totally disturbed, due to technology, government intervention, wars, crises, reorganization of the firms, change in customers' taste and preferences, but then again the firms return to stability. This period of multi-disequilibrium is a natural state, since economy is dynamic, and the disturbance is a launching pad to refreshment; in order to rebalance the market, these n firms must improve their technology, their structure, cut production cost, or else they exit the competition. "All the bad for the good", says a Romanian proverb, so disequilibrium brings later new blood into economy.

This cycle of multi-equilibrium - multi-disequilibrium repeats continuously.

Economics systems move from *multi-disequilibrium* to *multi-equilibrium* back and forth [this is hard science, since it is an economics invariant], but the movements/changes from one to another are not easy to predict when and how, nor to control [this is soft science, because of the small probability that we can calculate them with].

Concluding remarks

The idea of Cultural Advantages - while perhaps has been discussed elsewhere - is mostly treated only as sub-chapter in discussions concerning competitive advantage, or development economics studies. But most economics students keep on thinking in the framework of Ricardo-Adam Smith's Comparative Advantage or Porter's Competitive Advantage, i.e. a country should be able to offer goods at competitive prices (read 'low prices') to keep its competitive edge.

But in the framework of Cultural Advantages, these rules are now changing. While price keeps on being a determining factor, other factors also play critical roles, for instance willingness to learn new cultures, and to gain new (exotic) experience, which can be found by visiting other countries. This is the beginning of Cultural Advantage studies.

To summarize our ideas in this presentation, the cultural economics is possible mainly because modern consumers demand not only 'goods' (called 'mass products'), but also experience (learn each other's cultures, languages, etc.)

We acknowledge that this study is far from being complete, and therefore would like to invite others to contribute to its further development.

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Conditional probability of actually detecting a financial fraud – a neutrosophic extension to Benford's law

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Abstract

This study actually draws from and builds on an earlier paper (Kumar and Bhattacharya, 2002). Here we have basically added a *neutrosophic dimension* to the problem of determining the *conditional probability* that a financial fraud has been actually committed, given that no Type I error occurred while rejecting the null hypothesis **H_0:** The observed first-digit frequencies approximate a Benford distribution; and accepting the alternative hypothesis **H_1:** The observed first-digit frequencies do not approximate a Benford distribution. We have also suggested a conceptual model to implement such a neutrosophic fraud detection system.

Key Words

Benford's law, forensic accounting, probability distributions, neutrosophics

Re-visiting the problem of testing for manipulation in accounting data

In an earlier paper (Kumar and Bhattacharya, 2002), we had proposed a Monte Carlo adaptation of Benford's law. There has been some research already on the application of Benford's law to financial fraud detection (Carslaw, 1988 and Busta and Weinberg 1998). However, most of the practical work in this regard has been concentrated in detecting the first digit frequencies from the account balances selected on basis of some known audit sampling method and then directly comparing the result with the expected Benford frequencies (Raimi, 1976 and Hill, 1998). We have voiced our reservations about this technique in so far as that the Benford frequencies are necessarily *steady state frequencies* and may not therefore be truly reflected in the sample frequencies. As samples are always of finite sizes, it is therefore perhaps not entirely fair to arrive at any conclusion on the basis of such a direct comparison, as the sample frequencies won't be steady state frequencies.

However, if we draw digits randomly using the *inverse transformation technique* from within random number ranges derived from a cumulative probability distribution function based on the Benford frequencies then the problem boils down to running a *goodness of fit* kind of test to identify any significant difference between observed and simulated first-digit frequencies. This test may be conducted using a known sampling distribution like for example the *Pearson's* χ^2 *distribution*. The random number ranges for the Monte Carlo simulation are to be

drawn from a cumulative probability distribution function based on the following Benford probabilities given in Table I.

Table I

First Significant Digit	1	2	3	4	5	6	7	8	9
Benford Probability	0.301	0.176	0.125	0.097	0.079	0.067	0.058	0.051	0.046

The first-digit probabilities can be best approximated mathematically by the log-based formula Benford derived: **P (First significant digit = d) = $\log_{10}$ [1 + (1/d)]** (Benford, 1938).

Computational Algorithm:

1. Define a finite sample size n and draw a sample from the relevant account balances using a suitable audit sampling procedure

2. Perform a continuous Monte Carlo run of length $\lambda^* \approx (1/2\varepsilon)^{2/3}$ grouped in epochs of size n using a customized MS-Excel spreadsheet. Derivation of λ^* and other statistical issues have been discussed in detail in our earlier paper (Kumar and Bhattacharya, 2002)

3. Test for significant difference in sample frequencies between the first digits observed in the sample and those generated by the Monte Carlo

simulation by using a “goodness of fit” test using the χ^2 distribution. The null and alternative hypotheses are as follows:

H_0: The observed first digit frequencies approximate a Benford distribution

H_1: The observed first digit frequencies do not approximate a Benford distribution

This statistical test will not reveal whether or not a fraud has actually been committed. All it does is establishing at a desired level of confidence, that the accounting data has been manipulated (if H_0 is rejected).

However, given that H_1 is accepted and H_0 is rejected, it could imply any of the following events:

I. There is no manipulation - Type I error has occurred i.e. H_0 rejected when true.
II. There is manipulation *and* such manipulation *is definitely* fraudulent.
III. There is manipulation *and* such manipulation *may or may not be* fraudulent.
IV. There is manipulation *and* such manipulation *is definitely not* fraudulent.

Neutrosophic extension

Neutrosophic probabilities are a generalization of classical and fuzzy probabilities and cover those events that involve some degree of indeterminacy. It provides a better approach to quantifying uncertainty than classical or even fuzzy probability theory. Neutrosophic probability theory uses a subset-approximation for truth-value as well as

indeterminacy and falsity values. Also, this approach makes a distinction between "relative true event" and "absolute true event" the former being true in only some probability sub-spaces while the latter being true in all probability sub-spaces. Similarly, events that are false in only some probability sub-spaces are classified as "relative false events" while events that are false in all probability sub-spaces are classified as "absolute false events". Again, the events that may be hard to classify as either 'true' or 'false' in some probability sub-spaces are classified as "relative indeterminate events" while events that bear this characteristic over all probability sub-spaces are classified as "absolute indeterminate events". (Smarandache, 2001)

While in classical probability **n_sup ≤ 1**, in neutrosophic probability **n_sup ≤ 3^+** where n_sup is the upper bound of the probability space. In cases where the truth and falsity components are complimentary, i.e. there is no indeterminacy, the components sum to unity and neutrosophic probability is reduced to classical probability as in the tossing of a fair coin or the drawing of a card from a well-shuffled deck.

Coming back to our original problem of financial fraud detection, let E be the event whereby a Type I error has occurred and F be the event whereby a fraud is actually detected. Then the *conditional neutrosophic probability* **NP (F | E^c)** is defined over a probability space consisting of a triple of sets (T, I, U). Here, T, I and U are probability sub-spaces wherein event F is t% true, i% indeterminate and u% untrue respectively, given that no Type I error occurred.

The sub-space T within which t varies may be determined by factors such as past records of fraud in the organization, propensity to commit fraud by the employees concerned, and effectiveness of internal control systems. On the other hand, the sub-

space U within which u varies may be determined by factors like personal track records of the employees in question, the position enjoyed and the remuneration drawn by those employees. For example, if the magnitude of the embezzled amount is deemed too frivolous with respect to the position and remuneration of the employees involved. The sub-space I within which i varies is most likely to be determined by the mutual inconsistency in the circumstantial evidence (Zadeh, 1976) that might arise out of the effects of some of the factors determining T and U. For example, if an employee is for some reason really irked with the organization, then he or she may be inclined to commit fraud not so much to further his or her own interests as to harm the interests of the organization, although the act of actually committing the suspected fraud may in this case overtly appear inconsistent with the organizational status and remuneration enjoyed by that person.

A conceptual model to implement the neutrosophic fraud detection system

Modern technology has armed the investigative accountants with tools and techniques not only to track down the perpetrators of fraud more efficiently than was possible in the absence of those technologies but also to carry out a multi-faceted analytical inquiry into the nature of financial frauds and their perpetrators.

We propose classifying financial frauds in the modern corporate scenario using a systematic, multi-level categorization. The simplest one would of course be a two-level classification where one classifier is in terms of the *nature of fraudulent manipulation* and the other is in terms of *involvement of the perpetrators* e.g. fraud by an individual, a group of isolated individuals or a collusive fraud. A sub-

classification within this second categorization could be based on whether the group (isolated or collusive) consists of hierarchical positions or horizontal positions in the organizational structure. Then what may come up with is the following two-dimensional *manipulation-involvement* or *MI*-matrix:

Chart 1:
Increasing complexity of involvement (i)

	Manipulation				
Involvement	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}
	α_{21}	α_{22}	α_{23}	α_{24}	α_{25}
	α_{31}	α_{32}	α_{33}	α_{34}	α_{35}
	α_{41}	α_{42}	α_{43}	α_{44}	α_{45}
	α_{51}	α_{52}	α_{53}	α_{54}	α_{55}
	α_{61}	α_{62}	α_{63}	α_{64}	α_{65}
	α_{71}	α_{72}	α_{73}	α_{74}	α_{75}
	α_{81}	α_{82}	α_{83}	α_{84}	α_{85}

Increasing complexity of manipulation (j)

The row and column elements in this simple *MI*-matrix may be denoted as follows:

α_{ij}:

i = 8: Single fraud perpetrated by single individual

i = 7: Multiple frauds perpetrated by single individual

i = 6: Single fraud perpetrated by group of isolated individuals

i = 5: Multiple frauds perpetrated by group of isolated individuals

i = 4: Single fraud perpetrated by a horizontally collusive group

i = 3: Multiple frauds perpetrated by a horizontally collusive group

i = 2: Single fraud perpetrated by a vertically collusive group

i = 1: Multiple frauds perpetrated by a vertically collusive group

j = 5: Combination of 1, 2, 3 and 4

j = 4: Suppression or destruction of key transaction records

j = 3: Misrepresentation of the fundamental nature of transaction

j = 2: Falsification of transaction date/amount/particulars (double-entry basis)

j = 1: Falsification of transaction date/amount/particulars (single-entry basis)

Once a particular case has been objectively classified on the *MI*-matrix, the investigative accountant may start looking for incriminating evidence in the financial records *from the right perspective*. For example, the search perspective will definitely differ if there is fundamental alteration in the nature of a transaction as compared to a simple erring journal entry. The perspective will also differ if only a single individual is involved rather than a collusive group.

The two-dimensional *MI*-matrix is the simplest form of systematic multi-level classification which certainly may be conceptually expanded to include more than two levels – e.g. a three-dimensional *MI*-matrix could possibly incorporate the aspect of *fraud potentiality* in terms of factors like personal track records of the employees in question, the position enjoyed by them in the organization, their remuneration and entitlements at the time of the fraud etc. It would therefore be an ideal computational set-up for implementation of a neutrosophic system. The conceptual fraud classification scheme we proposed here may thus be effectively combined with the rules of neutrosophic probability into developing a handy *forensic accounting expert system* for the future!

Conclusion

No doubt then that the *theory of neutrosophic probability* opens up a new vista of analytical reasoning for the techno-savvy forensic accountant. In this paper, we have only posit that a combination of statistical testing of audit samples based on Benford's law together with a neutrosophic reasoning could help the forensic accountant in getting a better fix on the quantitative possibility of actually detecting a financial fraud. This is an emerging science and thus holds a vast potential of future research endeavours the ultimate objective of which will be to actually come up with a reliable, comprehensive computational methodology to actually track down financial frauds with a very low failure rate. We believe our present work is just an initial step towards that ultimate destination.

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Redesigning Decision Matrix Method with an indeterminacy-based inference process

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ABSTRACT

For academics and practitioners concerned with computers, business and mathematics, one central issue is supporting decision makers. In this paper, we propose a generalization of Decision Matrix Method (DMM), using Neutrosophic logic. It emerges as an alternative to the existing logics and it represents a mathematical model of uncertainty and indeterminacy. This paper proposes the Neutrosophic Decision Matrix Method as a more realistic tool for decision making. In addition, a de-neutrosophication process is included.

Keywords: Decision Matrix Method, Neutrosophic Decision Matrix Method, Neutrosophic Logic, Decision Making.

Mathematics Subject Classification: Neutrosophic Logic.

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1. INTRODUCTION

For academics and practitioners concerned with computers, business and mathematics, one central issue is supporting decision makers. In that sense, making coherent decisions requires knowledge about the current or future state of the world and the path to formulating a fit response (Zack, 2007).

The authors propose a generalization of Decision Matrix Method (DMM), or Pugh Method as sometimes is called, using Neutrosophic logic (Smarandache, 1999). The main strengths of this paper are two-folds: it provides a more realistic method that supports group decisions with several alternatives and it presents a de-neutrosophication process. We think this is an useful endeavour.

The remainder of this paper is structured as follows: Section 3 reviews Decision Matrix Method; Section 3 shows a brief overview of Neutrosophic Logic and proposes Neutrosophic Decision Matrix Method and de-neutrosophication process; the final section shows the paper's conclusions.

2. DECISION MATRIX METHOD BACKGROUND

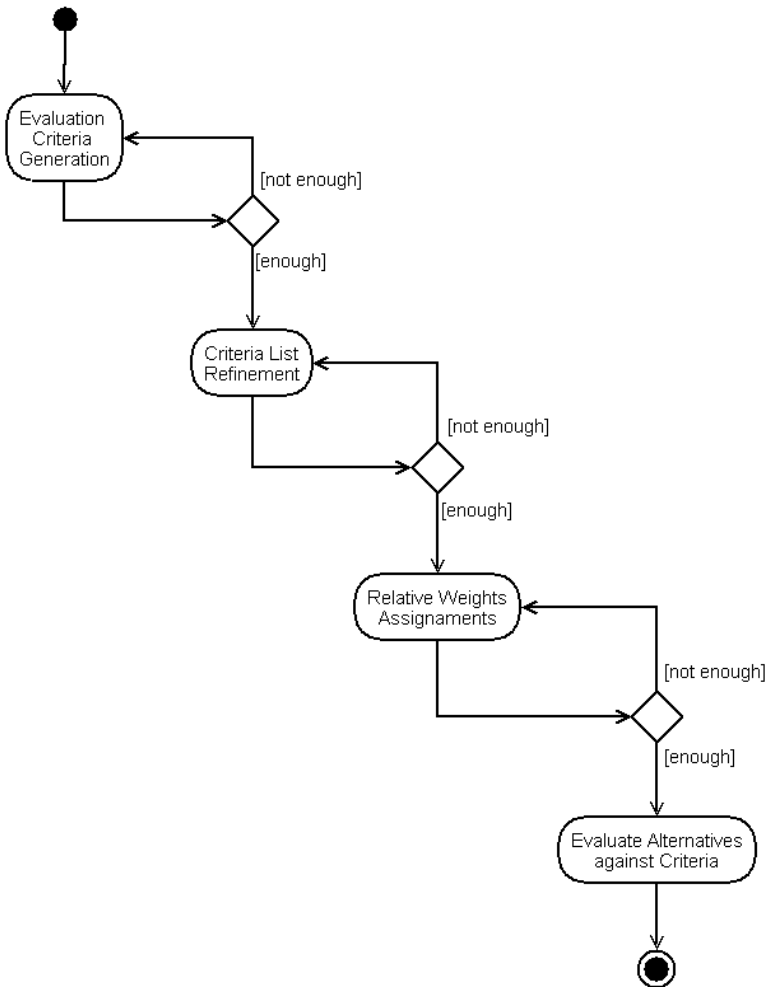
Decision Matrix Method (DMM) was developed by Stuart Pugh (1996) as an approach for selecting concept alternatives. DMM is a method (Murphy, 1979) that allows decision makers to systematically identify and analyze the strength of relationships between sets of information. This technique is especially interesting for looking at large numbers of factors and assessing each relative importance. Furthermore, DMM is a method for alternative selection using a scoring matrix. DMM is often used throughout planning activities to select product/service features and goals and to develop process stages and weight options.

DMM is briefly exposed. At the first time an evaluation team is established. Firstly, the team selects a list of weighted criteria and then evaluates each alternative against the previous criteria. That election could be done using any technique or mix of them (discussion meetings, brainstorming, and so on). This one must be refined in an iterative process.

The next step is to assign a relative weight to each criterion. Usually, ten points are distributed among the criteria. This assignment must be done by team consensus. In addition, each team member can assign weights by himself, then the numbers for each criterion are added for a composite criterion weighting.

Follow that, L-shaped matrix is drawn. This kind of matrix relates two groups of items to each other (or one group to itself). In the last step, the alternatives are scored relative to criteria.

Figure 1. Building a Decision Matrix



Some options are showed in Table 1.

Table 1. Assessing alternatives

	Method	Values range
1	Rating scale for each alternative.	For example {1=low, 2=medium, 3=high}
2	For each criterion, rank-order all alternatives according to each fits the criterion.	Order them with 1 being the option that is least fit to criterion.
3	Establish a reference. It may be one of the alternatives or any current product/service. For each criterion, rate each other alternative in comparison to the baseline.	For example: Scores of {-1=worse, 0=same, +1=better} Wider scales could be used.

At the end, multiply each alternative's rating by its weight. Add the points for each alternative. The alternative with the highest score will be the team's proposal.

Let C be the criteria vector of a DMM. $C = (c_1, c_2, ..., c_n)$ where c_j belongs to the criteria dominion of the problem and n is the total number of criteria.

Let W be the weights criteria vector of a DMM. $W = (w_1, w_2, ..., w_n)$ where $w_j \in [0, N) \mid N \neq \infty$.

Let A_i be the rating vector of i alternative. $A_i = (a_1, a_2, ..., a_n)$ where $a_m \in \{-1, 0, 1\}$.

Consider the matrix D be defined by $D = (a_{ij})$ where a_{ij} is the rating of alternative i to the criterion j, $a_{ij} \in \{-1, 0, 1\}$. D is called the rating matrix of the DMM.

Consider the vector S be defined by $S = W \times D$, being $D = (s_1, s_2, ..., s_m)$ where s_k is the product of weight i by alternative j and m is the number of alternatives.

$$\begin{pmatrix} s_1 & s_2 & \dots & s_m \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & \dots & w_n \end{pmatrix} \times \begin{pmatrix} a_{11} & \dots & a_{m1} \\ a_{12} & \dots & a_{m2} \\ \dots & \dots & \dots \\ a_{1n} & \dots & a_{mn} \end{pmatrix}$$

The highest s_k will be the team's proposal for the problem analyzed. Additionally, alternatives have been ranked by the team.

It is important to note that s_k measures only rate of alternative j respect to weight i, till now any scholar has not contemplated the indeterminacy of any relation between alternatives and criteria.

When we deal with unsupervised data, there are situations when team can not to determine any rate. Our proposal includes indeterminacy in DMM generating more realistic results. In our opinion, including indeterminacy in DMM is an useful endeavour.

3. NEUTROSOPHIC LOGIC FUNDAMENTALS

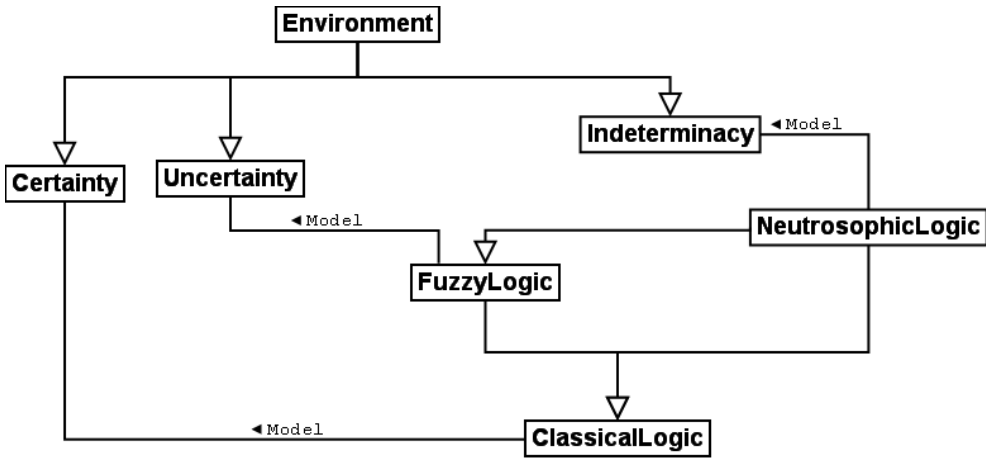
Neutrosophic Logic (Smarandache, 1999) emerges as an alternative to the existing logics and it represents a mathematical model of uncertainty, and indeterminacy. A logic in which each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F, is called Neutrosophic Logic. It uses a subset of truth (or indeterminacy, or falsity), instead of using a number, because in many cases, humans are not able to exactly determine the percentages of truth and of falsity but to approximate them: for example a proposition is between 30-40% true.

The subsets are not necessarily intervals, but any sets (discrete, continuous, open or closed or half-open/ half-closed interval, intersections or unions of the previous sets, etc.) in accordance with the given proposition. A subset may have one element only in special cases of this logic. It is imperative to mention

here that the Neutrosophic logic is a strait generalization of the theory of Intuitionist Fuzzy Logic.

According to Ashbacher (2002), Neutrosophic Logic is an extension of Fuzzy Logic (Zadeh, 1965) in which indeterminacy is included. It has become very essential that the notion of neutrosophic logic play a vital role in several of the real world problems like law, medicine, industry, finance, IT, stocks and share, and so on. Static context of Neutrosophic logic is showed in Figure 2.

Figure 2. Static context of Neutrosophic logic



Fuzzy theory measures the grade of membership or the non-existence of a membership in the revolutionary way but fuzzy theory has failed to attribute the concept when the relations between notions or nodes or concepts in problems are indeterminate. In fact one can say the inclusion of the concept of indeterminate situation with fuzzy concepts will form the neutrosophic concepts. In NL each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F.

We use a subset of truth (or indeterminacy, or falsity), instead of a number only, because in many cases we are not able to exactly determine the percentages of truth and of falsity but to approximate them: for example a proposition is between 30-40% true and between 60-70% false, even worst: between 30-40%

or 45-50% true (according to various analyzers), and 60% or between 66-70% false. The subsets are not necessary intervals, but any sets (discrete, continuous, open or closed or half-open/half-closed interval, intersections or unions of the previous sets, etc.) in accordance with the given proposition.

A subset may have one element only in special cases of this logic. Statically T, I, F are subsets, but dynamically they are functions/operators depending on many known or unknown parameters.

Constants (T, I, F) truth-values, where T, I, F are standard or non-standard subsets of the non-standard interval $]^{-}0,1^{+}[$, where n_{inf} = inf T + inf I + inf F ≥ $^{-}0$, and n_{sup} = sup T + sup I + sup F ≤ 3^{+}. Statically T, I, F are subsets, but dynamically T, I, F are functions/operators depending on many known or unknown parameters.

The NL is a formal frame trying to measure the truth, indeterminacy, and falsehood. The hypothesis is that no theory is exempted from paradoxes, because of the language imprecision, metaphoric expression, various levels or meta-levels of understanding/interpretation which might overlap.

3.1. Using indeterminacy in Decision Matrix Method

We propose a redesign of the DMM called Neutrosophic Decision Matrix Method (NDMM). This proposal includes indeterminacy in alternatives' rating and not is used to weights. It is because weights are the quantified value of criteria. They are selected by the team. Therefore, an indeterminacy weight has no sense. On the other hand, it is possible to consider indeterminacy to alternatives rating.

A Neutrosophic Decision Matrix is a neutrosophic matrix with neutrosophic values (alternatives ratings or indeterminacies as elements). Consider the matrix D be defined by $D = (a_{ij})$ where a_{ij} is the neutrosophic value of alternative i to the criterion j. D is called the rating matrix of the NDMM. In that

sense, $a_{ij} \subset [-1,1] \cup I$. We would interpret this expression as representing the total group of numbers as the union of two other groups. The first interval would start at -1 and proceed toward +1. The second would be an indeterminacy value.

The total set of numbers would be all those in the first group along with the indeterminacy value. Note that $I \in [-1,1]$, since it is an indeterminate value in that interval. In fact, we have that $a_{ij} = \{x | -1 \leq x \leq 1\}$.

In addition, we propose a de-neutrosophication process in NDMM. This one is based on max-min values of I. A neutrosophic value is transformed in an interval with two values, the maximum and the minimum value for I. In that sense, the neutrosophic scores will be an area, where the upper limit has $I = 1$ and the lower limit has $I = -1$. The solution set is $\chi = \bigcup_{j=1}^{n} s_j$, where j is the alternatives number and s is the score of each one. Any $s_k \mid k \neq j$ belongs to the complement of χ^c. Alternative selected is the global maximum in χ. It is an alternative A_m where $s_m^* \geq s_i \mid \forall i\,; i,m \in \chi$. De-neutrosophication process will be applied within the following application.

s_m^* is a line (y axis value fixed) represented the score of alternative A_m. It is possible that $s_i \in s_j$, since s_i is a line and s_j is an area after de-neutrosophication process. We select according to

$$\begin{cases} s_i & if \quad s_i > \dfrac{\max s_j - \min s_j}{2} + k \mid k \leq (\max s_j - s_i) \\ s_j & otherwise \end{cases}$$

3.2. An application

This example illustrates the improvements of NDMM versus DMM. NDMM proposal allows to represent indeterminacy in a decisional framework. Let C be

the criteria vector of a decision problem. $C = (c_1, c_2, c_3, c_4)$ where c_j belongs to the criteria dominion of the problem.

Let W be the weights criteria vector of a DMM. $W = (w_1, w_2, w_3, w_4)$. We have used a three-valued scale from 1 (less importance) to 3 (more importance).

$$\begin{cases} w_1 = 3 \\ w_2 = 3 \\ w_3 = 2 \\ w_4 = 1 \end{cases}$$

Three different alternatives are been considering. We call it A_i, where i is the order of each one.

Consider the following Neutrosophic Decision Matrix where alternatives and ratings are showed. Each column represents the ratings for an alternative and each row gives the criterion ratings for all the alternatives.

$$Neutrosophic\ Decision\ Matrix = \begin{pmatrix} 5 & 6 & 7 \\ 2 & I & 5 \\ 10 & 4 & I \\ 3 & 2 & 7 \end{pmatrix}$$

We have used a scale from 1 (less fit) to 10 (more fit). Indeterminacy is introduced in the second alternative (second criterion) and the third alternative (third criterion).

We show the S vector with the product of weight i by alternative j as a result.

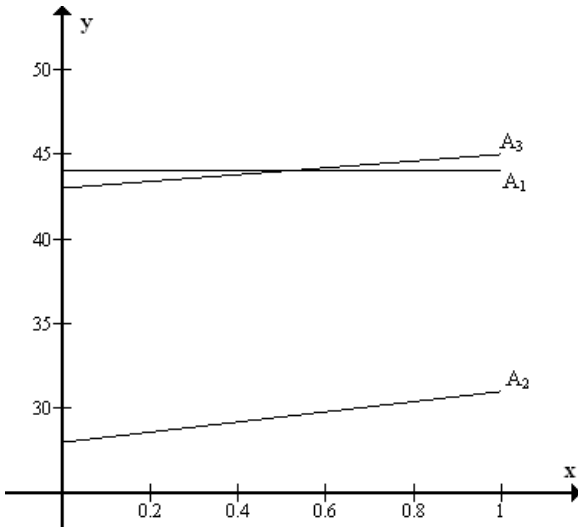
$$\begin{pmatrix} 49 & 28+3I & 40+2I \end{pmatrix} = \begin{pmatrix} 3 & 3 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 5 & 6 & 7 \\ 2 & I & 5 \\ 10 & 4 & I \\ 3 & 2 & 7 \end{pmatrix}$$

The neutrosophic score of each alternative is showed.

$$scores_n = \begin{cases} A_1 = 44 \\ A_2 = 28 + 3I \\ A_3 = 43 + 2I \end{cases}$$

If we consider scores got for second and third alternatives as equations the representation would be the showed in Figure 3. Obviously, A_1 is the best option.

Figure 3. Alternatives' neutrosophic scores

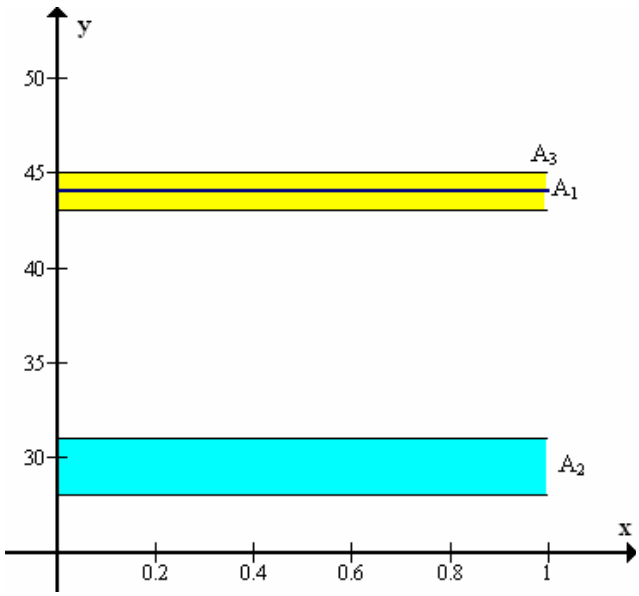


The next step is the de-neutrosophication process. We replace $I \in [0,1]$ both maximum and minimum values.

$$scores_d = \begin{cases} A_1 = 44 \\ A_2 = [28,31] \\ A_3 = [43,45] \end{cases}$$

Figure 4 shows the de-neutrosophic results. The results show alternatives 2 and 3 as areas. In this case $0 \le k \le 1$. A_3 will be selected if and only if $k > 0$. It is more realistic view from DMM.

Figure 4. Alternatives' de-neutrosophic scores



4. CONCLUSIONS

Numerous scientific publications address the issue of decision making in every fields. But, little efforts have been done for processing indeterminacy in this context. This paper shows a formal method for processing indeterminacy in Decision Matrix Method and include a de-neutrosophication process.

The main outputs of this paper are two-folds: it provides a neutrosophic tool for decision making and it also includes indeterminacy in a decision tool. In this paper a renewed Decision Matrix Method has been proposed. As a methodological support, we have used Neutrosophic Logic. This emerging logic extends the limits of information for supporting decision making and so on.

Using NDMM decision makers are not forced to select ratings when their knowledge is not enough for it. In that sense, NDMM is a more realistic tool since experts' judgements are focused on their expertise.

Anyway, more research is needed about Neutrosophic logic limit and applications. Incorporating the analysis of NDMM, the study proposes an innovative way for decision making.

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EDUCATION AND ADMINISTRATION

Elections and Evaluations of the American College/University Dean and Director

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Coming from Eastern Europe where, even under communism, the Dean and President were elected for a temporary period, I was surprised to see that in USA they are elected or appointed for permanent positions.

My **first proposal** would be to periodically elect the American College/University Dean and Director from the College/University pull of full-time faculty and full-time staff.
Each elected person should serve a 3-year term and be able to be re-elected one more term. Nobody should be allowed to serve more than two terms.

As an example, recently elected for a third consecutive term, Tony Blair, was asked by Britons to leave sooner and not apply for another term - despite the fact that UK economy is going well. But Englishmen got tired of the same person in the office!

It is better to have a periodically rotating leadership in order to avoid any trend towards dictatorship, favoritism, or retaliation.

A **second proposal** I had, after consulting with other people from our University of New Mexico at Gallup Campus, that each semester or at least once a year we, faculty and staff, be able to evaluate the UNM-G Dean of Instruction and the Director - similarly as students semesterly evaluate the faculty and also as we did a few years ago when we evaluated the former UNM-G Director Dr. R. Carlson.

Since this is actually a responsibility of the Ethics Committee, as Dr. Anthony Mansueto suggested to me in a previous e-mail, I hoped that Committee will take into consideration this proposal.

I received an answer from a faculty about the last assessment of the UNM-G Director:
"(…) it was organized by Faculty Senate's Operations Committee, but included Staff Senate, administrators and the UNM-G Advisory Committee. There was a core set of questions for everyone, but each group also had their own set of questions they felt best addressed the needs of that group. As president of Faculty Senate, I collected all the completed assessments and made sure they went to the Provost in UNM-Albuquerque."

The evaluation is intended to avoid favoritism to some people and retaliation against others from the part of the leadership, and also against discrimination against minorities that unfortunately continues to happen in this campus.
{For example, there are people who got release time just for not having enough students in class and the class was cancelled, while other people didn't ever get any release time for no matter what they have done!
There are people who got awarded for little thing, and others not even for being invited speakers at the prestigious NASA and NATO!}

The evaluation of American College/University Dean and Director should also be discussed in the Faculty Senate. The results' summary of the evaluation should be made available to the whole campus.

To deliver free preprint service for physical sciences

A proposal for further development &
Introduction to

www.sciprint.org

by

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December 28th, 2005

To deliver free preprint service for physical sciences

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2. Introduction
3. Background
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7. Fields of physics science
8. Proof of Concept
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10. Further steps, reports
11. Summary

(1) Purpose

This proposal is intended to develop a free digital preprint service for physical sciences to enable scientists/physicists publishing their preprint articles prior to submitting for formal publication in scientific journals, or perhaps they only want to see if their idea(s) received proper response prior to submitting it to journal editors.

(2) Introduction

Part of the infrastructure needed for doing science is a society of like-minded scientists. For physicists, it is essential to develop a society where they can share and discuss information related to their specific field.

In the meantime, the nature of scientific communication has changed significantly in response to rapid changes in the use of IT (information technology) in scientific publications [1][2][3].

The rapid growth of the Internet has also given scholars almost universal access to a communication medium that facilitates immediate sharing of results. In the last decade literally thousands of digital libraries have emerged, in conjunction with electronic journals which grow almost exponentially. One of the most significant advantage of digital library is to allow scientific communities to share information across institutional and geographic borders in no time [5].

An example of how digital preprint service could be very useful is given here. For physicists, due to some time required prior to get their paper published, sometimes they find it difficult to get publicizing their work quickly. Let suppose one physicist submits a paper for peer-reviewing. The peer-reviewing process would take sometime from six months up to one year (average), and after acceptance for publication the paper should wait in queue for the next six months (at least) before it could appear in the journal. By this time, new development in their corresponding field could take place which would make the paper already obsolete by the time of appearance in journal. This cycle goes on for almost each paper (in some cases, there are weekly and monthly journal to facilitate rapid communication).

This is why to some physicists [1]:

"*...implementation of peer-review -- an essential feature of scholarly communication -- is too rigid and sometimes acts to suppress new ideas, favor articles from prestigious institutions, and cause undue publication delays.*"

Therefore while the role of peer-review process remains to be instrumental to ensure quality of a scientific paper, it is essential to deliver a free preprint service where the authors may upload preprints, reprints, conference papers, pre-publication book chapters and author's lecture in multimedia format etc.

(3) Background

The origins of the Open Archives initiative were motivated by the growing number of electronic preprint archives. While some of these services began as informal vehicles for the dissemination of preliminary results and non-peer reviewed "gray literature", a number of them have evolved into an essential medium for sharing research results among the colleagues in a field.

Nonetheless, while similar preprint service is available for physicists, for instance by arXiv, CERN, SLAC,ICTP, etc., there is going concern among physicists that some of these archives were running with specific 'science policy' adopted by administrators. Unfortunately, authors and articles which do not support to this 'science policy' are likely to subject to excessive review

which appears unnecessary because these articles are only at 'preprint stage.'

1. Furthermore, this will assume that the archive administrators have the required knowledge to do such rapid review, while perhaps the proper reviewing known by scientific society is through *peer-reviewing* in scientific journals. No university, regardless its high reputability, should control the distribution of scientific papers via early review (which could be inappropriate), in particular where the ideas are very new to the scientific society.
2. This would mean a contradiction to the purpose of preprint service itself. They would impose unnecessary restriction for novel ideas, whereas their purpose is to deliver rapid publication new scientific preprints. There is worst condition that some *accepted papers* for publication in scientific journals were also rejected by archive administrators. This condition is at odd with the expected 'fair and free' posting for any preprint archive.
3. Because scientific paper for scientists/physicists could mean a sign of productivity, impossibility to get their work published in these 'open' archiving services deteriorates reputation to these authors. And it would also mean their works are in risk to become untracked by other peer, while otherwise sometimes others could find their work useful.
4. In other occasions, to include a review process into scientific 'preprint service' in practice could also mean unnecessary delay to get a paper appears in the preprint homepage.

(4) Advantages

By eliminating the rapid-review process, we could offer some obvious advantages:

1. ease-of-use in deployment, low-barrier for participation (only minimum Internet access is required);
2. self-publication of timely preprints. Other digital files could be considered are: seminar/conference proceedings, technical reports, book review, lecture summary, multimedia file (experimental test) etc.;
3. community diversity: the author could choose if they would put their articles into an 'open folder' accessible for all visitors, or to restricted folder where they could share to only few colleagues (private archive, resembles 'private deposit vault'). Let suppose an author has found significant breakthrough that he/she doesn't want announce yet publicly, but he/she wants a 'proof' that he/she has deposited earlier than anybody else, then he/she could use private archiving mode. Therefore, once the breakthrough has been confirmed experimentally, they could ask to retrieve the private archive to find out who has the first archiving time;
4. ability to submit in various formats: multimedia/video, lectures, public seminars etc.;
5. issue monthly notification of how much download to his/her paper & weekly newsletters.

(5) Deliverables

Expected features after improvement will include:

1. a homepage with functionalities to enable physicists/scientists upload their preprint files in almost any format (PDF, PS, PPT, Word, HTML, etc.) into specific folder, and get their preprint viewable by other colleagues immediately (real time);
2. text-based search throughout all content based on contextual search [4];
3. multimedia digital archives (deliver video from open lecture, seminars etc.);
4. citation/reference system;
5. auto-posting to other scientific database service;
6. notification to users new preprint based on subject of interests;
7. free newsletters;

8. user repository--> user can have his/her own special subfolder, like 'secure box' in bank, could be opened by his/her own password
9. monthly notification to user on download record of his/her paper in graphical chart format;
10. download tracker;
11. secure protection/antivirus;
12. online conference forum (using internet protocol);
13. posting news/breakthrough, broadcast new discovery;
14. posting digital multimedia archive: public lecture, free seminar etc.
15. support Open Archives Initiative's Protocol for Metadata Harvesting;
16. other features are to follow as per suggestion from users.

(6) Procedure:

1. To upload documents, scientists must register and login. The procedure is simple and self-explanatory and can be obtained by logging on to the sciprint.org homepage.He/she could upload in various formats: PDF, HTML, PS format, etc.
2. Guidelines to upload is also given in the sciprint.org homepage.
3. Registration of new member for the time being is carried out via email admin@sciprint.org.
4. Alternatively, the users could send their preprint articles via email to admin@sciprint.org.
5. The posted preprints will not be refereed or edited by anyone at sciprint.org or elsewhere, and only the authors are responsible for the preprint submitted.
6. It is required (recommended) to include a note where the preprint submitted should appear in journal.

(7) Fields in physics science:

At the time being, the preprints could be submitted according to this category of fields of physics science:

1. Alternative energy;
2. Astrophysics;
3. Biosciences;
4. Environmental science;
5. Gravitation and general relativity;
6. Hadronic physics (formerly elementary particle physics);
7. Mathematical physics;
8. Unification of physical theories.

(8) Proof of concept: recent experience

At the time being, sciprint.org has been opened for few months. It was based on our belief that only journal editors and peer-reviews could offer the proper judgment on the value of a preprint. Therefore there is no need to put such a restriction on preprint services. Furthermore, doing such restriction would only mean that preprint service cannot deliver rapid publication of preprint materials. This is the 'heart' of our service: '*Free preprint service without hassles*.'

For the time being, the administration service is carried out by a small team on voluntary basis. The result is summarized here:

1. Since first release in around June 2005 has grown to 93 papers (in Nov. 2005), and to 161 in Dec. 2005 in various subjects, including mathematical-physics, astrophysics etc.;
2. No special marketing effort except via peer-to-peer email by physicists themselves;
3. New features began at Sept. 2005. And from this time, number of paper download could be

monitored (see Appendix);

4. Number of paper download has grown at average 35 papers per week (5 paper per day), but has increased rapidly to more than 560 downloads by the end of Nov. 2005. By then end of January 2006, it has increased to more than 1000 times. Chart is shown at the Appendix.
5. Without almost no marketing effort (no advertising), it proves itself that physicists regard the *free-environment* where they can publish their preprints without any kind of restriction.

(9) **Timeline**

1. Expected Development & improvement stage: March 2006 – April 2006
2. Expected Internal test : April – May 2006
3. Expected Deployment : June 2006

(10) **Further steps**

1. Finding better dedicated hosting service;
2. Database re-design;
3. Develop new features based on existing applications;
4. Progress report will be prepared and delivered at six-months interval from the start of the project. These will assess the effectiveness of the management structure, a review of work to date, and an assessment of progress towards the project objectives.

(11) **Summary**

In order to deliver free preprint service for physics and natural sciences in various formats and also to enable multimedia digital archiving, it is proposed to enhance and improve the existing service of sciprint.org by introducing new features and using better dedicated servers. Multimedia archiving could also be expected as a result of such improvement. Key deliverables have been outlined, along with some basic requirements.

From the past few months, sciprint.org has proved to become a new alternative preprint service for scientific preprints, in particular physical sciences. And this will continue to the near future.

1st version November 28th, 2005. 1st revision: December 28th, 2005. 2nd revision: January 28th, 2006.

Contact email:

V. Christianto, email: admin@sciprint.org

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[6] http://arxiv.org/html/physics/0102004
[7] http://www.sciencemag.org/cgi/content/summary/283/5408/1610
[8] http://www.nature.com/cgi-taf/DynaPage.taf?file=/nature/journal/v402/n6759/full/402230b0_fs.html&content_filetype=pdf

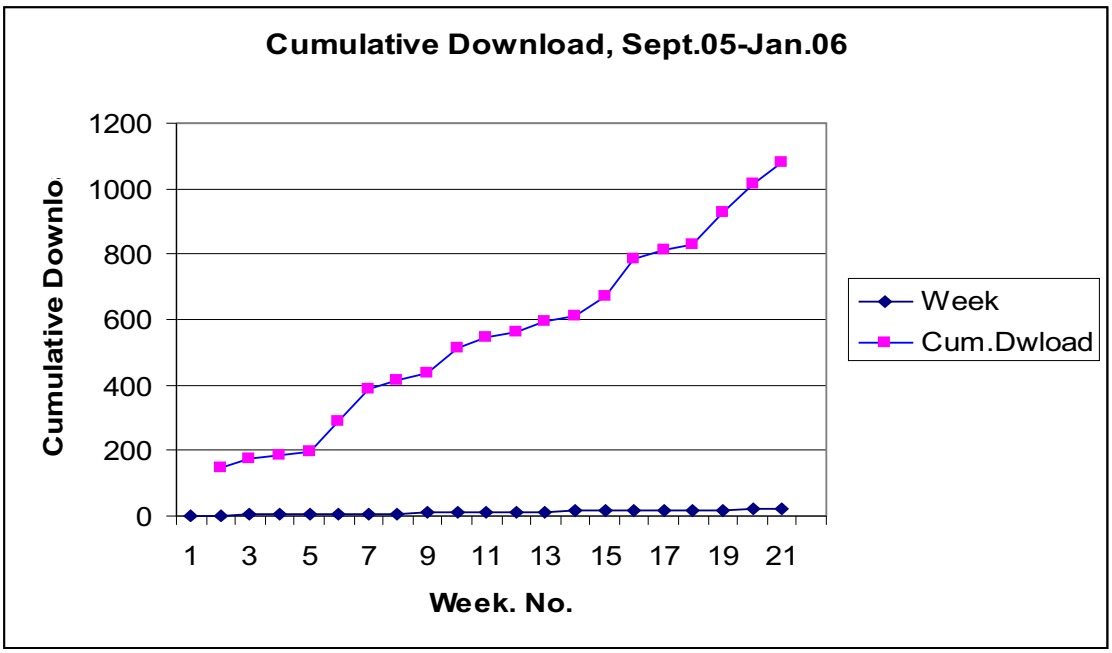
APPENDIX:

Recent experience in the past few months, September 2005 – January 2006

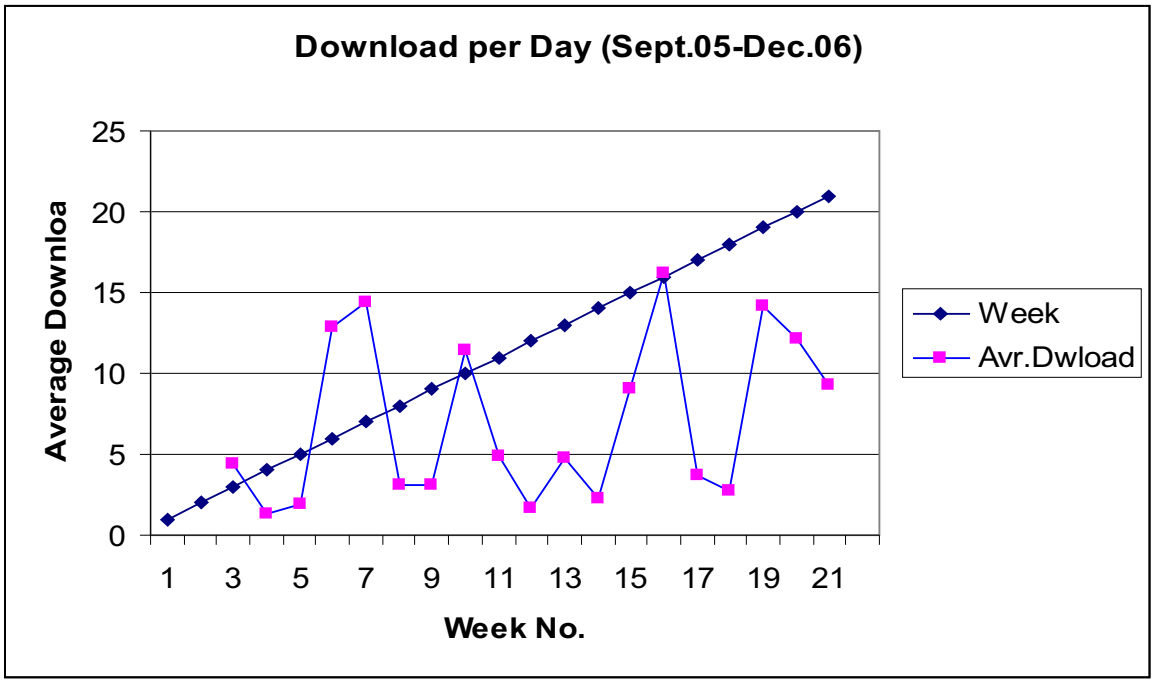
a. **Sciprint.org Download Number (September 2005- January 2006)**

Date	Week	Cum.Dwload	Week	Avr.Dwload	Week	% Growth
9-sept	1		1		1	
16-sept	2	146	2		2	
23-sept	3	177	3	4.43	3	
30-sept	4	186	4	1.29	4	5.08
17-oct	5	199	5	1.86	5	6.99
14-oct	6	289	6	12.86	6	45.23
21-oct	7	390	7	14.43	7	34.95
28-oct	8	412	8	3.14	8	5.64
4-nov	9	434	9	3.14	9	5.34
11-nov	10	514	10	11.43	10	18.43
18-nov	11	548	11	4.86	11	6.61
25-nov	12	560	12	1.71	12	2.19
2-dec	13	593	13	4.71	13	5.89
9-dec	14	609	14	2.29	14	2.70
16-dec	15	672	15	9.00	15	10.34
23-dec	16	785	16	16.14	16	16.82
30-dec	17	811	17	3.71	17	3.31
7-Jan	18	830	18	2.71	18	2.34
14-Jan	19	929	19	14.14	19	11.93
21-Jan	20	1014	20	12.14	20	9.15
28-Jan	21	1079	21	9.29	21	6.41
Average				**6.33**		**12.11**
Expected				**5**		

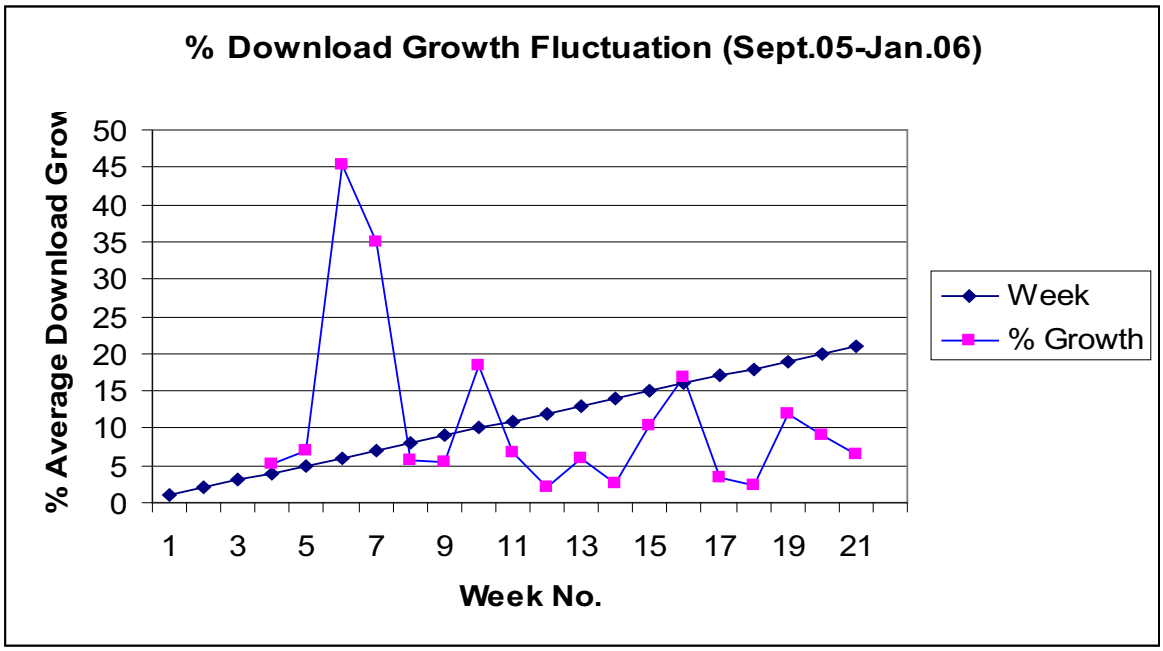
b. **Key indicator Cumulative Download:**



c. **Key indicator Average Download per Day:**



d. **Key indicator % download growth:**



GAME THEORY

A Group-Permutation Algorithm to Solve the Generalized SUDOKU

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Sudoku is a game with numbers, formed by a square with the side of 9, and on each row and column are placed the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, written only one time; the square is subdivided in 9 smaller squares with the side of 3×3, which, also, must satisfy the same condition, i.e. each square to contain all digits from 1 to 9 written only once.

The Japanese company Nikoli has popularized this game in 1986 under the name of *sudoku*, meaning "single number".

Sudoku can be generalized to squares whose dimensions are $n^2 \times n^2$, where $n \geq 2$, using various symbols (numbers, letters, mathematical symbols, etc.), written just one time on each row and on each column; and the large square is divided into n^2 small squares with the side $n \times n$ and each will contain all n^2 symbols written only once.

An elementary solution of one of these generalized Sudokus, with elements (symbols) from the set

$$S = \{s_1, s_2, ..., s_n, s_{n+1}, ..., s_{2n}, ..., s_{n^{\wedge}2}\}$$

(supposing that their placement represents the relation of total order on the set of elements S), is:

Row 1: all elements in ascending order

$$s_1, s_2, ..., s_n, s_{n+1}, ..., s_{2n}, ..., s_{n^{\wedge}2}$$

On the next rows we will use circular permutations, considering groups of n elements from the first row as follows:

Row 2:

$$s_{n+1}, s_{n+2}, ..., s_{2n}; s_{2n+1}, ..., s_{3n}; ..., s_{n^{\wedge}2}; s_1, s_2, ..., s_n$$

Row 3:

$$s_{2n+1}, ..., s_{3n}; ..., s_{n^{\wedge}2}; s_1, s_2, ..., s_n; s_{n+1}, s_{n+2}, ..., s_{2n}$$

……..

Row n:

$$s_{n^{\wedge}2-n+1}, ..., s_{n^{\wedge}2}; s_1, ..., s_n, s_{n+1}; s_{n+2}, ..., s_{2n}; ..., s_{3n}; ..., s_{n^{\wedge}2-n}.$$

Now we start permutations of the elements of row $n+1$ considering again groups of n elements.

Row $n+1$:

$$s_2, ..., s_n, s_{n+1}; s_{n+2}, ..., s_{2n}, s_{2n+1}; s_{n^{\wedge}2-n+2}, ..., s_{n^{\wedge}2}, s_1$$

Row $n+2$:

$$s_{n+2}, ..., s_{2n}, s_{2n+1}; s_{n^{\wedge}2-n+2,} ..., s_{n^{\wedge}2}, s_1; s_2, ..., s_n, s_{n+1}$$

……

Row $2n$:

$$s_{n\hat{}2-n+2}\ \ldots, s_{n\hat{}2},\ s_1;\ s_2,\ \ldots,\ s_n,\ s_{n+1};s_{n+2},\ \ldots,\ s_{2n},\ s_{2n+1}$$

Row $2n+1$:

$$s_3,\ \ldots,\ s_{n+2};s_{n+3},\ \ldots,\ s_{2n+2};\ s_{n\hat{}2+3},\ \ldots,\ s_{n\hat{}2}, s_1,\ s_2$$

and so on.

Replacing the set S by any permutation of its symbols, which we'll note by S', and applying the same procedure as above, we will obtain a new solution.

The classical Sudoku is obtained for $n=3$.

Below is an example of this group-permutation algorithm for the classical case:

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

For a **$4^2\times 4^2$ square** we use the following 16 symbols:

{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P}

and use the same group-permutation algorithm to solve this Sudoku.

From one solution to the generalized Sudoku we can get more solutions by simply doing permutations of columns or/and of rows of the first solution.

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
E	F	G	H	I	J	K	L	M	N	O	P	A	B	C	D
I	J	K	L	M	N	O	P	A	B	C	D	E	F	G	H
M	N	O	P	A	B	C	D	E	F	G	H	I	J	K	L
B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	A
F	G	H	I	J	K	L	M	N	O	P	A	B	C	D	E
J	K	L	M	N	O	P	A	B	C	D	E	F	G	H	I
N	O	P	A	B	C	D	E	F	G	H	I	J	K	L	M
C	D	E	F	G	H	I	J	K	L	M	N	O	P	A	B
G	H	I	J	K	L	M	N	O	P	A	B	C	D	E	F
K	L	M	N	O	P	A	B	C	D	E	F	G	H	I	J
O	P	A	B	C	D	E	F	G	H	I	J	K	L	M	N
D	E	F	G	H	I	J	K	L	M	N	O	P	A	B	C
H	I	J	K	L	M	N	O	P	A	B	C	D	E	F	G
L	M	N	O	P	A	B	C	D	E	F	G	H	I	J	K
P	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O

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GEOMETRY

Nine Solved and Nine Open Problems in Elementary Geometry

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In this paper we review nine previous proposed and solved problems of elementary *2D* geometry [4] and [6], and we extend them either from triangles to polygons or polyhedrons, or from circles to spheres (from *2D-space* to *3D-space*), and make some comments, conjectures and open questions about them.

Problem 1.

We draw the projections M_i of a point M on the sides A_iA_{i+1} of the polygon $A_1...A_n$. Prove that:

$$\|M_1A_1\|^2 + ... + \|M_nA_n\|^2 = \|M_1A_2\|^2 + ... + \|M_{n-1}A_n\|^2 + \|M_nA_1\|^2$$

Solution 1.

For all i we have:

$$\|MM_i\|^2 = \|MA_i\|^2 - \|A_iM_i\|^2 = \|MA_{i+1}\|^2 - \|A_{i+1}M_i\|^2$$

It results that:

$$\|M_iA_i\|^2 - \|M_iA_{i+1}\|^2 = \|MA_i\|^2 - \|MA_{i+1}\|^2$$

From where:

$$\sum_i \left(\|M_iA_i\|^2 - \|M_iA_{i+1}\|^2 \right) = \sum_i \left(\|MA_i\|^2 - \|MA_{i+1}\|^2 \right) = 0$$

Open Problem 1.

1.1. If we consider in a 3D-space the projections M_i of a point M on the *edges* A_iA_{i+1} of a polyhedron $A_1...A_n$ then what kind of relationship (similarly to the above) can we find?

1.2. But if we consider in a 3D-space the projections M_i of a point M on the *faces* F_i of a polyhedron $A_1...A_n$ with k≥4 faces, then what kind of relationship (similarly to the above) can we find?

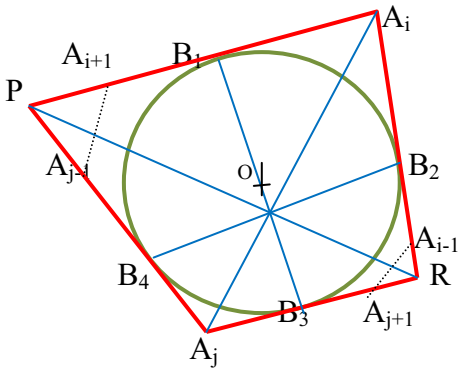
Problem 2.

Let's consider a polygon (which has at least 4 sides) circumscribed to a circle, and D the set of its diagonals and the lines joining the points of contact of two non-adjacent sides. Then D contains at least 3 concurrent lines.

Solution 2.

Let n be the number of sides. If $n=4$, then the two diagonals and the two lines joining the points of contact of two adjacent sides are concurrent (according to Newton's Theorem).

The case $n>4$ is reduced to the previous case: we consider any polygon $A_1...A_n$ (see the figure)



circumscribed to the circle and we choose two vertices A_i, A_j $(i \neq j)$ such that

$$A_j\ A_{j-1} \bigcap A_i\ A_{i+1} = P$$

and

$$A_j\ A_{j+1} \bigcap A_i\ A_{i-1} = R$$

.

Let B_h, $h \in \{1,2,3,4\}$ the contact points of the quadrilateral PA_jRA_i with the circle of center O. Because of the Newton's theorem, the lines $A_i\ A_j$, B_1B_3 and B_2B_4 are concurrent.

Open Problem 2.

2.1. In what conditions there are more than three concurrent lines?

2.2. What is the maximum number of concurrent lines that can exist (and in what conditions)?

2.3. What about an alternative of this problem: to consider instead of a circle an ellipse, and then a polygon *ellipsoscribed* (let's invent this word, *ellipso-scribed*, meaning a polygon whose all sides are tangent to an ellipse which inside of it): how many concurrent lines we can find among its diagonals and the lines connecting the point of contact of two non-adjacent sides?

2.4. What about generalizing this problem in a 3D-space: a sphere and a polyhedron circumscribed to it?

2.5. Or instead of a sphere to consider an ellipsoid and a polyhedron *ellipsoido-scribed* to it?

Of course, we can go by construction reversely: take a point inside a circle (similarly for an ellipse, a sphere, or ellipsoid), then draw secants passing through this point that intersect the

circle (ellipse, sphere, ellipsoid) into two points, and then draw tangents to the circle (or ellipse), or tangent planes to the sphere or ellipsoid) and try to construct a polygon (or polyhedron) from the intersections of the tangent lines (or of tangent planes) if possible.

For example, a regular polygon (or polyhedron) has a higher chance to have more concurrent such lines.

In the 3D space, we may consider, as alternative to this problem, the intersection of planes (instead of lines).

Problem 3.
In a triangle ABC let's consider the Cevians AA', BB' and CC' that intersect in P. Calculate the minimum value of the expressions:

$$E(P)=\frac{\|PA\|}{\|PA'\|}+\frac{\|PB\|}{\|PB'\|}+\frac{\|PC\|}{\|PC'\|}$$

and

$$F(P)=\frac{\|PA\|}{\|PA'\|}\cdot\frac{\|PB\|}{\|PB'\|}\cdot\frac{\|PC\|}{\|PC'\|}$$

where $A'\in[BC],\ B'\in[CA],\ C'\in[AB]$.

Solution 3.
We'll apply the theorem of Van Aubel three times for the triangle ABC, and it results:

$$\frac{\|PA\|}{\|PA'\|}=\frac{\|AC'\|}{\|C'B\|}+\frac{\|AB'\|}{\|B'C\|}$$
$$\frac{\|PB\|}{\|PB'\|}=\frac{\|BA'\|}{\|A'C\|}+\frac{\|BC'\|}{\|C'A\|}$$
$$\frac{\|PC\|}{\|PC'\|}=\frac{\|CA'\|}{\|A'B\|}+\frac{\|CB'\|}{\|B'A\|}$$

If we add these three relations and we use the notation

$$\frac{\|AC'\|}{\|C'B\|}=x>0,\ \frac{\|AB'\|}{\|B'C\|}=y>0,\ \frac{\|BA'\|}{\|A'C\|}=z>0$$

then we obtain:

$$E(P)=\left(x+\frac{1}{y}\right)+\left(x+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right)\geq 2+2+2=6$$

The minimum value will be obtained when $x=y=z=1$, therefore when P will be the gravitation center of the triangle.

When we multiply the three relations we obtain

$$F(P)=\left(x+\frac{1}{y}\right)\cdot\left(x+\frac{1}{y}\right)\cdot\left(z+\frac{1}{z}\right)\geq 8$$

Open Problem 3.

3.1. Instead of a triangle we may consider a polygon $A_1A_2\ldots A_n$ and the lines A_1A_1', A_2A_2', …, A_nA_n' that intersect in a point P.

Calculate the minimum value of the expressions:

$$E(P)=\frac{\|PA_1\|}{\|PA_1'\|}+\frac{\|PA_2\|}{\|PA_2'\|}+\ldots+\frac{\|PA_n\|}{\|PA_n'\|}$$

$$F(P)=\frac{\|PA_1\|}{\|PA_1'\|}\cdot\frac{\|PA_2\|}{\|PA_2'\|}\cdot\ldots\cdot\frac{\|PA_n\|}{\|PA_n'\|}$$

3.2. Then let's generalize the problem in the 3D space, and consider the polyhedron $A_1A_2\ldots A_n$ and the lines A_1A_1', A_2A_2', …, A_nA_n' that intersect in a point P. Similarly, calculate the minimum of the expressions E(P) and F(P).

Problem 4.

If the points A_1, B_1, C_1 divide the sides BC, CA respectively AB of a triangle in the same ratio $k>0$, determine the minimum of the following expression:

$$\|AA_1\|^2+\|BB_1\|^2+\|CC_1\|^2.$$

Solution 4.

Suppose $k>0$ because we work with distances.

$$\|BA_1\|=k\|BC\|,\ \|CB_1\|=k\|CA\|,\ \|AC_1\|=k\|AB\|$$

We'll apply tree times Stewart's theorem in the triangle ABC, with the segments AA_1, BB_1, respectively CC_1:

$$\|AB\|^2\cdot\|BC\|(1-k)+\|AC\|^2\cdot\|BC\|k-\|AA_1\|^2\cdot\|BC\|=\|BC\|^3(1-k)k$$

where

$$\|AA_1\|^2=(1-k)\|AB\|^2+k\|AC\|^2-(1-k)k\|BC\|^2$$

similarly,

$$\|BB_1\|^2=(1-k)\|BC\|^2+k\|BA\|^2-(1-k)k\|AC\|^2$$

$$\|CC_1\|^2=(1-k)\|CA\|^2+k\|CB\|^2-(1-k)k\|AB\|^2$$

By adding these three equalities we obtain:

$$\|AA_1\|^2+\|BB_1\|^2+\|CC_1\|^2=(k^2-k+1)\left(\|AB\|^2+\|BC\|^2+\|CA\|^2\right),$$

which takes the minimum value when $k=\frac{1}{2}$, which is the case when the three lines from the enouncement are the medians of the triangle.

The minimum is $\frac{3}{4}\left(\|AB\|^2+\|BC\|^2+\|CA\|^2\right)$.

Open Problem 4.

4.1. If the points A_1', A_2', …, A_n' divide the sides A_1A_2, A_2A_3, …, A_nA_1 of a polygon in the same ratio k>0, determine the minimum of the expression:

$$\|A_1A_1'\|^2+\|A_2A_2'\|^2+...+\|A_nA_n'\|^2.$$

4.2. Similarly question if the points A_1', A_2', …, A_n' divide the sides A_1A_2, A_2A_3, …, A_nA_1 in the positive ratios k_1, k_2, …, k_n respectively.

4.3. Generalize this problem for polyhedrons.

Problem 5.

In the triangle ABC we draw the lines AA_1, BB_1, CC_1 such that

$$\|A_1B\|^2+\|B_1C\|^2+\|C_1A\|^2=\|AB_1\|^2+\|BC_1\|^2+\|CA_1\|^2.$$

In what conditions these three Cevians are concurrent?

Partial Solution 5.

They are concurrent for example when A_1, B_1, C_1 are the legs of the medians of the triangle BCA. Or, as Prof. Ion Pătrașcu remarked, when they are the legs of the heights in an acute angle triangle BCA .

More general.

The relation from the problem can be written also as:

$$a\left(\|A_1B\|-\|A_1C\|\right)+b\left(\|B_1C\|-\|C_1A\|\right)+c\left(\|C_1A\|-\|C_1B\|\right)=0,$$

where a, b, c are the sides of the triangle.

We'll denote the three above terms as α, β, and respective γ, such that $\alpha+\beta+\gamma=0$.

$$\alpha=a\left(\|A_1B\|-\|A_1C\|\right)\Leftrightarrow\frac{\alpha}{a}=\|A_1B\|-\|A_1C\|-2\|A_1C\|$$

where

$$\frac{\alpha}{a^2}=\frac{a-2\|A_1C\|}{a}\Leftrightarrow\frac{a^2}{a^2-\alpha}=\frac{a}{2\|A_1C\|}\Leftrightarrow\frac{a}{2\|A_1C\|}=\frac{2a^2}{a^2-\alpha}\Leftrightarrow\frac{2a^2-a^2+\alpha}{a^2-\alpha}=\frac{a-\|A_1C\|}{\|A_1C\|}$$

Then

$$\frac{\|A_1B\|}{\|A_1C\|}=\frac{a^2+\alpha}{a^2-\alpha}.$$

Similarly:

$$\frac{\|B_1C\|}{\|B_1A\|}=\frac{b^2+\beta}{b^2-\beta} \text{ and } \frac{\|C_1A\|}{\|C_1B\|}=\frac{c^2+\gamma}{c^2-\gamma}$$

In conformity with Ceva's theorem, the three lines from the problem are concurrent if and only if:

$$\frac{\|A_1B\|}{\|A_1C\|}\cdot\frac{\|B_1C\|}{\|B_1A\|}\cdot\frac{\|C_1A\|}{\|C_1B\|}=1\Leftrightarrow\left(a^2+\alpha\right)\left(b^2+\beta\right)\left(c^2+\gamma\right)=\left(a^2-\alpha\right)\left(b^2-\beta\right)\left(c^2-\gamma\right)$$

Unsolved Problem 5.

Generalize this problem for a polygon.

Problem 6.

In a triangle we draw the Cevians $AA_1,\ BB_1,\ CC_1$ that intersect in P. Prove that

$$\frac{PA}{PA_1}\cdot\frac{PB}{PB_1}\cdot\frac{PC}{PC_1}=\frac{AB\cdot BC\cdot CA}{A_1B\cdot B_1C\cdot C_1A}$$

Solution 6.

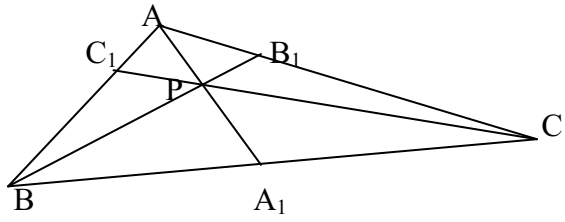
In the triangle ABC we apply the Ceva's theorem:

$$AC_1\cdot BA_1\cdot CB_1=-AB_1\cdot CA_1\cdot BC_1 \qquad (1)$$

In the triangle AA_1B, cut by the transversal CC_1, we'll apply the Menelaus' theorem:

$$AC_1\cdot BC\cdot A_1P=AP\cdot A_1C\cdot BC_1 \qquad (2)$$

In the triangle BB_1C, cut by the transversal AA_1, we apply again the Menelaus' theorem:



$$BA_1\cdot CA\cdot B_1P=BP\cdot B_1A\cdot CA_1 \qquad (3)$$

We apply one more time the Menelaus' theorem in the triangle CC_1A cut by the transversal BB_1:

$$AB\cdot C_1P\cdot CB_1=AB_1\cdot CP\cdot C_1B \qquad (4)$$

We divide each relation (2), (3), and (4) by relation (1), and we obtain:

$$\frac{PA}{PA_1}=\frac{BC}{BA_1}\cdot\frac{B_1A}{B_1C} \tag{5}$$

$$\frac{PB}{PB_1}=\frac{CA}{CB_1}\cdot\frac{C_1B}{C_1A} \tag{6}$$

$$\frac{PC}{PC_1}=\frac{AB}{AC_1}\cdot\frac{A_1C}{A_1B} \tag{7}$$

Multiplying (5) by (6) and by (7), we have:

$$\frac{PA}{PA_1}\cdot\frac{PB}{PB_1}\cdot\frac{PC}{PC_1}=\frac{AB\cdot BC\cdot CA}{A_1B\cdot B_1C\cdot C_1A}\cdot\frac{AB_1\cdot BC_1\cdot CA_1}{A_1B\cdot B_1C\cdot C_1A}$$

but the last fraction is equal to 1 in conformity to Ceva's theorem.

Unsolved Problem 6.

Generalize this problem for a polygon.

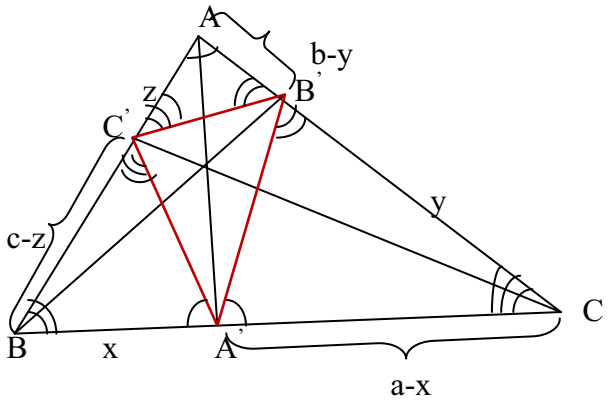
Problem 7.

Given a triangle ABC whose angles are all acute (acute triangle), we consider $A'B'C'$, the triangle formed by the legs of its altitudes.

In which conditions the expression:

$$\|A'B'\|\cdot\|B'C'\|+\|B'C'\|\cdot\|C'A'\|+\|C'A'\|\cdot\|A'B'\|$$

is maximum?



Solution 7.

We have

$$\Delta ABC\sim\Delta A'B'C'\sim\Delta AB'C\sim\Delta A'BC' \tag{1}$$

We note

$$\|BA'\|=x,\ \|CB'\|=y,\ \|AC'\|=z\,.$$

It results that

$$\|A'C\| = a - x,\ \|B'A\| = b - y,\ \|C'B\| = c - z$$

$$\overset{\wedge}{BAC} = \overset{\wedge}{B'A'C} = \overset{\wedge}{BA'C'};\ \overset{\wedge}{ABC} = \overset{\wedge}{AB'C'} = \overset{\wedge}{A'B'C'};\ \overset{\wedge}{BCA} = \overset{\wedge}{BC'A'} = \overset{\wedge}{B'C'A}$$

From these equalities it results the relation (1)

$$\Delta A'BC' \sim \Delta A'B'C \Rightarrow \frac{\|A'C'\|}{a-x} = \frac{x}{\|A'B'\|} \tag{2}$$

$$\Delta A'B'C \sim \Delta AB'C' \Rightarrow \frac{\|A'C'\|}{z} = \frac{c-z}{\|B'C'\|} \tag{3}$$

$$\Delta AB'C' \sim \Delta A'B'C \Rightarrow \frac{\|B'C'\|}{y} = \frac{b-y}{\|A'B'\|} \tag{4}$$

From (2), (3) and (4) we observe that the sum of the products from the problem is equal to:

$$x(a-x) + y(b-y) + z(c-z) = \frac{1}{4}(a^2+b^2+c^2) - \left(x - \frac{a}{2}\right)^2 - \left(y - \frac{b}{2}\right)^2 - \left(z - \frac{c}{2}\right)^2$$

which will reach its maximum as long as $x = \frac{a}{2},\ y = \frac{b}{2},\ z = \frac{c}{2}$, that is when the altitudes' legs are in the middle of the sides, therefore when the ΔABC is equilateral. The maximum of the expression is $\frac{1}{4}(a^2+b^2+c^2)$.

C**onclusion**[1]: If we note the lengths of the sides of the triangle ΔABC by ||AB|| = *c*, ||BC|| = *a*, ||CA|| = *b*, and the lengths of the sides of its orthic triangle ΔA`B`C` by ||A`B`|| = *c*`, ||B`C`|| = *a*`, ||C`A`|| = *b*`, then we proved that:

$$4(a`b` + b`c` + c`a`) \le a^2 + b^2 + c^2.$$

Unsolved Problems 7.

7.1. Generalize this problem to polygons. Let $A_1A_2\ldots A_m$ be a polygon and P a point inside it. From P, which is called a pedal point, we draw perpendiculars on each side A_iA_{i+1} of the polygon and we note by A_i' the intersection between the perpendicular and the side A_iA_{i+1}. Let's extend the definition of pedal triangle to a **pedal polygon** in a straight way: i.e. the polygon formed by the orthogonal projections of a pedal point on the sides of the polygon. The pedal polygon $A_1'A_2'\ldots A_m'$ is formed. What properties does this pedal polygon have?

7.2. Generalize this problem to polyhedrons. Let $A_1A_2\ldots A_n$ be a polyhedron and P a point inside it. From P we draw perpendiculars on each edge A_iA_j of the polyhedron and we note by A_{ij}' the intersection between the perpendicular and the side A_iA_{ij}. Let's name the

[1] This is called the **Smarandache's Orthic Theorem** (see [2], [3]).

new formed polyhedron an **edge pedal polyhedron**, $A_1'A_2'\ldots A_n'$. What properties does this edge pedal polyhedron have?

7.3. Generalize this problem to polyhedrons in a different way. Let $A_1A_2\ldots A_n$ be a poliyhedron and P a point inside it. From P we draw perpendiculars on each polyhedron face F_i and we note by A_i' the intersection between the perpendicular and the side F_i. Let's call the new formed polyhedron a **face pedal polyhedron**, which is $A_1'A_2'\ldots A_p'$, where p is the number of polyhedron's faces. What properties does this face pedal polyhedron have?

Problem 8.

Given the distinct points $A_1,...,A_n$ on the circumference of a circle with the center in O and of ray R.

Prove that there exist two points A_i, A_j such that $\left\|\overrightarrow{OA_i}+\overrightarrow{OA_j}\right\|\geq 2R\cos\frac{180^o}{n}$

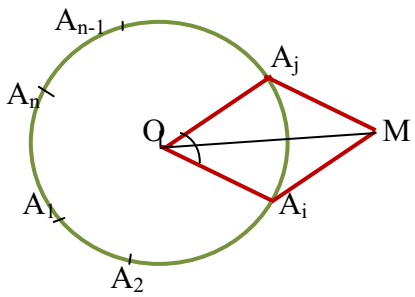
Solution 8.

Because

$$\sphericalangle A_1OA_2+\sphericalangle A_2OA_3+\ldots+\sphericalangle A_{n-1}OA_n+\sphericalangle A_nOA_1=360^o$$

and $\forall i\in\{1,2,...,n\}$, $\sphericalangle A_iOA_{i+2}>0^o$, it result that it exist at least one angle $\sphericalangle A_iOA_j\leq\frac{360^o}{n}$

(otherwise it follows that $S>\frac{360^0}{n}\cdot n=360^o$).



$$\overrightarrow{OA_i}+\overrightarrow{OA_j}=\overrightarrow{OM}\Rightarrow\left\|\overrightarrow{OA_i}+\overrightarrow{OA_j}\right\|=\left\|\overrightarrow{OM}\right\|$$

The quadrilateral OA_iMA_j is a rhombus. When α is smaller, $\left\|\overrightarrow{OM}\right\|$ is greater. As $\alpha\leq\frac{360^o}{n}$, it results that: $\left\|\overrightarrow{OM}\right\|=2R\cos\frac{\alpha}{2}\geq 2R\cos\frac{180^o}{n}$.

Open Problem 8:

Is it possible to find a similar relationship in an ellipse? (Of course, instead of the circle's radius R one should consider the ellipse's axes a and b.)

Problem 9:

Through one of the intersecting points of two circles we draw a line that intersects a second time the circles in the points M_1 and M_2 respectively. Then the geometric locus of the point M which divides the segment M_1M_2 in a ratio k (i.e. $M_1M = k \cdot MM_2$) is the circle of center O (where O is the point that divides the segment of line that connects the two circle centers O_1 and respectively O_2 into the ratio k, i.e. $O_1O = k \cdot OO_2$) and radius OA, without the points A and B.

Proof

Let $O_1E \perp M_1M_2$ and $O_2F \perp M_1M_2$. Let $O \in O_1O_2$ such that $O_1O = k \cdot OO_2$ and $M \in M_1M_2$, where $M_1M = k \cdot MM_2$.

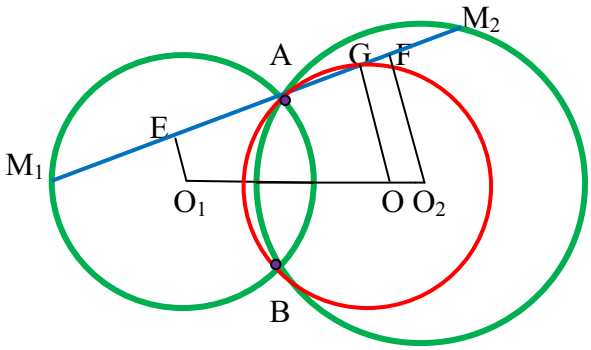


Fig. 1.

We construct $OG \perp M_1M_2$ and we make the notations: $M_1E \equiv EA = x$ and $AF \equiv FM_2 = y$.

Then, $AG \equiv GM$, because

$$AG = EG - EA = \frac{k}{k+1}(x+y) - x = \frac{-x+ky}{k+1}$$

and

$$GM = M_1M - M_1A - AG = \frac{k}{k+1}(2x+2y) - 2x - \frac{-x+ky}{k+1} = \frac{-x+ky}{k+1}.$$

Therefore we also have $OM \equiv OA$.

The geometric locus is a circle of center O and radius OA, without the points A and B (the red circle in Fig. 1).

Conversely.

If $M \in (GO, OA) \setminus \{A, B\}$, the line AM intersects the two circles in M_1 and M_2 respectively.

We consider the projections of the points O_1, O_2, O on the line M_1M_2 in E, F, G respectively. Because $O_1O = k \cdot OO_2$ it results that $EG = k \cdot GF$.

Making the notations: $M_1E \equiv EA = x$ and $AF \equiv FM_2 = y$ we obtain that

$$M_1M = M_1A + AM = M_1A + 2AG = 2x + 2(EG - EA) =$$

$$= \left[2x + 2\frac{k}{k+1}(x+y) - x \right] = \frac{k}{k+1}(2x + 2y) = \frac{k}{k+1} M_1M.$$

For $k = 2$ we find the Problem IV from [5].

Open Problem 9.

9.1. The same problem if instead of two circles one considers two ellipses, or one ellipse and one circle.

9.2. The same problem in *3D,* considering instead of two circles two spheres (their surfaces) whose intersection is a circle $\mathcal{C}$. Drawing a line passing through the circumference of $\mathcal{C}$, it will intersect the two spherical surfaces in other two points M_1 and respectively M_2.
Conjecture: The geometric locus of the point M which divides the segment M_1M_2 in a ratio k (i.e. $M_1M = k \cdot MM_2$) includes the spherical surface of center O (where O is the point that divides the segment of line that connects the two sphere centers O_1 and respectively O_2 into the ratio k, i.e. $O_1O = k \cdot OO_2$) and radius OA, without the intersection circle $\mathcal{C}$.

A partial proof is this: if the line M_1M_2 which intersect the two spheres is the same plane as the line O_1O_2 then the *3D* problem is reduce to a *2D* problem and the locus is a circle of radius OA and center O defined as in the original problem, where the point A belongs to the circumference of $\mathcal{C}$ (except two points). If we consider all such cases (infinitely many actually), we get a sphere of radius OA (from which we exclude the intersection circle $\mathcal{C}$) and centered in O (A can be any point on the circumference of intersection circle $\mathcal{C}$).

The locus has to be investigated for the case when M_1M_2 and O_1O_2 are in different planes.

9.3. What about if instead of two spheres we have two ellipsoids, or a sphere and an ellipsoid?

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Limits of Recursive Triangle and Polygon Tunnels

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Abstract.
In this paper we present unsolved problems that involve infinite tunnels of recursive triangles or recursive polygons, either in a decreasing or in an increasing way. The "nedians or order i in a triangle" are generalized to "nedians of ratio r" and "nedians of angle α" or "nedians at angle β", and afterwards one considers their corresponding "nedian triangles" and "nedian polygons".
This tunneling idea came from physics. Further research would be to construct similar tunnel of 3-D solids (and generally tunnels of n-D solids).

A) Open Question 1 (Decreasing Tunnel).

1. Let ΔABC be a triangle and let $\Delta A_1B_1C_1$ be its **orthic triangle** (i.e. the triangle formed by the feet of its altitudes) and H_1 its orthocenter (the point on intersection of its altitudes).
   Then, let's consider the triangle $\Delta A_2B_2C_2$, which is the orthic triangle of triangle $\Delta A_1B_1C_1$, and H_2 its orthocenter.
   And the recursive tunneling process continues in the same way.
   Therefore, let's consider the triangle $\Delta A_nB_nC_n$, which is the orthic triangle of triangle $\Delta A_{n-1}B_{n-1}C_{n-1}$, and H_n its orthocenter.

a) What is the locus of the orthocenter points $H_1, H_2, \ldots, H_n, \ldots$? {Locus means the set of all points satisfying some condition.}

b) Is this limit:

$$\lim_{n \to \infty} \Delta A_nB_nC_n$$

convergent to a point? If so, what is this point?

c) Calculate the sequences

$$\alpha_n = \frac{area(\Delta A_nB_nC_n)}{area(\Delta A_{n-1}B_{n-1}C_{n-1})} \text{ and } \beta_n = \frac{perimeter(\Delta A_nB_nC_n)}{perimeter(\Delta A_{n-1}B_{n-1}C_{n-1})}.$$

d) We generalize the problem from triangles to polygons. Let $AB\ldots M$ be a polygon with $m \geq 4$ sides. From A we draw a perpendicular on the next polygon's side BC, and note its intersection with this side by A_1. And so on. We get another polygon $A_1B_1\ldots M_1$.
We continue the recursive construction of this tunnel of polygons and we get the polygon sequence $A_nB_n\ldots M_n$.
d1) Calculate the limit:

$$\lim_{n \to \infty} \Delta A_n B_n \ldots M_n$$

d2) *A*nd the ratios of areas and perimeters as in question *c).*

e) A version of this polygonal extension *d)* would be to draw a perpendicular from *A* not necessarily on the next polygon's side, but on another side (say, for example, on the third polygon's side) – and keep a similar procedure for the next perpendiculars from all polygon vertices *B, C*, etc.

In order to tackle the problem in a easier way, one can start by firstly studying particular initial triangles ΔABC, such as the equilateral and then the isosceles.

B) Open Question 2 (Decreasing Tunnel).

2. Same problem as in Open Question 1, but replacing "orthic triangle" by "medial triangle", and respectively "orthocenter" by "center of mass (geometric centroid)", and "altitude" by "median". Therefore:

   Let ΔABC be a triangle and let $\Delta A_1B_1C_1$ be its **medial triangle** (i.e. the triangle formed by the feet of its medians on the opposite sides of the triangle ΔABC) and H_1 its center of mass (or geometric centroid) (the point on intersection of its medians).
   Then, let's consider the triangle $\Delta A_2B_2C_2$, which is the medial triangle of triangle $\Delta A_1B_1C_1$, and H_2 its center of mass.
   And the recursive tunneling process continues in the same way.
   Therefore, let's consider the triangle $\Delta A_nB_nC_n$, which is the medial triangle of triangle $\Delta A_{n-1}B_{n-1}C_{n-1}$, and H_n its center of massy.

a) What is the locus of the center of mass points $H_1, H_2, \ldots, H_n, \ldots$?
   {This has an easy answer; all H_i will coincide with H_1 (FS, IP).}

b) Is this limit:
$$\lim_{n \to \infty} \Delta A_n B_n C_n$$
   convergent to a point? If so, what is this point?
   {Same response; the limit is equal to H_1 (FS, IP).}

c) Calculate the sequences
$$\alpha_n = \frac{area(\Delta A_n B_n C_n)}{area(\Delta A_{n-1} B_{n-1} C_{n-1})} \text{ and } \beta_n = \frac{perimeter(\Delta A_n B_n C_n)}{perimeter(\Delta A_{n-1} B_{n-1} C_{n-1})}.$$

d) We generalize the problem from triangles to polygons. Let $AB\ldots M$ be a polygon with $m \geq 4$ sides. From A we draw a line that connects A with the midpoint of BC, and note its intersection with this side by A_1. And so on. We get another polygon $A_1B_1\ldots M_1$.
   We continue the recursive construction of this tunnel of polygons and we get the polygon sequence $A_nB_n\ldots M_n$.

d1) Calculate the limit:

$$\lim_{n \to \infty} \Delta A_n B_n \ldots M_n.$$

d2) And the ratios of areas and perimeters of two consecutive polygons as in question *c).*

e) A version of this polygonal extension *d)* would be to draw a line that connects *A* not necessarily on the midpoint of the next polygon's side, but with the midpoint of another side (say, for example, of the third polygon's side) – and keep a similar procedure for the next lines from all polygon vertices *B, C*, etc.

C) Open Questions 3-7 (Decreasing Tunnels).

3. Same problem as in Open Question 1, but considering a tunnel of **incentral triangles** and their incentral points, and their interior angles' bisectors.
   Incentral triangle is the triangle whose vertices are the intersections of the interior angle bisectors of the reference triangle ΔABC with the respective opposite sides of ΔABC.
4. Same problem as in Open Question 1, but considering a tunnel of **contact triangles** (intouch triangles) and their incircle center points, and their interior angles' bisectors.
   A contact triangle is a triangle formed by the tangent points of the triangle sides to its incircle.
5. Same problem as in Open Question 1, but considering a tunnel of **pedal triangles** and a fixed point *P* in the plane of triangle ΔABC.
   A pedal triangle of P is formed by the feet of the perpendiculars from P to the sides of the triangle ΔABC.
6. Same problem as in Open Question 1, but considering a tunnel of **symmedial triangles.**
   "The **symmedial triangle** $\Delta K_A K_B K_C$ (a term coined by E.W. Weisstein [4]), is the triangle whose vertices are the intersection points of the symmedians with the reference triangle ΔABC."
7. Same problem as in Open Question 1, but considering a tunnel of **cyclocevian triangles.**
   A cyclocevian triangle of triangle ΔABC with respect to the planar point *P* is the Cevian triangle of the cyclocevian conjugate of P.

D) Open Questions 8-12 (Increasing Tunnels).

8. Similar problem as in Open Question 1, but considering a tunnel of **anticevian triangles** of the triangle ΔABC with respect to the same planar point *P*. For question *c)* and *d1)* only.
   The anticevian triangle of the given triangle ΔABC with respect to the given point *P* is the triangle of which ΔABC is the Cevian triangle with respect to *P*.
9. Similarly, but considering a tunnel of **tangential triangles.**
   The tangential triangle to the given triangle ΔABC is a triangle formed by the tangents to the circumcircle of ΔABC at its vertices.
10. Similarly, but considering a tunnel of **antipedal triangles.**

The antipedal triangle of the given triangle ΔABC with respect to the given point P is the triangle of which ΔABC is the pedal triangle with respect to P.

11. Similarly, but considering a tunnel of **excentral triangles.**
The excentral triangle (or tritangent triangle) of the triangle ΔABC is the triangle with vertices corresponding to the excenters of ΔABC.
12. Similarly, but considering a tunnel of **anticomplementary triangles.**
The anticomplementary (or antimedian) triangle of the triangle ΔABC is the triangle formed by the parallels drew through the vertices of the triangle ΔABC to the opposite sides.

E) Open Questions Involving Nedians 13-14.

a) One calls nedians of order *i* [see 4] of the triangle ΔABC the lines that pass through each of the vertices of the triangle ΔABC and divide the opposite site of the triangle into the ratio *i/n*, for $1 \leq i \leq n-1$.
Let's generalize this to **nedians of ratio *r***, which means lines that pass through each of the vertices of the given triangle ΔABC and divide the opposite site of the triangle into the ratio *r*.
We introduce the notion of *nedian triangles*, first the interior nedian triangle of order *i* (or more general interior nedian triangle of ratio *r*), which is the triangle formed by the three points of intersections of the three nedians of order *i* (or respectively of the three nedians of ratio *r*), taken two by two;
and that of exterior nedian triangle of order i (or more general exterior nedian triangle of ratio r), which is the triangle $\Delta A'B'C'$ such that $A' \in BC$, $B' \in CA$, and $C' \in AB$ - where AA', BB', and CC' are nedians of order *i* (respectively of ratio *r*) in the triangle ΔABC.

b) Another notion to introduce: **nedians of angle *α*** (or ***α*-nedians**), which are nedians that each of them forms the same angle α with its respective side of the triangle, i.e.

$$\measuredangle(AA', AB) = \measuredangle(BB', BC) = \measuredangle(CC', CA) = \boldsymbol{\alpha}.$$

And associated with this we have interior ***α*-nedian triangle** and exterior α-nedian triangle.

c) And one more derivative to introduce now: nedians at angle *β* to the opposite side (or nedians-*β*), which are of course nedians that form with the opposite side of the triangle ΔABC the same angle *β*.
{As a particular case we have the altitudes, which are nedians at an angle of 90° or 90°-nedians.}
And associated with this we have interior nedian-*β* triangle and exterior nedian-*β* triangle.

d) All these notions about nedians in a triangle can be extended to **nedians in a polygon**, and to the formation of corresponding **nedian polygons**.

Then:

13. Let ΔABC be a triangle and let $\Delta A_1B_1C_1$ be its **interior nedian triangle of ratio r.** Then, let's consider the triangle $\Delta A_2B_2C_2$, which is the interior nedian triangle of order i of triangle $\Delta A_1B_1C_1$.
And the recursive tunneling process continues in the same way.

Therefore, let's consider the triangle $\Delta A_nB_nC_n$, which is the interior nedian triangle of ordedr i of triangle the triangle $\Delta A_{n-1}B_{n-1}C_{n-1}$.
Same questions b)-e) as in Open Question 1.

14. Similar questions for **exterior nedian triangle of ratio r**.
15-16. Similar questions for **interior α-nedian triangle** and **exterior α-nedian triangle**.
16-17. Similar questions for **interior nedian-β triangle** and **exterior nedian-β triangle**.
18-23. Similar questions as the above 13-17 for the corresponding **nedian polygons**.

F) More Open Questions.

The reader can exercise his or her research on other types of decreasing or increasing tunnels of special triangles (if their construction may work), such as the: extangential triangle, cotangential triangle, antisuplementary triangle, automedial triangle, altimedial triangle, circumpedal triangle, antiparalel triangle, Napoléon triangles, Vecten triangles, Sharygin triangles, Brocard triangles, Smarandache-Pătrașcu triangles (or orthohomological triangles[i]), Carnot triangle, Fuhrmann triangle, Morley triangle, Țițeica triangle, Lucas triangle, Lionnet triangle, Schroeter triangle, Grebe triangle, etc.
{We don't present their definitions since the reader can find them in books of *Geometry of Triangle* or in mathematical encyclopedias, see for examples [1] and [6].}

G) Construction.

Further research would be to construct similar tunnels of 3-D solids (and, more general, **tunnels of n-D solids** in R^n).

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4. Florentin Smarandache, *Généralisations et généralités*, Ed. Nouvelle, Fès, Morocco, 1984, http://fs.gallup.unm.edu//Generalisations.pdf.
5. Viorel Gh. Vodă, *Surprize în matematica elementară*, Ed. Albatros, Bucharest, 1981.
6. Eric W. Weisstein, *MathWorld* -- A Wolfram Web Resource, http://mathworld.wolfram.com/.

[i] We call two triangles, which are simultaneously orthological and homological, *orthohomological triangles (*or *Smarandache-Pătrașcu triangles* [2]*)*; for example: if the triangle ΔABC is given and P is a point inside it such that its pedal triangle $\Delta A_1B_1C_1$ is homological with ΔABC, then we say that the triangles ΔABC and $\Delta A_1B_1C_1$ are orthohomological.

A Theorem about Simultaneous Orthological and Homological Triangles

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Abstract. In this paper we prove that if P_1, P_2 are isogonal points in the triangle ABC, and if $A_1B_1C_1$ and $A_2B_2C_2$ are their corresponding pedal triangles such that the triangles ABC and $A_1B_1C_1$ are homological (the lines AA_1, BB_1, CC_1 are concurrent), then the triangles ABC and $A_2B_2C_2$ are also homological.

Introduction.

In order for the paper to be self-contained, we recall below the main definitions and theorems needed in solving this theorem.

Also, we introduce the notion of *Orthohomological Triangles*, which means two triangles that are simultaneously orthological and homological.

Definition 1

In a triangle ABC the Cevians AA_1 and AA_2 which are symmetric with respect to the angle's BAC bisector are called isogonal Cevians.

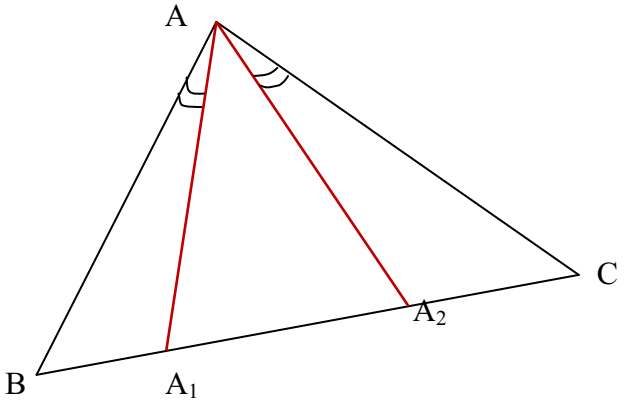

Fig. 1

Observation 1

If $A_1, A_2 \in BC$ and AA_1, AA_2 are isogonal Cevians then $\sphericalangle BAA_1 \equiv \sphericalangle BAA_2$. (See Fig.1.)

Theorem 1 (Steiner)

If in the triangle ABC, AA_1 and AA_2 are isogonal Cevians, A_1, A_2 are points on BC then:

$$\frac{A_1B}{A_1C}\cdot\frac{A_2B}{A_2C}=\left(\frac{AB}{AC}\right)^2$$

Proof

We have:

$$\frac{A_1B}{A_1C}=\frac{area\Delta BAA_1}{area\Delta CAA_1}=\frac{\frac{1}{2}AB\cdot AA_1\sin\left(\sphericalangle BAA_1\right)}{\frac{1}{2}AC\cdot AA_1\sin\left(\sphericalangle CAA_1\right)} \tag{1}$$

$$\frac{A_2B}{A_2C}=\frac{area\Delta BAA_2}{area\Delta CAA_2}=\frac{\frac{1}{2}AB\cdot AA_2\sin\left(\sphericalangle BAA_2\right)}{\frac{1}{2}AC\cdot AA_2\sin\left(\sphericalangle CAA_2\right)} \tag{2}$$

Because $\sin\left(\sphericalangle BAA_1\right)=\sin\left(\sphericalangle BAA_2\right)$ and $\sin\left(\sphericalangle BAA_2\right)=\sin\left(\sphericalangle CAA_1\right)$ by multiplying the relations (1) and (2) side by side we obtain the Steiner relation:

$$\frac{A_1B}{A_1C}\cdot\frac{A_2B}{A_2C}=\left(\frac{AB}{AC}\right)^2 \tag{3}$$

Theorem 2

In a given triangle, the isogonal Cevians of the concurrent Cevians are concurrent.

Proof

We'll use the Ceva's theorem which states that the triangle's ABC Cevians AA_1, BB_1, CC_1 ($A_1\in BC$, $B_1\in AC$, $C_1\in AB$) are concurrent if and only if the following relation takes place:

$$\frac{A_1B}{A_1C}\cdot\frac{B_1C}{B_1A}\cdot\frac{C_1A}{C_1B}=1 \tag{4}$$

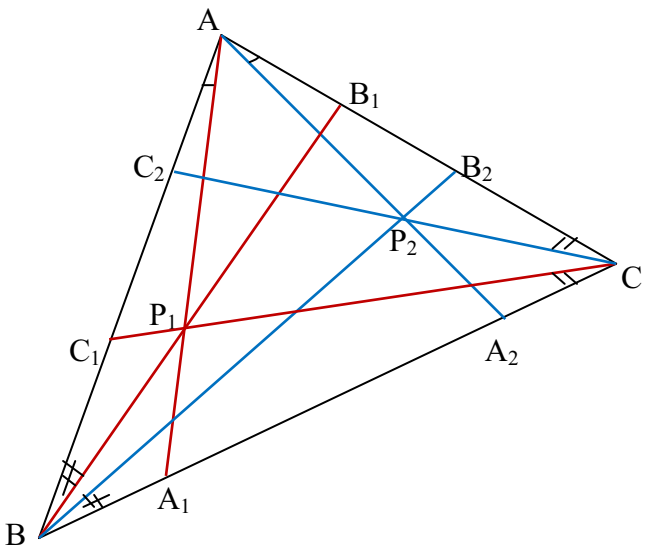

Fig. 2

We suppose that $AA_1,\ BB_1,\ CC_1$ are concurrent Cevians in the point P_1 and we'll prove that their isogonal $AA_2,\ BB_2,\ CC_2$ are concurrent in the point P_2. (See Fig. 2).

From the relations (3) and (4) we find:

$$\frac{A_2B}{A_2C}=\left(\frac{AB}{AC}\right)^2\cdot\frac{A_1C}{A_1B} \tag{5}$$

$$\frac{B_2C}{B_2A}=\left(\frac{BC}{AB}\right)^2\cdot\frac{B_1A}{B_1C} \tag{6}$$

$$\frac{C_2A}{C_2B}=\left(\frac{AC}{BC}\right)^2\cdot\frac{C_1B}{C_1A} \tag{7}$$

By multiplying side by side the relations (5), (6) and (7) and taking into account the relation (4) we obtain:

$$\frac{A_2B}{A_2C}\cdot\frac{B_2C}{B_2A}\cdot\frac{C_2A}{C_2B}=1,$$

which along with Ceva's theorem proves the proposed intersection.

Definition 2

The intersection point of certain Cevians and the point of intersection of their isogonal Cevians are called isogonal conjugated points or isogonal points.

Observation 2

The points P_1 and P_2 from Fig. 2 are isogonal conjugated points.

In a non right triangle its orthocenter and the circumscribed circle's center are isogonal points.

Definition 3

If P is a point in the plane of the triangle ABC, which is not on the triangle's circumscribed circle, and A', B', C' are the orthogonal projections of the point P respectively on BC, AC, and AB, we call the triangle $A'B'C'$ the pedal triangle of the point P.

Definition 4

The pedal triangle of the center of the inscribed circle in the triangle is called the contact triangle of the given triangle.

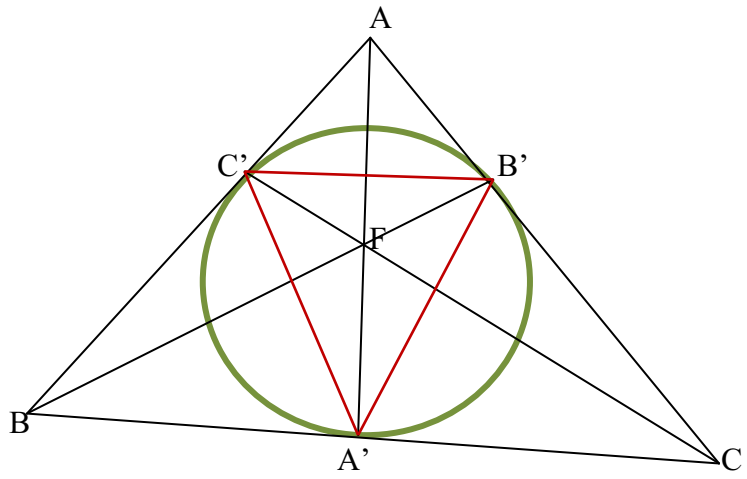

Fig. 3

Observation 3

In figure 3, $A'B'C'$ is the contact triangle of the triangle ABC. The name is connected to the fact that its vertexes are the contact points (of tangency) with the sides of the inscribed circle in the triangle ABC.

Definition 5

The pedal triangle of the orthocenter of a triangle is called orthic triangle.

Definition 6

Two triangles are called orthological if the perpendiculars constructed from the vertexes of one of the triangle on the sides of the other triangle are concurrent.

Definition 7

The intersection point of the perpendiculars constructed from the vertexes of a triangle on the sides of another triangle (the triangles being orthological) is called the triangles' orthology center.

Theorem 3 (The Orthological Triangles Theorem)

If the triangles ABC and $A'B'C'$ are such that the perpendiculars constructed from A on $B'C'$, from B on $A'C'$ and from C on $A'B'$ are concurrent (the triangles ABC and $A'B'C'$ being orthological), then the perpendiculars constructed from A' on BC, from B' on AC, and from C' on AB are also concurrent.

To prove this theorem firstly will prove the following:

Lemma 1 (Carnot)

If ABC is a triangle and A_1, B_1, C_1 are points on BC, AC, AB respectively, then the perpendiculars constructed from A_1 on BC, from B_1 on AC and from C_1 on AB are concurrent if and only if the following relation takes place:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0 \qquad (8)$$

Proof

If the perpendiculars in A_1, B_1, C_1 are concurrent in the point M (see Fig. 4), then from Pythagoras theorem applied in the formed right triangles we find:

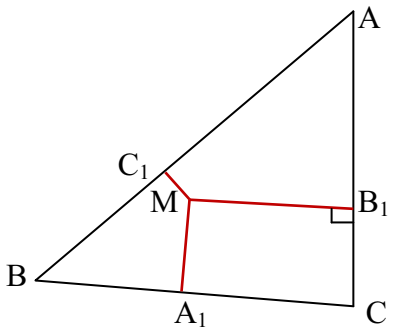

Fig. 4

$$A_1B^2 = MB^2 - MA_1^2 \tag{9}$$

$$A_1C^2 = MC^2 - MA_1^2 \tag{10}$$

hence

$$A_1B^2 - A_1C^2 = MB^2 - MC^2 \tag{11}$$

Similarly it results

$$B_1C^2 - B_1A^2 = MC^2 - MA^2 \tag{12}$$

$$C_1A^2 - C_1B^2 = MA^2 - MC^2 \tag{13}$$

By adding these relations side by side it results the relation (8).

Reciprocally

We suppose that relation (8) is verified, and let's consider the point M being the intersection of the perpendiculars constructed in A_1 on BC and in B_1 on AC. We also note with C' the projection of M on AC. We have that:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 + C'A^2 - C'B^2 = 0 \tag{14}$$

Comparing (8) and (14) we find that

$$C_1A^2 - C_1B^2 = C'A^2 - C'B^2$$

and

$$(C_1A - C_1B)(C_1A + C_1B) = (C'A - C'B)(C'A + C'B)$$

and because

$$C_1A - C_1B = C'A + C'B = AB$$

we obtain that $C' = C_1$, therefore the perpendicular in C_1 passes through M also.

Observation 4

The triangle ABC and the pedal triangle of a point from its plane are orthological triangles.

The proof of Theorem 3

Let's consider ABC and $A'B'C'$ two orthological triangles (see Fig. 5). We note with M the intersection of the perpendiculars constructed from A on $B'C'$, from B on $A'C'$ and from C on $A'B'$, also we'll note with A_1, B_1, C_1 the intersections of these perpendiculars with $B'C'$, $A'C'$ and $A'B'$ respectively.

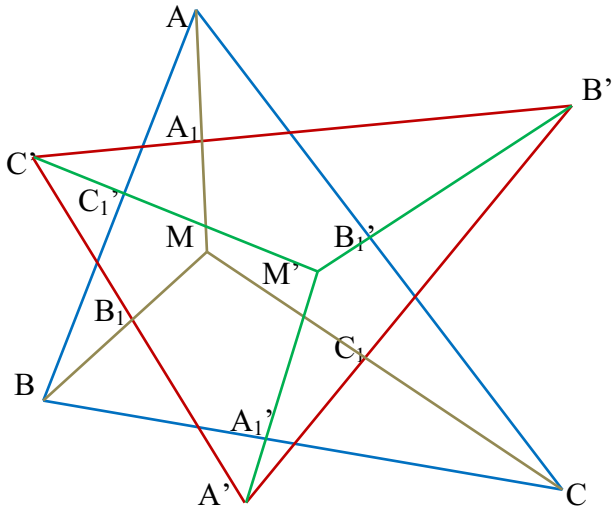

Fig. 5

In conformity with lemma 1, we have:

$$A_1B'^2 - A_1C'^2 + B_1C'^2 - B_1A'^2 + C_1A'^2 - C_1B'^2 = 0 \qquad (15)$$

From this relation using the Pythagoras theorem we obtain:

$$B'A^2 - C'A^2 + C'B^2 - A'B^2 + A'C^2 - B'C^2 = 0 \qquad (16)$$

We note with A_1', B_1', C_1' the orthogonal projections of A', B', C' respectively on BC, CA, AB. From the Pythagoras theorem and the relation (16) we obtain:

$$A_1'B^2 - A_1'C^2 + B_1'C^2 - B_1'A^2 + C_1'A^2 - C_1'B^2 = 0 \qquad (17)$$

This relation along with Lemma 1 shows that the perpendiculars drawn from A' on BC, from B' on AC and from C' on AB are concurrent in the point M'.

The point M' is also an orthological center of triangles $A'B'C'$ and ABC.

Definition 8

The triangles ABC and $A'B'C'$ are called bylogical if they are orthological and they have the same orthological center.

Definition 9

Two triangles ABC and $A'B'C'$ are called homological if the lines AA', BB', CC' are concurrent. Their intersection point is called the homology point of triangles ABC and $A'B'C'$

Observation 6

In figure 6 the triangles AA', BB', CC' are homological and the homology point being O

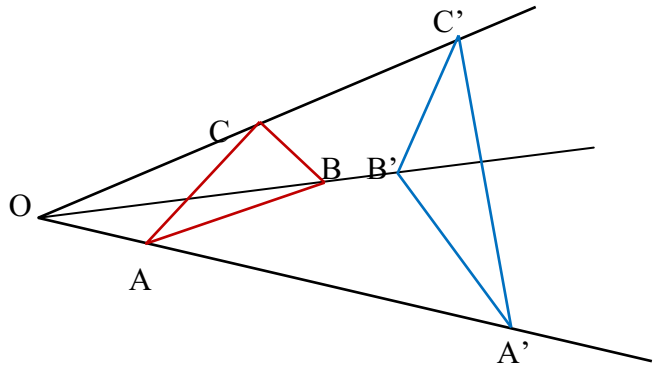

Fig.6

If ABC is a triangle and $A'B'C'$ is its pedal triangle, then the triangles ABC and $A'B'C'$ are homological and the homology center is the orthocenter H of the triangle ABC

Definition 10

A number of n points ($n \geq 3$) are called concyclic if there exist a circle that contains all of these points.

Theorem 5 (The circle of 6 points)

If ABC is a triangle, P_1, P_2 are isogonal points on its interior and $A_1B_1C_1$ respectively

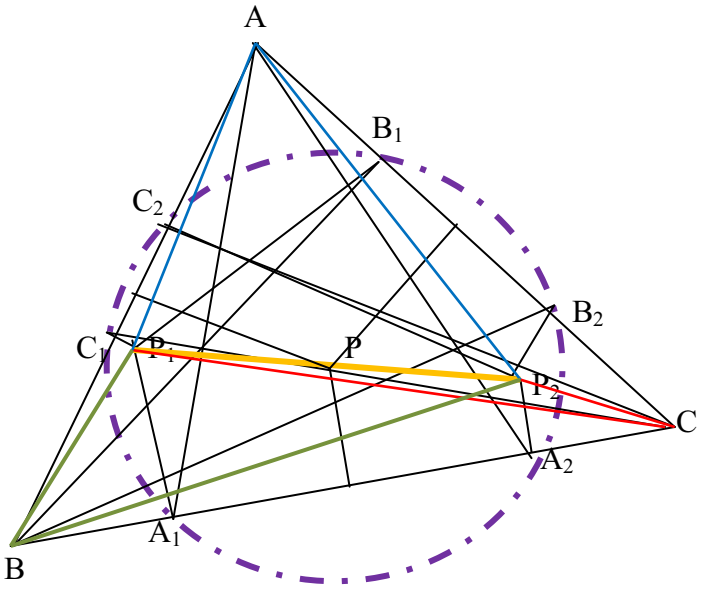

Fig. 7

$A_2B_2C_2$ the pedal triangles of P_1 and P_2, then the points $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic.

Proof

We will prove that the 6 points are concyclic by showing that these are at the same distance of the middle point P of the line segment P_1P_2.

It is obvious that the medians of the segments $(A_1A_2),(B_1B_2),(C_1C_2)$ pass through the point P, which is the middle of the segment (P_1P_2).The trapezoid $A_1P_1P_2A_2$ is right angle and the mediator of the segment (A_1A_2) will be the middle line, therefore it will pass through P,(see Fig. 7).

Therefore we have:

$$PA_1 = PA_2,\ PB_1 = PB_2,\ PC_1 = PC_2 \tag{18}$$

We'll prove that $PB_1 = PC_2$ by computing the length of these segments using the median's theorem applied in the triangles $P_1B_1P_2$ and $P_1C_2P_2$.

We have:

$$4PB_1^2 = 2\left(P_1B_1^2 + P_2B_1^2\right) - P_1P_2^2 \tag{19}$$

We note

$$AP_1 = x_1,\ AP_2 = x_2,\ m\left(\sphericalangle BAP_1\right) = m\left(\sphericalangle CAP_2\right) = \alpha .$$

In the right triangle $P_2B_2B_1$ applying the Pythagoras theorem we obtain:

$$P_2B_1^2 = P_2B_2^2 + B_1B_2^2 \tag{20}$$

From the right triangle AB_2P_2 we obtain:

$$P_2B_2 = AP_2\sin\alpha = x_2\sin\alpha \text{ and } AB_2 = x_2\cos\alpha$$

From the right triangle AP_1B_1 it results $AB_1 = AP_1\cos\left(A-\alpha\right)$, therefore

$$AB_1 = x_1\cos\left(A-\alpha\right) \text{ and } P_1B_1 = x_1\sin\left(A-\alpha\right),$$

thus

$$B_1B_2 = \left|AB_2 - AB_1\right| = \left|x_2\cos\alpha - x_1\cos\left(A-\alpha\right)\right| \tag{21}$$

Substituting back in relation (17), we obtain:

$$P_2B_1^2 = x_2^2\sin^2\alpha + \left[x_2\cos\alpha - x_1\cos\left(A-\alpha\right)\right]^2 \tag{22}$$

From the relation (16), it results:

$$4PB_1^2 = 2\left[x_1^2 + x_2^2 - 2x_1x_2\cos\alpha\cos\left(A-\alpha\right)\right]P_1P_2^2 \tag{23}$$

The median's theorem in the triangle $P_1C_2P_2$ will give:

$$4PC_2^2 = 2\left(P_1C_2^2 + P_2C_2^2\right) - P_1P_2^2 \tag{24}$$

Because $P_1C_1 = x_1\sin\alpha$, $AC_1 = x_1\cos\alpha$, $AC_2 = x_2\cos\left(A-\alpha\right)$, $P_1C_2^2 = P_1C_1^2 + C_1C_2^2$, we find that

$$4PC_2^2 = 2\left[x_1^2 + x_2^2 - 2x_1x_2\cos\alpha\cos\left(A-\alpha\right)\right] - P_1P_2^2 \tag{25}$$

The relations (23) and (25) show that

$$PB_1 = PC_2 \tag{26}$$

Using the same method we find that :

$$PA_1 = PC_1 \tag{27}$$

The relations (18), (26) and (27) imply that:

$$PA_1 = PA_2 = PB_1 = PB_2 = PC_1 = PC_2$$

From which we can conclude that $A_1, A_2, B_1, B_2, C_1, C_2$ are concyclic.

Lemma 2 (The power of an exterior point with respect to a circle)

If the point A is exterior to circle $C(O,r)$ and d_1, d_2 are two secants constructed from A that intersect the circle in the points B, C respectively E, D, then:

$$AB \cdot AC = AE \cdot AD = cons. \tag{28}$$

Proof

The triangles ADB and ACE are similar triangles (they have each two congruent angles respectively), it results:

$$\frac{AB}{AE} = \frac{AD}{AC}$$

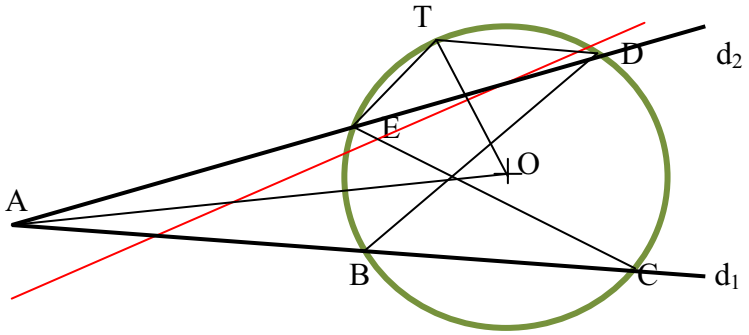

Fig. 8

and from here:

$$AB \cdot AC = AE \cdot AD \tag{29}$$

We construct the tangent from A to circle $C(O,r)$ (see Fig. 8). The triangles ATE and ADT are similar (the angles from the vertex A are common and $\sphericalangle ATE \equiv \sphericalangle ADT = \frac{1}{2} m\left(\widehat{TE}\right)$).

We have:

$$\frac{AE}{AT} = \frac{AT}{AD},$$

it results

$$AE \cdot AD = AT^2 \tag{30}$$

By noting $AO = a$, from the right triangle ATO (the radius is perpendicular on the tangent in the contact point), we find that:

$$AT^2 = AO^2 - OT^2,$$

therefore

$$AT^2 = a^2 - r^2 = const. \tag{31}$$

The relations (29), (30) and (31) are conducive to relation (28).

Theorem 6 (Terquem)

If AA_1, BB_1, CC_1 are concurrent Cevians in the triangle ABC and A_2, B_2, C_2 are intersections of the circle circumscribed to the triangle A_1,B_1,C_1 cu (BC), (CA), (AB), then the lines AA_2, BB_2, CC_2 are concurrent.

Proof

Let's consider F_1 the concurrence point of the Cevians AA_1, BB_1, CC_1.

From Ceva's theorem it results that:

$$A_1B \cdot B_1C \cdot C_1A = A_1C \cdot B_1A \cdot C_1B \tag{32}$$

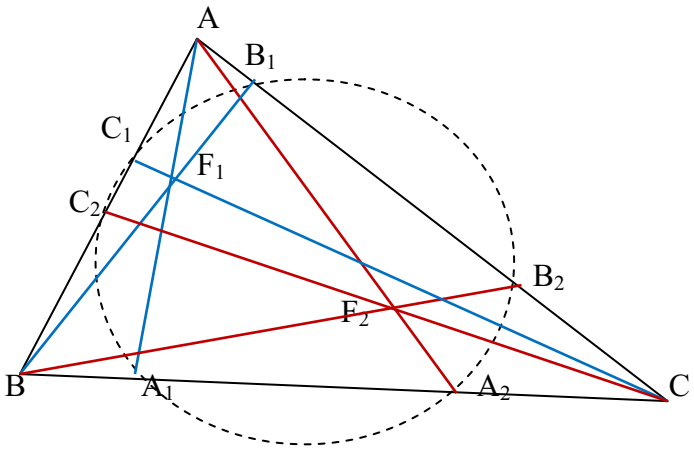

Fig 9

Considering the vertexes A, B, C's power with respect to the circle circumscribed to the triangle $A_1B_1C_1$, we obtain the following relations:

$$AC_1 \cdot AC_2 = AB_1 \cdot AB_2 \tag{33}$$

$$BA_1 \cdot BA_2 = BC_1 \cdot BC_2 \tag{34}$$

$$CB_1 \cdot CB_2 = CA_1 \cdot CA_2 \tag{35}$$

Multiplying these relations side by side and taking into consideration the relation (32), we obtain

$$AC_2 \cdot BA_2 \cdot CB_2 = AB_2 \cdot BC_2 \cdot CA_2 \tag{36}$$

This relation can be written under the following equivalent format

$$\frac{A_2B}{A_2C} \cdot \frac{B_2C}{B_2A} \cdot \frac{C_2A}{C_2B} = 1 \tag{37}$$

From Ceva's theorem and the relation (37) we obtain that the lines AA_2, BB_2, CC_2 are concurrent in a point noted in figure 9 by F_2.

Note 1

The points F_1 and F_2 have been named the Terquem's points by Candido of Pisa – 1900.

For example in a non right triangle the orthocenter H and the center of the circumscribed circle O are Terquem's points.

Definition 11

Two triangles are called orthohomological if they are simultaneously orthological and homological.

Theorem 7[1]

If P_1, P_2 are two conjugated isogonal points in the triangle ABC, and $A_1B_1C_1$ and $A_2B_2C_2$ are their respectively pedal triangles such that the triangles ABC and $A_1B_1C_1$ are homological, then the triangles ABC and $A_2B_2C_2$ are also homological.

Proof

Let's consider that F_1 is the concurrence point of the Cevians AA_1, BB_1, CC_1 (the center of homology of the triangles ABC and $A_1B_1C_1$). In conformity with Theorem 6 the circumscribed circle to triangle $A_1B_1C_1$ intersects the sides (BC), (CA), (AB) in the points A_2, B_2, C_2, these points are exactly the vertexes of the pedal triangle of P_2, because if two circles have in common three points, then the two circles coincide; practically, the circle circumscribed to the triangle $A_1B_1C_1$ is the circle of the 6 points (Theorem 5).

Terquem's theorem implies the fact that the triangles ABC and $A_2B_2C_2$ are homological. Their homological center is F_2, the second Terquem's point of the triangle ABC.

Observation 7

If the points P_1 and P_2 isogonal conjugated in the triangle ABC coincide, then the triangles ABC and $A_2B_2C_2$, the pedal of $P_1 = P_2$ are homological.

Proof

From $P_1 = P_2$ and the fact that P_1, P_2 are isogonal conjugate, it results that $P_1 = P_2 = I$ - the center of the inscribed circle in the triangle ABC. The pedal triangle of I is the contact triangle. In this case the lines AA_1, BB_1, CC_1 are concurrent in Γ, Gergonne's point, which is the homological center of these triangles.

Observation 8

The reciprocal of Theorem 7 for orthohomological triangles is not true.

To prove this will present a counterexample in which the triangle ABC and the pedal triangles $A_1B_1C_1$, $A_2B_2C_2$ of the points P_1 and P_2 are homological, but the points P_1 and P_2 are not isogonal conjugated; for this we need several results.

Definition 12

In a triangle two points on one of its side and symmetric with respect to its middle are called isometrics.

[1] This theorem was called the *Smarandache-Pătraşcu Theorem of Orthohomological Triangles* (see [3], [4]).

Definition 13

The circle tangent to a side of a triangle and to the other two sides' extensions of the triangle is called exterior inscribed circle to the triangle.

Observation 9

In figure 10 we constructed the extended circle tangent to the side (BC). We note its center with I_a. A triangle ABC has, in general, three exinscribed circles

Definition 14

The triangle determined by the contact points with the sides (of a triangle) of the exinscribed circle is called the cotangent triangle of the given triangle.

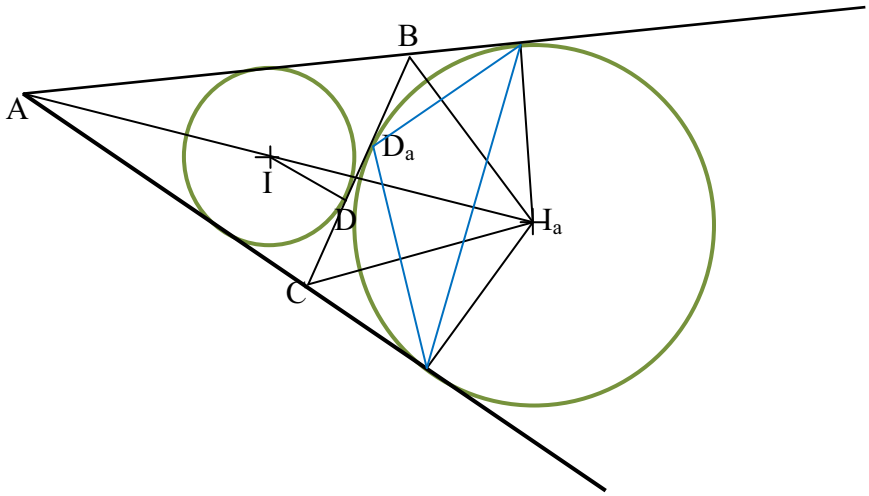

Fig. 10

Theorem 8

The isometric Cevians of the concurrent Cevians are concurrent.

The proof of this theorem results from the definition 14 and Ceva's theorem

Definition 15

The contact points of the Cevians and of their isometric Cevians are called conjugated isotomic points.

Lemma 3

In a triangle ABC the contact points with a side of the inscribed circle and of the exinscribed circle are isotomic points.

Proof

The proof of this lemma can be done computational, therefore using the tangents' property constructed from an exterior point to a circle to be equal, we compute the CD and BD_a (see Fig. 10) in function of the length a,b,c of the sides of the triangle ABC.

We find that $CD = p - c = BD_a$, which shows that the Cevians AD and AD_a are isogonal (p is the semi-perimeter of triangle ABC, $2p = a+b+c$).

Theorem 9

The triangle ABC and its cotangent triangle are isogonal.

We'll use theorem 8 and taking into account lemma 3, and the fact that the contact triangle and the triangle ABC are homological, the homological center being the Gergonne's point.

Observation 10

The homological center of the triangle ABC and its cotangent triangle is called Nagel's point (N).

Observation 11

The Gergonne's point (Γ) and Nagel's point (N) are isogonal conjugated points.

Theorem 10

The perpendiculars constructed on the sides of a triangle in the vertexes of the cotangent triangle are concurrent.

The proof of this theorem results immediately using lema1 (Carnot)

Definition 12

The concurrence point of the perpendiculars constructed in the vertexes of the cotangent triangle on the sides of the given triangle is called the Bevan's point (V).

We will prove now that the reciprocal of the theorem of the orthohomological triangles is false

We consider in a given triangle ABC its contact triangle and also its cotangent triangle. The contact triangle and the triangle ABC are homological, the homology center being the Geronne's point (Γ). The given triangle and its cotangent triangle are homological, their homological center being Nagel's point (N). Beven's point and the center of the inscribed circle have as pedal triangles the cotangent triangles and of contact, but these points are not isogonal conjugated (the point I is its own isogonal conjugate).

References:

1. C. Mihalescu, *Geometria elementelor remarcabile*, Ed. Tehnică, Bucureşti, 1959.
2. R.A. Johnson, *Modern Geometry: An Elementary Treatise on the Geometry of the Triangle and the Circle*, 1929.
3. Mihai Dicu, *The Smarandache-Pătrașcu Theorem of Orthohomological Triangles*, http://www.scribd.com/doc/28311880/Smarandache-Patrascu-Theorem-of-Orthohomological-Triangles, 2010.
4. Claudiu Coandă, *A Proof in Barycentric Coordinates of the Smarandache-Pătrașcu Theorem*, Sfera journal, 2010.

An Application of a Theorem of Orthohomological Triangles

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Abstract.
In this note we prove a problem given at a Romanian student mathematical competition, and we obtain an interesting result by using a *Theorem of Orthohomological Triangles*[1].

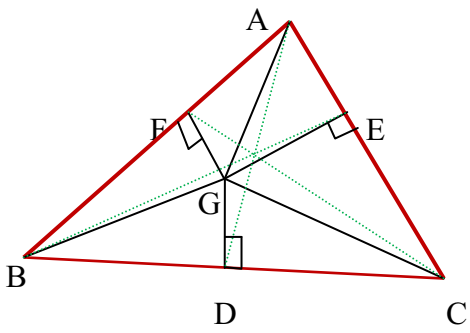
Problem L. 176 (from [1])

Let D, E, F be the projections of the centroid G of the triangle ABC on the lines BC, CA, and respectively AB. Prove that the Cevian lines AD, BE, and CF meet in an unique point if and only if the triangle is isosceles. {Proposed by Temistocle Bîrsan.}

Proof

Applying the generalized Pythagorean theorem in the triangle BGC, we obtain:

$$CG^2 = BG^2 + BC^2 - 2BD \cdot BC \qquad (1)$$



Because $CG = \frac{2}{3} m_c$, $BG = \frac{2}{3} m_b$ and from the median's theorem it results:

$$4m_b^2 = 2(a^2 + c^2) - b^2 \text{ and } 4m_c^2 = 2(a^2 + b^2) - c^2$$

From (1) we get: $BD = \frac{3a^2 - b^2 + c^2}{6a}$.

[1] It has been called the *Smarandache-Pătraşcu Theorem of Orthohomological Triangles* (see [2], [3], [4]).

From $BC = a$ and $BC = BD + DC$, we get that:

$$DC = \frac{3a^2 + b^2 - c^2}{6a}$$

Similarly we find:

$$CE = \frac{3b^2 - c^2 + a^2}{6b},\ EA = \frac{3b^2 + c^2 - a^2}{6b}$$

$$FA = \frac{3c^2 - a^2 + b^2}{6c},\ FB = \frac{3c^2 + a^2 - b^2}{6c}.$$

Applying Ceva's theorem it results that AD, BE, CF are concurrent if and only if

$$\left(3a^2 - b^2 + c^2\right)\left(3b^2 - c^2 + a^2\right)\left(3c^2 - a^2 + b^2\right) = \left(3a^2 + b^2 - c^2\right)\left(3b^2 + c^2 - a^2\right)\left(3c^2 + a^2 - b^2\right) \quad (2)$$

Let's consider the following notations:

$$a^2 + b^2 + c^2 = T,\ 2a^2 - 2b^2 = \alpha,\ 2b^2 - 2c^2 = \beta,\ 2c^2 - 2a^2 = \gamma$$

From (2) it results:

$$(T + \alpha)(T + \beta)(T + \gamma) = (T - \alpha)(T - \beta)(T - \gamma).$$

And from here:

$$T^3 + (\alpha + \beta + \gamma)T^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)T + \alpha\beta\gamma = T^3 - (\alpha + \beta + \gamma)T^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)T - \alpha\beta\gamma.$$

Because $\alpha + \beta + \gamma = 0$, we obtain that $2\alpha\beta\gamma = 0$, therefore $\alpha = 0$ or $\beta = 0$ or $\gamma = 0$, thus $a = b$ or $b = c$ or $a = c$; consequently the triangle ABC is isosceles.

The reverse: If ABC is an isosceles triangle, then it is obvious that AD, BE, and CF are concurrent.

Observations

1. The proved problem asserts that:
   "A triangle ABC and the pedal triangle of its weight center are orthomological triangles if and only if the triangle ABC is isosceles."

2. Using the previous result and the Smarandache-Pătrăşcu Theorem (see [2], [3], [4]) we deduce that:
   "A triangle ABC and the pedal triangle of its simedian center are orthomological triangles if and only if the triangle ABC is isosceles."

References

[1] Temistocle Bîrsan, Training problems for mathematical contest, B. College Level – L. 176, Recreaţii Matematice journal, Iași, Romania, Year XII, No. 1, 2010.

[2] Ion Pătrașcu & Florentin Smarandache, *A Theorem about Simultaneous Orthological and Homological Triangles*, in arXiv.org, Cornell University, NY, USA.

[3] Mihai Dicu, *The Smarandache- Pătrașcu Theorem of Orthohomological Triangles*, http://www.scribd.com/doc/28311880/Smarandache-Patrascu-Theorem-of-Orthohomological-Triangles.
[4] Claudiu Coandă, *A Proof in Barycentric Coordinates of the Smarandache-Pătrașcu Theorem*, Sfera journal, 2010.

A Multiple Theorem with Isogonal and Concyclic Points

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Let's consider A', B', C' three points on the sides (BC), (CA), (AB) of triangle ABC such that simultaneously are satisfied the following conditions:

i. $A'B^2 + B'C^2 + C'A^2 = A'C^2 + B'A^2 + C'B^2$

ii. The lines AA', BB', CC' are concurrent.

Prove that:

a) The perpendiculars drawn in A' on BC, in B' on AC, and in C' on AB are concurrent in a point P.

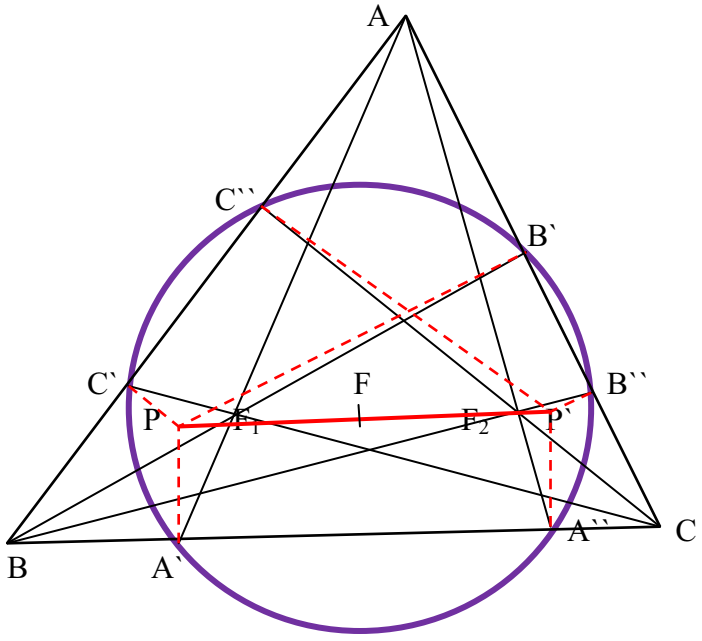
b) The perpendiculars drawn in A' on $B'C'$, in B' on $A'C'$, and in C' on $A'B'$ are concurrent in a point P'.

c) The points P and P' are isogonal.

d) If A'', B'', C'' are the projections of P' on BC, CA, respective AB, then the points A', A'', B', B'', C', C'', are concyclic points.

e) The lines AA'', BB'', CC'' are concurrent.

Proof:



a) Let P be the intersection of the perpendicular drawn in A' on BC with the perpendicular drawn in B' on AC. We have:

$$PB^2 - PC^2 = A'B^2 - A'C^2$$
$$PC^2 - PA^2 = B'C^2 - B'A^2.$$

By adding side by side these two relations, it results

$$PB^2 - PA^2 = A'B^2 - A'C^2 + B'C^2 - B'A^2. \quad (1)$$

If we note with C_1 the projection of P on AB, we have:

$$PB^2 - PA^2 = C_1B^2 - C_1A^2. \quad (2)$$

From the relations (1), (2), and (i) we obtain that $C_1 \equiv C'$, therefore P has as ponder triangle the triangle $A'B'C'$.

b) Let A_1, B_1, C_1 respective the orthogonal projections of the points A, B, C on $B'C', C'A'$ respectively $A'B'$.

We have

$$A_1C'^2 - A_1B'^2 = C'A^2 - B'A^2,$$
$$B_1C'^2 - B_1A'^2 = C'B^2 - A'B^2,$$
$$C_1A'^2 - C_1B'^2 = A'C^2 - B'C^2.$$

From these relations we deduct

$$A_1C'^2 + B_1A'^2 + C_1B'^2 = A_1B'^2 + B_1C'^2 + C_1B'^2$$

therefore, a relation of the same type as (i) for the triangle $A'B'C'$. By using a similar method it results that $A_1B_1C_1$ is the triangle ponder of a point P'.

c) The quadrilateral $AB'PC'$ is inscribable, therefore $\sphericalangle APB' \equiv \sphericalangle AC'B'$, and because these angles are the complements of the angles $\sphericalangle C'AP$ and $\sphericalangle B'AP'$, it results that these angles are congruent, therefore the Cevians AP and AP' are isogonal, similarly we can show that the Cevians BP and BP' are isogonal and also the Cevians CP and CP' are isogonal.

d) It is obvious that the medians of the segments $(A'A''), (B'B'')$ and $(C'C'')$ pass through F, which is the middle of the segment (PP'). We have to prove that F is the center of the circle that contains the given points of the problem.

We will use the median's theorem on the triangles $C'PP'$ and $B'PP'$ to compute $C'F$ and $B'F$.

We note $m\left(\overset{\frown}{P'AC}\right) = m(\sphericalangle PAB) = \alpha$, $AP = x$, $AP' = x'$;

then we have

$$4C'F^2 = 2\left(PC'^2 + P'C'^2\right) - PP'^2$$
$$4B'F^2 = 2\left(PB'^2 + P'B'^2\right) - PP'^2$$

$PC' = x\sin\alpha$, $P'C'^2 = P'C''^2 + C''C'^2$, $P'C'' = x'\sin(A-\alpha)$

$AC'' = x'\cos(A-\alpha)$, $AC' = x\cos\alpha$,

$P'C'^2 = x'^2 + \sin^2(A-\alpha) + \left(x'\cos(A-\alpha) - x\cos\alpha\right)^2 =$

$$= x'^2 + x^2\cos^2\alpha - 2xx'\cos\alpha\cos(A-\alpha)$$

$$4C'F^2 = 2\left[x'^2 + x^2\cos^2\alpha - 2xx'\cos\alpha\cos(A-\alpha)\right] - PP'^2$$

$$4C'F^2 = 2\left[x'^2 + x^2 - 2xx'\cos\alpha\cos(A-\alpha)\right] - PP'^2$$

Similarly we determine the expression for $4B'F^2$, and then we obtain that $C'F = B'F$, therefore the points C', C'', B'', B' are concyclic.

We'll follow the same method to prove that $C'F = A'F$ which leads to the fact that the points C', C'', A', A'' are also concyclic, and from here to the requested statement.

e) From (ii) it results (from Ceva's theorem) that:

$$A'B\cdot B'C\cdot C'A = A'C\cdot B'A\cdot C'B. \qquad (3)$$

Let's consider the points' A, B, C power respectively in rapport to the circle determined by the points $A', A'', B', B'', C', C''$,
we have

$$AB'\cdot AB'' = AC'\cdot AC''$$
$$BA'\cdot BA'' = BC'\cdot BC''$$
$$CA'\cdot CA'' = CB'\cdot CB''.$$

Multiplying these relations we obtain:

$$A'B\cdot BA''\cdot B'C\cdot BC''\cdot C'A\cdot AC'' = C'B\cdot BC''\cdot B'A\cdot AB''\cdot A'C\cdot CA''. \qquad (4)$$

Taking into account the relation in (3), it results

$$BA''\cdot CB''\cdot AC'' = BC''\cdot AB''\cdot CA''.$$

This last relation along with Ceva's theorem will lead us to the conclusion that the lines AA'', BB'', CC'' are concurrent.

Reference:

F. Smarandache, *Problèmes avec et sans ... problèmes!*, Somipress, Fès, Morocco, 1983.

Properties of a Hexagon Circumscribed to a Circle

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In this paper we analyze and prove two properties of a hexagon circumscribed to a circle:

Property 1.
If $ABCDEF$ is a hexagon circumscribed to a circle with the center in O, tangent to the sides AB, BC, CD, DE, EF, FA respectively in A', B', C', D', E', F', and if the lines of the triplet formed from two lines that belong to the set $\{AD, BE, CF\}$ and a line that belongs to the set $\{A'D', B'E', C'F'\}$ are concurrent, then the lines AD, BE, CF, $A'D'$, $B'E'$, $C'F'$ are concurrent.

Property 2.
If $ABCDEF$ is a hexagon circumscribed to a circle with the center in O, tangent to the sides AB, BC, CD, DE, EF, FA respectively in A', B', C', D', E', F', such that the hexagon $A'B'C'D'E'F'$ is circumscribable, then the lines AD, BE, CF, $A'D'$, $B'E'$, $C'F'$ are concurrent.

To prove these propositions we'll use:

Lemma 1 (Brianchon's Theorem)
If $ABCDEF$ is a hexagon circumscribable then the lines AD, BE, CF are concurrent.

Lemma 2
If $ABCDEF$ is a hexagon circumscribed to a circle tangent to the sides AB, BC, CD, DE, EF, FA respectively in A', B', C', D', E', F', such that $A'D' \cap C'F' = \{A_o\}$, $B'E' \cap A'D' = \{B_o\}$, $C'F' \cap B'E' = \{C_o\}$, then $A_o \in AD$, $B_o \in BE$, $C_o \in CF$.

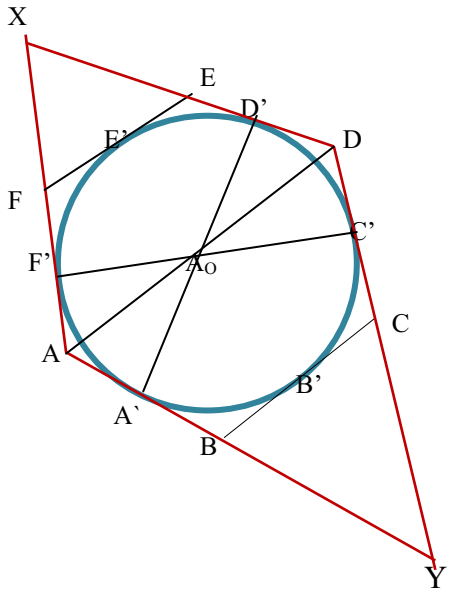

Fig. 1

Proof of Lemma 2

We note $\{X\} = AF \bigcap DE$ and $\{Y\} = AB \bigcap DC$ (see figure 1).

In the quadrilateral $XAYD$ circumscribed, the Newton's theorem gives that the lines AD, $A'D'$, $C'F'$ and XY are concurrent, therefore $A_O \in AD$.

Similarly, is proven that $B_O \in BE$ and that $C_O \in CF$

Proof of Property 1

We suppose that AD, BE and $A'D'$ are concurrent in the point I (see fig. 2).

We denote $\{X\} = AF \bigcap DE$ and $\{Y\} = AB \bigcap DC$, we apply Newton's theorem in the quadrilateral $XAYD$, it results that the line $C'F'$ also passes through I.

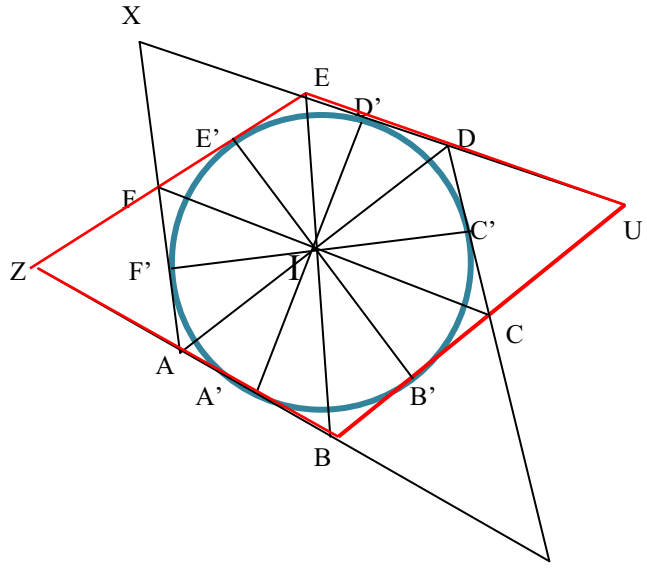

Fig. 2

On the other side from Lemma 1 it results that CF passes through I.

We note $\{Z\} = EF \cap AB$ and $\{U\} = BC \cap ED$ in the circumscribed quadrilateral $EZBU$. Newton's theorem shows that the lines BE, ZU, $B'E'$ and $A'D'$ are concurrent. Because BE and $A'D'$ pass through I, it results that also $B'E'$ passes through I, and the proof is complete.

Observation

There exist circumscribable hexagons $ABCDEF$ in which the six lines from above are concurrent (a banal example is the regular hexagon).

Proof of Property 2

From Lemma 1 we obtain that $AD \cap BE \cap CF = \{I\}$ and $A'D' \cap B'E' \cap C'F' = \{I'\}$. From Lemma 2 it results that $I' \in AD$ and $I' \in BE$, because $AD \cap BE = \{I\}$, we obtain that $I = I'$ and consequently all six lines are concurrent.

Reference:

Florentin Smarandache, "Problems with and without…problems!", Somipress, Fés, Morocco, 1983.

A GENERALIZATION OF A LEIBNIZ GEOMETRICAL THEOREM

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Abstract:

In this article we present a generalization of a Leibniz's theorem in geometry and an application of this.

Leibniz's theorem. Let M be an arbitrary point in the plane of the triangle ABC, then $MA^2 + MB^2 + MC^2 = \frac{1}{3}(a^2 + b^2 + c^2) + 3MG^2$, where G is the centroid of the triangle. We generalize this theorem:

Theorem. Let's consider $A_1, A_2, ..., A_n$ arbitrary points in space and G the centroid of this points system; then for an arbitrary point M of the space is valid the following equation:

$$\sum_{i=1}^{n} MA_i^2 = \frac{1}{n} \sum_{1 \le i < j \le n} A_i A_j^2 + n \cdot MG^2 .$$

Proof. First, we interpret the centroid of the n points system in a recurrent way.

If $n = 2$ then is the midpoint of the segment.

If $n = 3$, then it is the centroid of the triangle.

Suppose that we found the centroid of the $n-1$ points created system. Now we join each of the n points with the centroid of the $n-1$ points created system; and we obtain n bisectors of the sides. It is easy to show that these n medians are concurrent segments. In this manner we obtain the centroid of the n points created system. We'll denote G_i the centroid of the A_k, $k = 1, 2, ..., i-1, i+1, ..., n$ points created system. It can be shown that $(n-1)A_iG = GG_i$. Now by induction we prove the theorem.

If $n = 2$ the $MA_1^2 + MA_2^2 = \frac{1}{2}A_1A_2^2 + 2MG^2$

or

$$MG^2 = \frac{1}{4}\left(2\left(MA_1^2 + MA_2^2\right)\right),$$

where G is the midpoint of the segment A_1A_2. The above formula is the side bisector's formula in the triangle MA_1A_2. The proof can be done by Stewart's theorem, cosines

theorem, generalized theorem of Pythagoras, or can be done vectorial. Suppose that the assertion of the theorem is true for $n = k$. If $A_1, A_2, ..., A_k$ are arbitrary points in space, G_0 is the centroid of this points system, then we have the following relation:

$$\sum_{i=1}^{k} MA_i^2 = \frac{1}{k} \sum_{1 \le i < j \le k} A_i A_j^2 + k \cdot MG_0^k .$$

Now we prove for $n = k+1$.

Let $A_{k+1} \notin \{A_1, A_2, ..., A_k, G_0\}$ be an arbitrary point in the space and let G be the centroid of the $A_1, A_2, ..., A_k, A_{k+1}$ points system. Taking into account that G is on the segment $A_{k+1}G_0$ and $k \cdot A_{k+1}G = GG_0$, we apply Stewart's theorem to the points M, G_0, G, A_{k+1}, from where:

$$MA_{k+1}^2 \cdot GG_0 + MG_0^2 \cdot GA_{k+1} - MG^2 \cdot A_{k+1}G_0 = GG_0 \cdot GA_{k+1} \cdot A_{k+1}G_0 .$$

According to the previous observation $A_{k+1}G = \frac{k}{k+1} A_{k+1}G_0$

and $GG_0 = \frac{k}{k+1} A_{k+1}G_0$.

Using these, the above relation becomes:

$$MA_{k+1}^2 + k \cdot MG_0^2 = \frac{k}{k+1} A_{k+1}G_0^2 + (k+1)MG^2 .$$

From here

$$k \cdot MG_0^2 = \sum_{i=1}^{k} MA_i^2 - \frac{1}{k} \sum_{1 \le i < j \le k} A_i A_j^2 .$$

From the supposition of the induction, with $M \equiv A_{k+1}$ as substitution, we obtain

$$\sum_{i=1}^{k} A_i A_j^2 = \frac{1}{k} \sum_{1 \le i < j \le k} A_i A_j^2 + k \cdot A_{k+1}G_0^2$$

and thus

$$\frac{k}{k+1} A_{k+1}G_0^2 = \frac{1}{k+1} \sum_{i=1}^{k} A_i A_{k+1}^2 - \frac{1}{k(k+1)} \sum_{1 \le i < j \le k} A_i A_j^2 .$$

Substituting this in the above relation we obtain that

$$\sum_{i=1}^{k+1} MA_i^2 = \left(\frac{1}{k} - \frac{1}{k(k+1)} \right) \sum_{1 \le i < j \le k} A_i A_j^2 + \frac{1}{k+1} \sum_{i=1}^{k} A_i A_{k+1}^2 + (k+1)MG^2 =$$

$$= \frac{1}{k+1} \sum_{1 \le i < j \le k+1} A_i A_j^2 + (k+1) MG^2 .$$

With this we proved that our assertion is true for $n = k+1$. According to the induction, it is true for every $n \ge 2$ natural numbers.

Application 1. If the points $A_1, A_2, ..., A_n$ are on the sphere with the center O and radius R, then using in the theorem the substitution $M \equiv O$ we obtain the identity:

$$OG^2 = R^2 - \frac{1}{n^2} \sum_{1 \le i < j \le n} A_i A_j^2 .$$

In case of a triangle: $OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right)$.

In case of a tetrahedron: $OG^2 = R^2 - \frac{1}{16}\left(a^2 + b^2 + c^2 + d^2 + e^2 + f^2\right)$.

Application 2. If the points $A_1, A_2, ..., A_n$ are on the sphere with the center O and radius R, then $\sum_{1 \le i < j \le n} A_i A_j^2 \le n^2 R^2$.

The equality holds if and only if $G \equiv O$. In case of a triangle: $a^2 + b^2 + c^2 \le 9R^2$, in case of a tetrahedron: $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 \le 16R^2$.

Application 3. Using the arithmetic and harmonic mean inequality, from the previous application, it results the following inequality:

$$\sum_{1 \le i < j \le n} \frac{1}{A_i A_j^2} \ge \frac{(n-1)^2}{4R^2}.$$

In the case of a triangle: $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge \frac{1}{R^2}$, in case of a tetrahedron:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} + \frac{1}{f^2} \ge \frac{9}{4R^2}.$$

Application 4. Considering the Cauchy-Buniakowski-Schwarz inequality from the Application 2, we obtain the following inequality:

$$\sum_{1 \le i < j \le n} A_i A_j^2 \le nR\sqrt{\frac{n(n-1)}{2}}.$$

In case of a triangle: $a + b + c \le 3\sqrt{3R}$, in case of a tetrahedron:

$$a + b + c + d + e + f \le 4\sqrt{6R}.$$

Application 5. Using the arithmetic and harmonic mean inequality, from the previous application we obtain the following inequality

$$\sum_{1 \le i < j \le n} \frac{1}{A_i A_j^2} \ge \frac{(n-1)\sqrt{n(n-1)}}{2R\sqrt{2}}.$$

In case of a triangle: $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{\sqrt{3}}{R}$, in case of a tetrahedron:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f} \ge \frac{3}{R}\sqrt{\frac{3}{2}}.$$

Application 6. Considering application 3, we obtain the following inequality:

$$\frac{n^2(n-1)^2}{4} \le \left(\sum_{1 \le i < j \le n} A_i A_j^k\right)\left(\sum_{1 \le i < j \le n} \frac{1}{A_i A_j^k}\right) \le$$

$$\leq \begin{cases} \dfrac{(M+m)^2 n^2 (n-1)^2}{16M \cdot m} & \text{if } \dfrac{n(n-1)}{2} \text{ is even,} \\ \dfrac{(M+m)^2 n^2 (n-1)^2 - 4(M-m)^2}{16M \cdot m} & \text{if } \dfrac{n(n-1)}{2} \text{ is odd} \end{cases}$$

where $m = \min \left\{ A_i A_j^k \right\}$ and $M = \max \left\{ A_i A_j^k \right\}$. In case of a triangle:

$$9 \leq \left(a^k + b^k + c^k\right)\left(a^{-k} + b^{-k} + c^{-k}\right) \leq \frac{2M^2 + 5M \cdot m + 2m^2}{M \cdot m},$$

in case of a tetrahedron:

$$36 \leq \left(a^k + b^k + c^k + d^k + e^k + f^k\right)\left(a^{-k} + b^{-k} + c^{-k} + d^{-k} + e^{-k} + f^{-k}\right) \leq \frac{9\left(M+m\right)^2}{M \cdot m}.$$

Application 7. Let $A_1, A_2, ..., A_n$ be the vertexes of the polygon inscribed in the sphere with the center O and radius R. First we interpret the orthocenter of the inscribable polygon $A_1 A_2 ... A_n$. For three arbitrary vertexes, corresponds one orthocenter. Now we take four vertexes. In the obtained four orthocenters of the triangles we construct the circles with radius R, which have one common point. This will be the orthocenter of the inscribable quadrilateral. We continue in the same way. The circles with radius R that we construct in the orthocenters of the $n-1$ sides inscribable polygons have one common point. This will be the orthocenter of the n sides, inscribable polygon. It can be shown that O, H, G are collinear and $n \cdot OG = OH$. From the first application

$$OH^2 = n^2 R^2 - \sum_{1 \leq i < j \leq n} A_i A_j^2$$

and

$$GH^2 = \left(n-1\right)^2 R^2 - \left(1 - \frac{1}{n}\right)^2 \sum_{1 \leq i < j \leq n} A_i A_j^2.$$

In case of a triangle $OH^2 = 9R^2 - \left(a^2 + b^2 + c^2\right)$ and $GH^2 = 4R^2 - \frac{4}{9}\left(a^2 + b^2 + c^2\right)$.

Application 8. In the case of an $A_1 A_2 ... A_n$ inscribable polygon $\sum_{1 \leq i < j \leq n} A_i A_j^2 = n^2 R^2$ if and only if $O \equiv H \equiv G$. In case of a triangle this is equivalent with an equilateral triangle.

Application 9. Now we compute the length of the midpoints created by the $A_1, A_2, ..., A_n$ space points system. Let $S = \{1, 2, ..., i-1, i+1, ..., n\}$ and G_0 be the centroid of the A_k, $k \in S$, points system. By substituting $M \equiv A_i$ in the theorem, for the length of the midpoints we obtain the following relation:

$$A_i G_0^2 = \frac{1}{n-1} \sum_{k \in S} A_i A_k^2 - \frac{1}{\left(n-1\right)^2} \sum_{u,v \in S : u \neq v} A_u A_v^2.$$

Application 10. In case of a triangle $m_a^2 = \frac{2\left(b^2 + c^2\right) - a^2}{4}$ and its permutations. From here:

$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}\left(a^2 + b^2 + c^2\right),$$

$$m_a^2 + m_b^2 + m_c^2 \le \frac{27}{4} R^2,$$

$$m_a + m_b + m_c \le \frac{9}{2} R.$$

Application 11. In case of a tetrahedron $m_a^2 = \frac{1}{9}\left(3\left(a^2 + b^2 + c^2\right) - \left(d^2 + e^2 + f^2\right)\right)$ and its permutations.

From here:

$$\sum m_a^2 = \frac{4}{9}\left(\sum a^2\right),$$

$$\sum m_a^2 \le \frac{64}{9} R^2,$$

$$\sum m_a \le \frac{16}{3} R.$$

Application 12. Denote $m_{a,f}$ the length of the segments, which join midpoint of the a and f skew sides of the tetrahedron (bimedian). In the interpretation of the application $9m_{a,f}^2 = \frac{1}{4}\left(b^2 + c^2 + d^2 + e^2 - a^2 - f^2\right)$ and its permutations.

From here

$$m_{a,f}^2 + m_{b,d}^2 + m_{c,e}^2 = \frac{1}{4}\left(\sum a^2\right),$$

$$m_{a,f}^2 + m_{b,d}^2 + m_{c,e}^2 \le 4R^2,$$

$$m_{a,f} + m_{b,d} + m_{c,e} \le 2R\sqrt{3}.$$

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GENERALIZATION OF THE THEOREM OF MENELAUS USING A SELF-RECURRENT METHOD

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Abstract.
This generalization of the Theorem of Menelaus from a triangle to a polygon with *n* sides is proven by a self-recurrent method which uses the induction procedure and the Theorem of Menelaus itself.

The **Theorem of Menelaus for a Triangle** is the following:

If a line *(d)* intersects the triangle $\Delta A_1A_2A_3$ sides A_1A_2, A_2A_3, and A_3A_1 respectively in the points M_1, M_2, M_3, then we have the following equality:

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{M_2A_2}{M_2A_3}\cdot\frac{M_3A_3}{M_3A_1}=1$$

where by M_1A_1 we understand the (positive) length of the segment of line or the distance between M_1 and A_1; similarly for all other segments of lines.

Let's generalize the Theorem of Menelaus for any *n-gon* (a polygon with *n* sides), where $n \geq 3$, using our Recurrence Method for Generalizations, which consists in doing an induction and in using the Theorem of Menelaus itself.

For $n = 3$ the theorem is true, already proven by Menelaus.

The **Theorem of Menelaus for a Quadrilateral**.

Let's prove it for $n = 4$, which will inspire us to do the proof for any *n*.

Suppose a line *(d)* intersects the quadrilateral $A_1A_2A_3A_4$ sides A_1A_2, A_2A_3, A_3A_4, and A_4A_1 respectively in the points M_1, M_2, M_3, *and* M_4, while its diagonal A_2A_4 into the point M [see *Fig. 1* below].

We split the quadrilateral $A_1A_2A_3A_4$ into two disjoint triangles (*3-gons*) $\Delta A_1A_2A_4$ and $\Delta A_4A_2A_3$, and we apply the Theorem of Menelaus in each of them, respectively getting the following two equalities:

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{MA_2}{MA_4}\cdot\frac{M_4A_4}{M_4A_1}=1$$

and

$$\frac{MA_4}{MA_2}\cdot\frac{M_2A_2}{M_2A_3}\cdot\frac{M_3A_3}{M_3A_4}=1.$$

Now, we multiply these last two relationships and we obtain the Theorem of Menelaus for $n = 4$ (a quadrilateral):

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{M_2A_2}{M_2A3}\cdot\frac{M_3A_3}{M_3A_4}\cdot\frac{M_4A_4}{M_4A_1}=1.$$

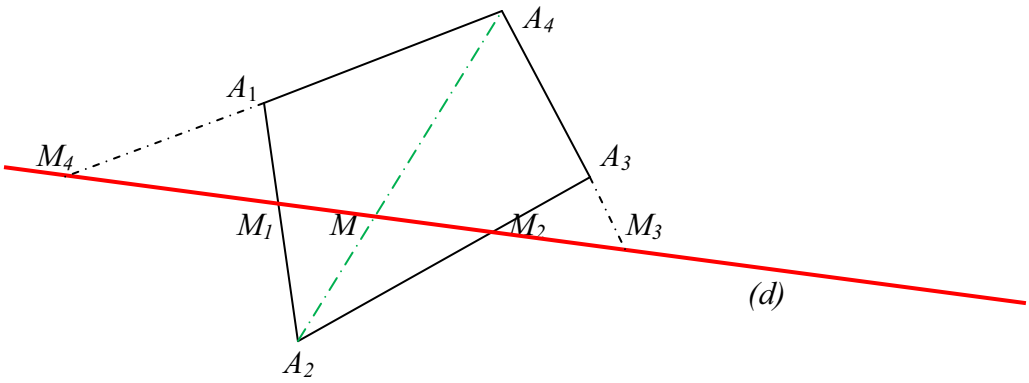

Fig. 1

Let's suppose by induction upon $k \geq 3$ that the Theorem of Menelaus is true for any *k-gon* with $3 \leq k \leq n-1$, and we need to prove it is also true for $k = n$.

Suppose a line *(d)* intersects the n-gon $A_1A_2...A_n$ sides A_iA_{i+1} in the points M_i, while its diagonal A_2A_n into the point M {of course by A_nA_{n+1} one understands A_nA_1} – see *Fig. 2*.

We consider the *n-gon* $A_1A_2...A_{n-1}A_n$ and we split it similarly as in the case of quadrilaterals in a *3-gon* $\Delta A_1A_2A_n$ and an *(n-1)-gon* $A_nA_2A_3...A_{n-1}$ and we can respectively apply the Theorem of Menelaus according to our previously hypothesis of induction in each of them, and we respectively get:

$$\frac{M_1A_1}{M_1A_2}\cdot\frac{MA_2}{MAn}\cdot\frac{MnAn}{MnA_1}=1$$

and

$$\frac{MAn}{MA_2}\cdot\frac{M_2A_2}{M_2A_3}\cdot\ldots\cdot\frac{M_{n-2}A_{n-2}}{M_{n-2}A_{n-1}}\cdot\frac{M_{n-1}A_{n-1}}{M_{n-1}A_n}=1$$

whence, by multiplying the last two equalities, we get

the **Theorem of Menelaus for any *n-gon***:

$$\prod_{i=1}^{n}\frac{M_iA_i}{M_iA_{i+1}}=1\,.$$

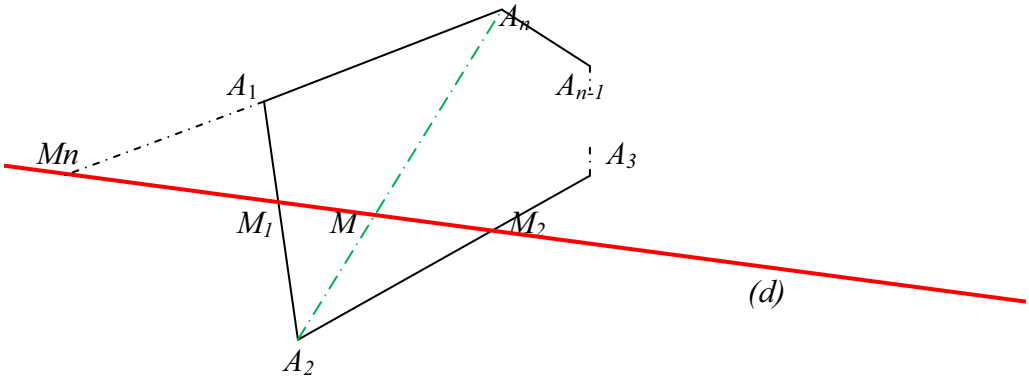

Fig. 2

Conclusion.

We hope the reader will find useful this self-recurrence method in order to generalize known scientific results by means of themselves!

{Translated from French by the Author.}

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THE DUAL THEOREM RELATIVE TO THE SIMSON'S LINE

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Abstract

In this article we elementarily prove some theorems on the poles and polars theory, we present the transformation using duality and we apply this transformation to obtain the dual theorem relative to the Samson's line.

I. POLE AND POLAR IN RAPPORT TO A CIRCLE

Definition 1. Considering the circle $C(O,R)$, the point P in its plane $P \neq O$ and the point P' such that $\overrightarrow{OP} \cdot \overrightarrow{OP'} = R^2$. It is said about the perpendicular p constructed in the point P' on the line OP that it is the point's P **polar**, and about the point P that it is the line's p **pole**.

Observations

1. If the point P belongs to the circle, its polar is the tangent in P to the circle $C(O,R)$.

Indeed, the relation $\overrightarrow{OP} \cdot \overrightarrow{OP'} = R^2$ gives that $P' = P$.

2. If P is interior to the circle, its polar p is a line exterior to the circle.

3. If P and Q are two points such that $\mathrm{m}(POQ) = 90^\circ$, and p, q are their polars, from the definition results that $p \perp q$.

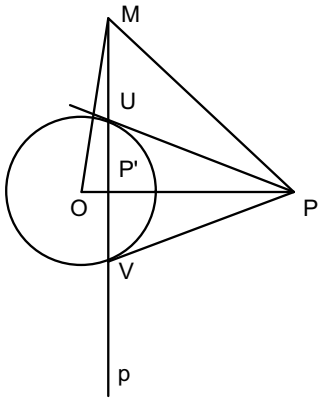

Fig. 1

Proposition 1.

If the point P is external to the circle $C(O,R)$, its polar is determined by the contact points with the circle of the tangents constructed from P to the circle

Proof

Let U and V be the contact points of the tangents constructed from P to the circle $C(O,R)$ (see fig.1). In the right triangle OUP, if $P"$ is the orthogonal projection of U on OP, we have $OU^2 = OP" \cdot OP$ (the cathetus theorem), but also $OP' \cdot OP = R^2$, it results that $P" = P'$ and therefore U belongs to the polar of P. Similarly V belongs to the polar, therefore UV is the polar of P.

Observation

From the Proposition 1 it results the construction's method of the polar of an exterior point to a circle.

Theorem 1. (The Polar Characterization)

The point M belongs to the polar of the point P in rapport to the circle $C(O,R)$ if and only if

$$MO^2 - MP^2 = 2R^2 - OP^2.$$

Proof

If M is an arbitrary point on the polar of the point P in rapport to the circle $C(O,R)$, then

$$MP' \perp OP$$

(see fig. 1) and

$$MO^2 - MP^2 = \left(P'O^2 + P'M^2\right) - \left(P'P^2 + P'M^2\right) = P'O^2 -$$

$$-P'P^2 = OU^2 - P'U^2 + P'U^2 - PU^2 = R^2 - \left(OP^2 - R^2\right) = 2R^2 - OP^2.$$

Reciprocally, if M is in the circle's plane such that

$$MO^2 - MP^2 = 2R^2 - OP^2.$$

We denote with M' the projection of M on OP, and we have

$$M'O^2 - M'P^2 = \left(MO^2 - M'M^2\right) - \left(MP^2 - \ -M'M^2\right) = MO^2 - MP^2 = 2R^2 - OP^2.$$

On the other side

$$P'O^2 - P'P^2 = 2R^2 - OP^2.$$

From

$$M'O^2 - M'P^2 = P'O^2 - P'P^2$$

it results that

$$M' = P',$$

therefore M belongs to the polar of the point P.

Theorem 2. (Philippe de la Hire)

If P,Q,R are points that don't belong to the circle $C(O,R)$ and p,q,r are their polars in rapport to the circle, then

1° $P \in q \Leftrightarrow Q \in p$ (If a point belongs to the polar of another point, then also a second point belongs to the polar of the first point in rapport to a circle)

2° $r = PQ \Leftrightarrow R \in p \bigcap q$ (The pole of a line that passes through two points is the intersection of the polars of the two points).

Proof:

1° From the theorem 1 we have

$$P \in q \Leftrightarrow PO^2 - PQ^2 = 2R^2 - OQ^2 .$$

Then

$$QO^2 - OP^2 = 2R^2 - OP^2 \Leftrightarrow Q \in p .$$

2° Let's consider $R \in p \bigcap q$; from 1° results $P \in r$ and $Q \in r$ therefore

$$r = PQ .$$

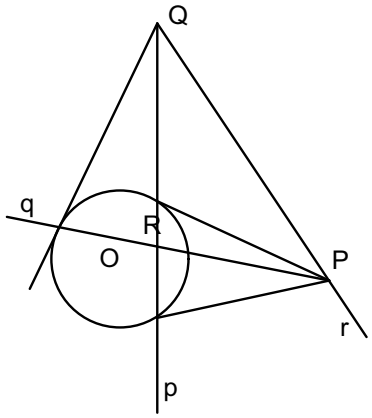

Fig. 2

Observations

1. From the theorem 2 we retain:
The polar of a point which is the intersection of two given lines is the line determined by the poles of those lines.
2. The poles of some concurrent lines are collinear and reciprocally, the polars of some collinear points are concurrent.

II. THE TRANSFORMATION THROUGH DUALITY

The duality in rapport with a circle $C(O,R)$ is the geometric transformation that associates to any point $P \neq O$ its polar, and which associates to a line from plane its pole.

Through duality, practically, the role of the lines and of the points are permutated, such that to a figure F formed of points and lines, through duality it corresponds to it a new figure F' formed from lines (the figure's F polars) and of points (the poles of the figure's F lines) in rapport to a given circle.

The duality was introduced by the French mathematician Victor Poncelet in 1822.

When the figure F is formed of points, lines and eventually a circle, transforming it through duality in rapport with the circle we will still be in the elementary geometry

domain and we obtain a new figure F', its elements and properties being duals to those of the figure F.

From the proved theorems we retain:
- If a point is located on a line, through its duality it will correspond its polar that passes through the line's pole in rapport to the circle.

- To the line determined by two points it corresponds, through duality in rapport with a circle, the intersection point of the polars of those two points.

- To the intersection's point of two lines it corresponds, through duality in rapport with a circle, the line determined by the poles of these lines.

III. THE DUAL THEOREM RELATIVE TO THE SIMSON'S LINE

Theorem 3. (The Simson's Line)

If $A'B'C'$ is a triangle inscribed in the circle with the center in O and A_1', B_1', C_1' are the orthogonal projections of a point M from the circle respectively on $B'C', C'A'$ and $A'B'$, then the points A_1', B_1', C_1' are collinear.

We leave to the reader's attention the proof of this known theorem.

We transform through duality in rapport to the circumscribed circle to the triangle $A'B'C'$ the configuration of this theorem. To the points A', B', C' correspond through duality their polars a, b, c, which are the tangents in A', B', C' to the circle (see fig. 3), and to the point M corresponds its polar m, the tangent in M to circle.

To the line $A'B'$ it is associated through duality its pole $\{C\} = a \cap b$, similarly to the line $A'C'$ corresponds the point $\{B\} = a \cap c$ and to the line $B'C'$ corresponds $\{A\} = b \cap c$.

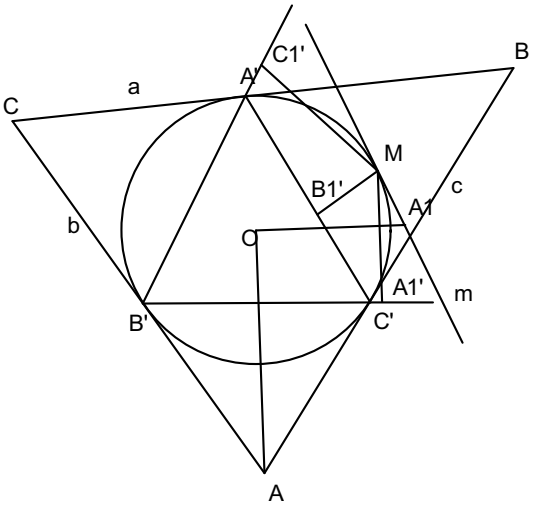

Fig. 3

Because $MA_1' \perp B'C'$ it results that their poles are situated on perpendicular lines that pass through O, therefore, if we denote with A_1 the line's pole MA_1' we will find A_1 as the intersection of the perpendicular constructed in O on AO with the tangent m. Similarly we obtain the points B_1 and C_1.

Through the considered duality, to the point A_1' corresponds its polar, which is AA_1 (because the pole of MA_1' is A_1 and the pole of $B'C'$ is A).

Similarly to the point $B_1^{'}$ corresponds BB_1 and to the point $C_1^{'}$ through duality corresponds CC_1. Because the points $A_1^{'}, B_1^{'}, C_1^{'}$ are collinear (the Simson's line) it results that their polars AA_1, BB_1, CC_1 are concurrent in a point S (the Simson's line's pole).

We observe that the circumscribed circle to the triangle $A'B'C'$ becomes inscribed circle in the triangle ABC, and therefore we can formulate the following:

Theorem 4 (The Dual Theorem of the Simson's Line).

If ABC is any triangle and A_1, B_1, C_1 are, respectively, the intersections of the perpendiculars constructed in the center I of the inscribed circle in triangle on AI, BI, CI with a tangent constructed to the inscribed circle, then the lines AA_1, BB_1, CC_1 are concurrent.

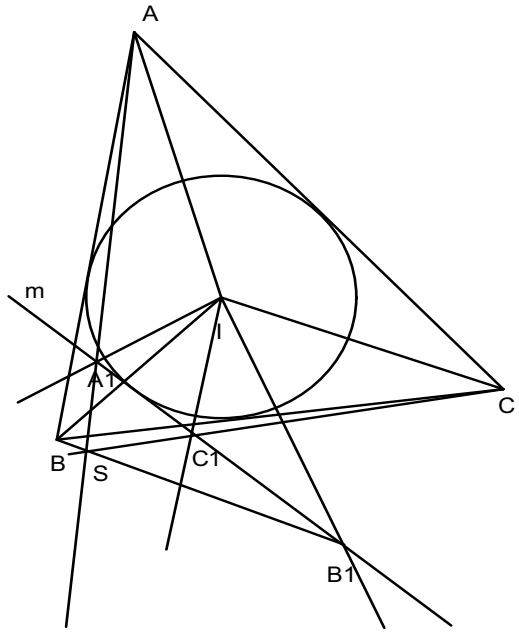

Fig. 4

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DE LONGCHAMPS' POINT, LINE AND CIRCLE

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The purpose of this article is to familiarize the reader with these notions, emphasizing on connections between them.

Lemma

The circles drawn on the sides of an obtuse triangle ABC, as diameters, and on the medians of this triangle, as diameters, have the same radical circle and that it is the circle with the center in the orthocenter of the triangle ABC and the ray $\rho = 2R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$ (this circle is called the polar circle of the triangle ABC).

Proof

Let's consider A′, B′, C′ the altitudes' base points of the obtuse triangle ABC and A_1, B_1, C_1 its sides centers (see fig. 1).

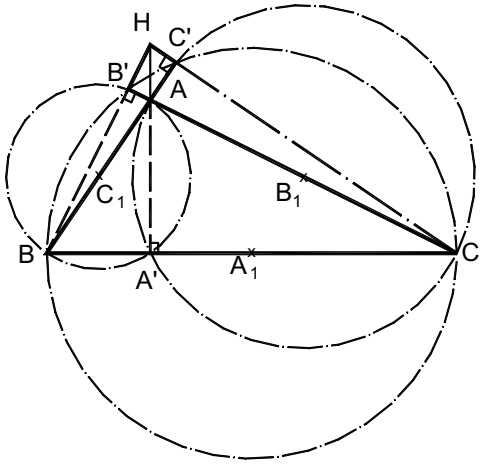

Fig. 1

The circle drawn on [AB] as diameter passes through A′ and B′, the circle drawn on [BC] as diameter passes through B′ and C′, and the circle drawn on [AC] as diameter passes through A′ and C′. We notice that these circles have as common chords (radical axes) the altitudes of the triangle and because these are concurrent in H, it results that the orthocenter is the radical center of the considered circles.

On the other side, the circle which has the median $[AA_1]$ as diameter passes through A′. We have:

$$HA \cdot HA' = HB \cdot HB' = HC \cdot HC',$$

which shows that the power of H in respect to the constructed circle on $[AA_1]$ as diameter is equal to the power of H in respect to the constructed circles on the sides of the triangle ABC as diameters. Therefore, H is the radical center of the constructed circles on medians as diameters.

From the relation: $HA \cdot HA' = \rho^2$ we'll determine the ray ρ of the polar circle.

It is known that $AH = -2R\cos A$; from the triangle BHA' we obtain:

$$HA' = BH \cdot \cos\left(\hat{BHA'}\right) = BH \cdot \cos C .$$

Taking into account that $BH = 2R\cos B$,
we obtain:

$$\rho = 2R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$$

Definition 1.

If ABC is an obtuse triangle we say that the **De Longchamps' circle** of the triangle ABC is the circle that is orthogonal to the circles that have their centers in the triangle's vertexes and as rays the opposed sides of these vertexes (Casey – 1886).

Theorem 1.

The De Longchamps' circle of the obtuse triangle ABC has the ray given by the formula:

$$R_L = 4R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$$

Proof

The circle $\mathcal{C}(A; BC)$ intersects the circle $\mathcal{C}(B; AC)$ in the points P and M and the circle $\mathcal{C}(A; BC)$ intersects the circle $\mathcal{C}(C; AB)$ in the points N and M.

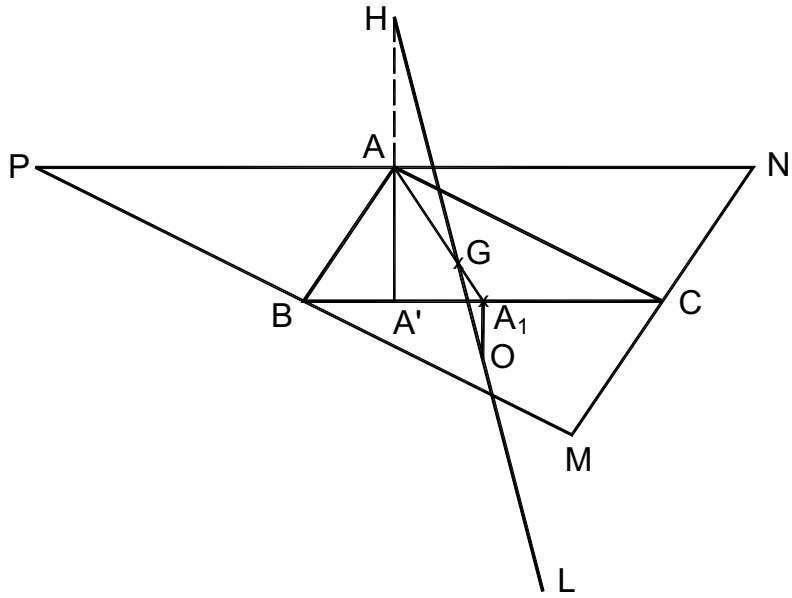

Fig. 2

The M, N and P are the vertex of the triangle **anti-complementary** of the triangle ABC (the triangle with the sides parallel to the sides of the given triangle, drawn through the vertexes of the given triangle).

We observe that the quadrilaterals ACBP, ACMB and ABCN are parallelograms and that the circles from the theorem's enunciation are the circles drawn on the sides of the anti-complementary triangle as diameters.

Applying the lemma it results that it exists an orthogonal circle to these circles, which this has as center the orthocenter L of the triangle MNP. Because the triangle MNP is the anti-complementary triangle of the triangle ABC and is similar to it, the similarity rapport being equal to 2, it results that the ray of the De Longchamps' circle will be the double of the polar circle's ray of the triangle ABC, therefore, $R_L = 2\rho$, thus:

$$R_L = 4R\sqrt{-\cos A \cdot \cos B \cdot \cos C}$$

Definition 2.

We call **power circles** of a triangle ABC the three circles with the centers in the middle points A_1, B_1, C_1 of the triangle's sides and which pass, respectively, through the opposite vertexes A, B and C.

Theorem 2.

If ABC is an obtuse triangle, the De Longchamps's circle is the radical circle of the power circles of the triangle ABC.

Proof

Let MNP the anti-complementary triangle of the triangle ABC, the power circle with the center A_1 and the ray A_1A passes through M (see fig. 3), similarly, this circle intersects for the second time NP in the altitude's base point from M of the anti-complementary triangle.

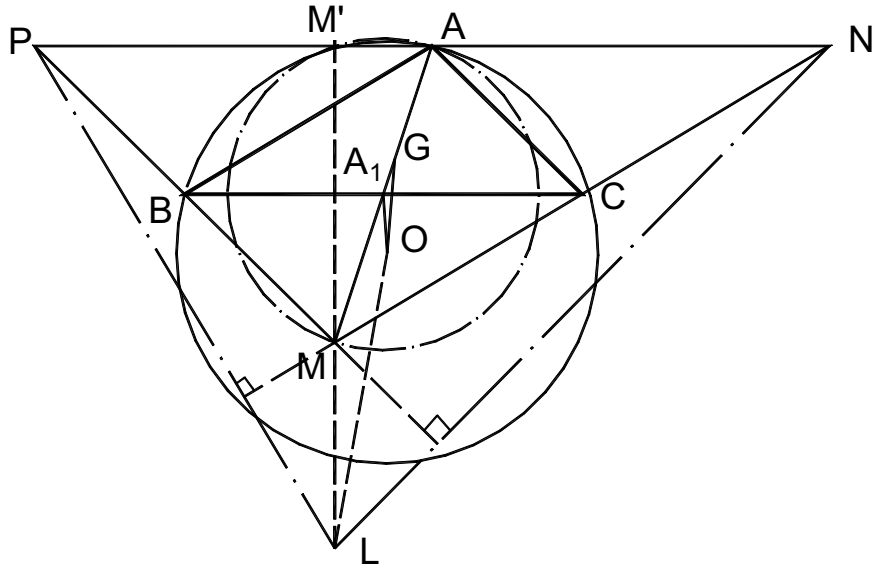

Fig. 3

The circle constructed on [MP] as diameter intersects with the above mentioned power circle on the altitude MM′ and also trough the points M and M′ passes the circle constructed on [MN] as diameter. These circles have as ortho-central radical center L of the triangle MNP. Repeating this reasoning we obtain that L is the radical center of the power circles of the triangle ABC.

Observation 1.

The De Longchamps's circle of a triangle is defined only if the triangle is obtuse.

Definition 3.

The **De Longchamps' point** of a triangle is the radical center of the power circles of the triangle.

Theorem 3.

The De Longchamps' point L, of the triangle ABC, is the symmetric of the orthocenter H of the triangle in rapport to the center O of the circumscribed circle of the triangle.

Proof

The anti-complementary triangle MNP of the triangle ABC and the triangle ABC are homothetic through the homotopy of the pole G and of rapport 2; the same are the De Longchamps's circles and the polar circle of the triangle ABC, it follows that the points L, G, H

are collinear and $LG = 2GH$. On the other side the points O, G, H are collinear (the Euler's line) and $GH = 2GO$.

We have:

$$LG = LO + OG\ ;\ OG = \frac{1}{2}GH = \frac{1}{3}OH.$$

$$LO = LG - OG = 2GH - \frac{1}{2}GH = \frac{3}{2}GH.$$

We obtain:

$$LO = OH.$$

Definition 4.

The **De Longchamps'** line is defined as the radical axes of the De Longchamps' circle and of the circumscribed circle of a triangle.

Theorem 4.

The De Longchamps's line of a triangle is the radical axes of the circumscribed circle to the triangle and of the circle circumscribed to the anti-complementary triangle of the given triangle.

Proof

The center of the circle circumscribed to the anti-complementary triangle of the triangle ABC is the orthocenter H of the triangle ABC and its ray is 2R. We'll denote with Q the intersection between the De Longchamps' line and the Euler's line (see fig. 4).

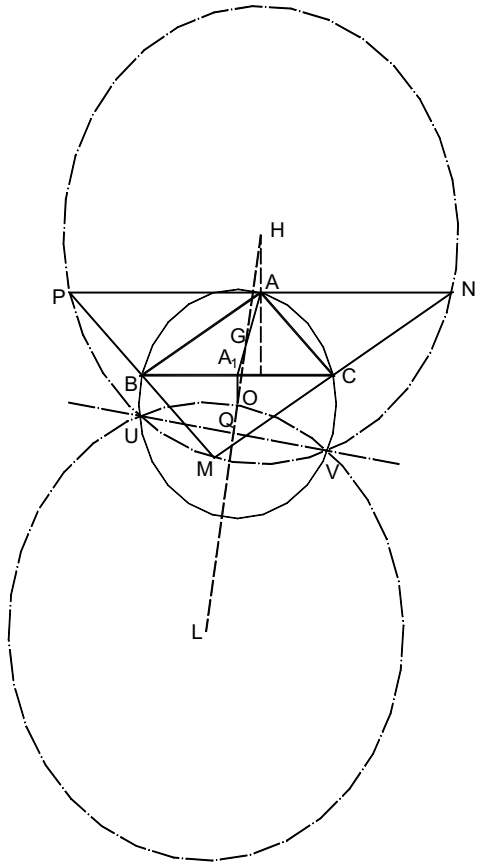

Fig. 4

We have:

$$R^2 - OQ^2 = R_L^2 - LQ^2$$

$$OQ = LO - LQ = HO - LQ$$
$$R^2 - (HO - LQ)^2 = 4\rho^2 - LQ^2,$$

it results:

$$LQ = \frac{4\rho^2 - R^2 + OH^2}{2HO}.$$

Because

$$HO^2 = R^2 \cdot (1 - 8\cos A \cdot \cos B \cdot \cos C) = R^2 + 2\rho^2,$$

we obtain:

$$LQ = \frac{3\rho^2}{HO} \text{ şi } R^2 - OQ^2 = 4\rho^2 - LQ^2 = \rho^2\left(4 - \frac{9\rho^2}{HO^2}\right)$$

$$4R^2 - HQ^2 = 4R^2 - (HL - LQ)^2 = 4R^2 - 2(HO - LQ)^2 = \rho^2\left(4 - \frac{9\rho^2}{HO^2}\right).$$

Therefore

$$4R^2 - HQ^2 = R^2 - OQ^2,$$

thus the radical axes of the circumscribed circles to the anti-complementary MNP and ABC is the De Longchamps' line.

Theorem 5.

The De Longchamps' line of a triangle is the polar of the triangle's we sight center in rapport to the De Longchamps' circle.

Proof

We have $LG = \frac{4}{3}HO$ and $LQ \cdot LG = 4\rho^2$, then $LQ \cdot LG = R_L^2$. It results that GV (see fig. 4) is tangent to the De Longchamps' circle. Therefore, the line UV (the polar of G) is the De Longchamps' line.

Definition 5.

It is called **reciprocal transversal** of a transversal M, N, P in the triangle ABC the line M′, N′, P′ formed by the symmetric points of the points M, N, P in rapport to the centers of the sides BC, and AB.

Observation 2.

a) In figure 5 the sides M, N, P and M′, N′, P′ are reciprocal transversals.

b) The notion of reciprocal transversal was **introduced by G. De Longchamps** in 1866.

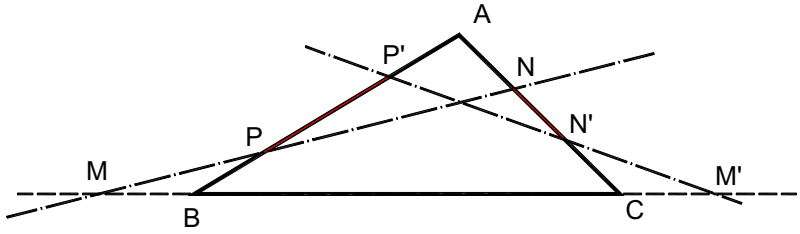

Fig. 5

Definition 6.

The Lemoine's line of a triangle ABC is the line that contains the intersections with the opposite sides of the triangle of the tangents constructed on the triangle's vertexes to its circumscribed circle.

Theorem 6.

The De Longchamps' line is the reciprocal transversal of the Lemoine's line.

Proof

Let S be the intersection of the tangent constructed from A to the circumscribed circle of the triangle ABC with the side BC (see fig. 6).

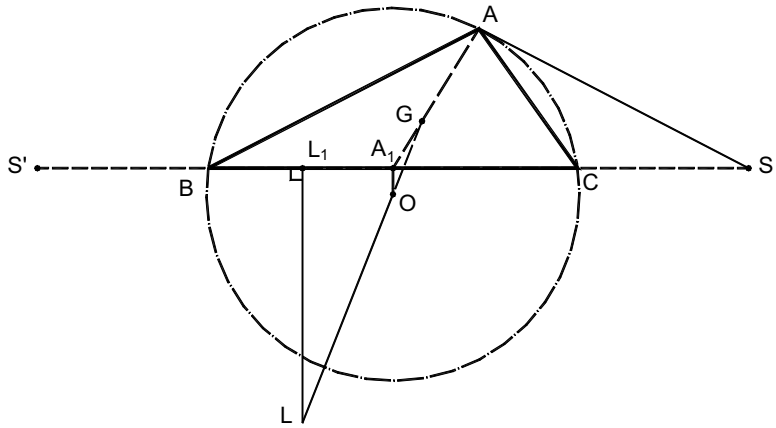

Fig. 6

The point S is, practically, the base of the external simediane from A. It is known that $\frac{SC}{SB}=\frac{b^2}{c^2}$, and we find $SC=\frac{ab^2}{c^2-b^2}$.

We will proof that the symmetrical point of S in rapport to the middle of BC, S′, belongs to the radical axes of the De Longchamps's circle and of the circumscribed circle (the De Longchamps' line).

We have that $S'B=\frac{a\,b^2}{c^2-b^2}$.

Let L_1 be the orthogonal projection of L on BC. We'll proof that

$$S'L^2-R_L^2=S'O^2-R^2,$$

L is the radical center, then

$$LB^2-b^2=LC^2-c^2=R_L^2.$$

We obtain that $LB^2-LC^2=b^2-c^2$

and also

$$L_1B^2-L_1C^2=b^2-c^2.$$

$S'L^2=LL_1^2+S'L_1^2$ şi $S'O^2=S'A_1^2+OA_1^2$ (A_1 the middle of (BC))

$$S'L^2-S'O^2=LL_1^2+(S'B+BL_1)^2-S'A_1^2-OA_1^2=LL_1^2+BL_1^2+S'B^2+2S'B\cdot BL_1-$$

$$-(S'B+BA_1)^2-OA_1^2$$

We find that

$$S'L^2-S'O^2=LL_1^2+BL_1^2+2S'B\cdot BL_1-2S'B\cdot BA_1-R^2=$$

$$=LB^2+2S'B(BL_1-BA_1)-R^2.$$

We substitute $S'B=\frac{a\,b^2}{c^2-b^2}$ and $L_1B=\frac{a^2+b^2-c^2}{2a}$ and we obtain

$$S'L^2-S'O^2=R_L^2-R^2.$$

Similarly it can be shown that the symmetric points in rapport to the middle points of the sides (AC) and (AB) of the base of the exterior simediane constructed from B and C, belong to the De Longchamps' line.

Application

Let ABC be an acute triangle and A_1 the middle of (BC). The circles $\mathcal{C}(A;BC)$ and $\mathcal{C}(A_1;AA_1)$ have a common chord A′A". Similarly, we define the line segments B′B" and C′C". Prove that the line segments A′A", B′B", C′C" are concurrent.

(Problem given at the test for training the 2008 team)

Solution

We denote $\mathcal{C}(A;BC) \cap \mathcal{C}(B;AC) = \{P, P'\}$

$\mathcal{C}(A;BC) \cap \mathcal{C}(C;AB) = \{N, N'\}$

$\mathcal{C}(B;AC) \cap \mathcal{C}(C;AB) = \{M, M'\}$

The triangle MNP will be the anti-complementary triangle of the triangle ABC while the triangle M′N′P′ will be the orthic triangle of the triangle MNP.

It results that $\mathcal{C}(A;BC)$, $\mathcal{C}(B;AC)$ and $\mathcal{C}(C_1;CC_1)$ have as the radical axis the altitude PP′ of the triangle MNP.

The circles $\mathcal{C}(A;BC)$, $\mathcal{C}(C;AB)$ and $\mathcal{C}(B_1;BB_1)$ have as radical axis the altitude NN′, while the circles $\mathcal{C}(B;AC)$, $\mathcal{C}(C;AB)$ and $\mathcal{C}(A_1;AA_1)$ have as radical axis the altitude MM′ of the triangle MNP.

Let $\{L\} = MM' \cap NN' \cap PP'$ be the orthocenter of the triangle MNP.

We have that L is the radical center of the circles $\mathcal{C}(A;BC)$, $\mathcal{C}(B;AC)$ and $\mathcal{C}(C;AB)$. L is the radical center of the circles $\mathcal{C}(A_1;AA_1)$, $\mathcal{C}(B_1;BB_1)$ and $\mathcal{C}(C_1;CC_1)$.

Also, L is the radical center of the circles $\mathcal{C}(A;BC)$, $\mathcal{C}(A_1;AA_1)$ and $\mathcal{C}(C;AB)$.

Indeed, the radical axis of the circles $\mathcal{C}(A;BC)$ and $\mathcal{C}(C;AB)$ is the altitude NN′, and the radical axis of the circles $\mathcal{C}(A_1;AA_1)$ and $\mathcal{C}(C;AB)$ is the altitude MM′.

It will result that the radical axis of the circles $\mathcal{C}(A;BC)$ and $\mathcal{C}(A_1;AA_1)$, that is the chord A′A" passes through L.

Similarly, it results that B′B" and C′C" pass through L. The concurrence point is L, the orthocenter of the anti-complementary triangle of the triangle ABC, therefore the De Longchamps' point of the triangle ABC.

Reference

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THE DUAL OF THE ORTHOPOLE THEOREM

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Abstract

In this article we prove the theorems of the orthopole and we obtain, through duality, its dual, and then some interesting specific examples of the dual of the theorem of the orthopole.

The transformation through duality was introduced in 1822 by the French mathematician Victor Poncelet. By the duality in rapport with a given circle to the points correspond lines (their polars), and to the straight lines correspond points (their poles).

Given a figure *F* formed of lines, points and, eventually, a circle, by applying to it the transformation through duality in rapport with the circle, we obtain a new figure F', which is formed of lines that are the polars of the figure's *F* points in rapport with the circle and from points that are the poles of the figure's *F* lines in rapport with the circle. Also, through duality to a given theorem corresponds a new theorem called its dual. After this introduction, we'll obtain the dual of the orthopole theorem.

The Orthopole Theorem (Soons – 1886).

If *ABC* is a triangle, *d* a line in its plane and A', B', C' the vertexes' projections of A, B, C on *d*, then the perpendiculars from A', B', C' on the sides BC, CA, AB are concurrent (the concurrence point is called the triangle's orthopole, in rapport to the line *d*).

In order to proof the orthopole's theorem will be using the following:

Theorem (L. Carnot - 1803)

The necessary and sufficient condition that the perpendiculars drawn on the sides BC, CA, AB of the triangle ΔABC, through the points A_1, B_1, C_1 that belong to these sides, to be concurrent is:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0 .$$

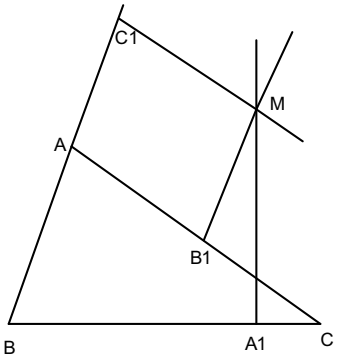

Fig. 1

Proof:

The condition is **necessary**: Let *M* be the concurrent point of the perpendiculars drawn in

A_1, B_1, C_1 on the sides of the triangle ΔABC (see Fig. 1).

We have

$$A_1B^2 - A_1C^2 = MB^2 - MA_1^2 - MC^2 + MA_1^2 = MB^2 - MC^2$$
$$B_1C^2 - B_1A^2 = MC^2 - MB_1^2 - MB_1^2 - MA^2 = MC^2 - MA^2$$
$$C_1A^2 - C_1B^2 = MA^2 - MC_1^2 + MC_1^2 - MB^2 = MA^2 - MB^2$$

Adding member by member these three relations it is obtained the relation from the above theorem.

The condition is **sufficient**: Let M be the intersection of the perpendiculars in A_1 on BC and in B_1 on AC, şi C_1' the projection of M on AB.

We have:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1'A^2 - C_1'B^2 = 0,$$

and from hypothesis:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0.$$

We obtain:

$$C_1'A^2 - C_1'B^2 = C_1A^2 - C_1B^2,$$

from which we find: $C_1' = C_1$, and therefore, the perpendiculars drawn in A_1, B_1, C_1 on the triangle's sides are concurrent.

The proof of the Orthopole Theorem

Let's note A_1, B_1, C_1 the projections of the points A_1', B_1', C_1' on BC, CA, AB (see Fig. 2).

We have:

$$A_1B^2 - A_1C^2 = A'B^2 - A'C^2 = BB'^2 + A'B'^2 - CC'^2 - A'C'^2 \quad (1).$$

Similarly, we obtain:

$$B_1C^2 - B_1A^2 = B'C^2 - B'A^2 = B'C'^2 + CC'^2 - A'B'^2 - AA'^2 \quad (2).$$

$$C_1A^2 - C_1B^2 = C'A^2 - C'B^2 = AA'^2 + A'C'^2 - B'C'^2 - BB'^2 \quad (3).$$

From the relations (1), (2) and (3), we obtain:

$$A_1B^2 - A_1C^2 + B_1C^2 - B_1A^2 + C_1A^2 - C_1B^2 = 0,$$

relation that in conformity to the Carnot's Theorem implies the concurrency of the lines $A'A_1, B'B_1, C'C_1$.

We denote with O the orthopole of the line d in rapport to the triangle ΔABC. We'll apply now a duality in rapport to the circle $C(O,r)$ to the corresponding configuration of the orthopole theorem. Then, to the points A, B, C will correspond their polars a, b, c. To the line AB corresponds its pole, which we'll note C' and it is $a \cap b$,

similarly, we'll obtain the poles B' and A' of the lines AC and BC. To the line *d* will correspond, through the considered duality, its pole, which we'll note with *P*.

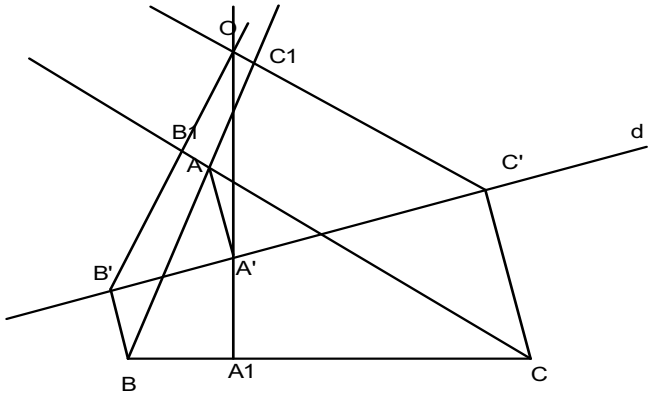

Fig. 2

If we denote with A_1', B_1', C_1' respectively, the intersections of line *P* with the sides of the triangle ΔABC, through the considered duality to these points correspond the lines $A'P, B'P$ and $C'P$ respectively. Because the lines AA' and *d* are perpendicular, their poles P_1 and P will be placed such that $\mathrm{m}(P_1OP) = 90^\circ$, therefore P_1 is the intersection of the perpendicular in O on OP with $B'C' = a$. Similarly, the pole of the perpendicular BB' on *d* will be P_2 the intersection with $b = A'C'$ of the perpendicular drawn in O on OP and at the perpendicular's intersection in O on OP with $c = A'B'$ we will find P_3 the pole of CC'.

To the perpendicular drawn in A' on BC corresponds, through duality, its pole A_1 which is located at the intersection of the perpendicular in O on $A'O$ with PP_1. Similarly we construct the points B_1, C_1 corresponding to the perpendiculars drawn from B' on AC and from C' on AB. Because these last perpendiculars are concurrent în the line's orthopole, their poles A_1, B_1, C_1 are collinear points (they belong to the orthopole's polar).

Selecting certain points, we can formulate the following:

The Dual Theorem of the Orthopole

If ABC is a triangle, O and P two distinct point in its plane such that the perpendicular in O on OP intersects BC, CA, AB respectively in the points P_1, P_2, P_3, and the perpendiculars drawn in the point O on OA, OB, OC intersect respectively the lines PP_1, PP_2, PP_3 in the points A_1, B_1, C_1, then the points A_1, B_1, C_1 are collinear.

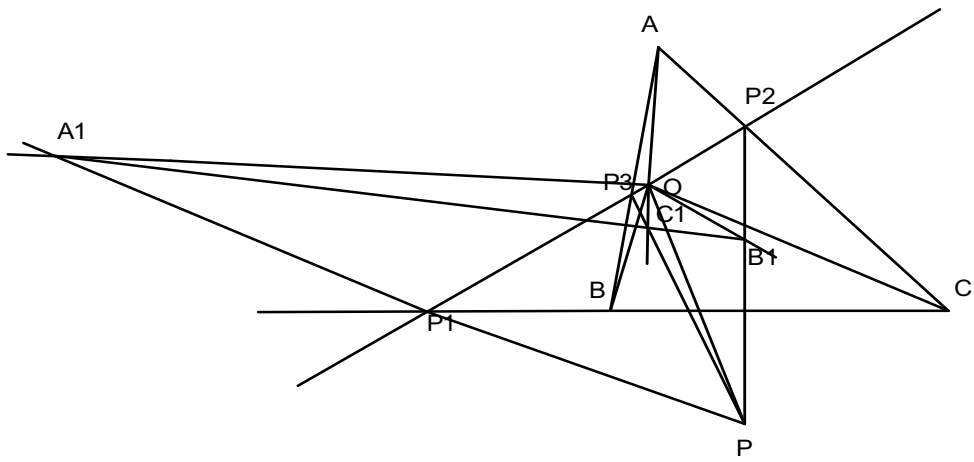

Fig. 3

Observation:

By inversing the solutions of O and P will find, following the same constructions indicated in the dual theorem of the orthopole, other collinear points A_1', B_1', C_1'.

Next, will point out several particular cases of the dual theorem of the orthopole.

1. Theorem of Bobillier

If ABC is a triangle and O is an arbitrary point in its plane, the perpendiculars drawn in O on AO, BO, CO intersect respectively BC, AC, AB into the collinear points A_1, B_1, C_1.

Proof

We apply the dual theorem of the orthopole in the particular case $P = A$: then the point P_1 coincides with A_1 because PP_1 becomes AP_1 (the point P_1 belongs to the line BC), similarly, the points B_1 and C_1 belong to AC respectively AB, it results that A_1, B_1, C_1 are collinear.

Remark

The Bobillier's Theorem was obtained transforming through duality in rapport with a circle O the theorem relative to a triangle's altitudes' concurrence.

2. Theorem

If ABC is a triangle and P a point on its circumscribed circle with the center O, the tangents in P to the circle intersect the sides BC, CA, AB respectively in P_1, P_2, P_3.

Will denote with $A^{'},B^{'},C^{'}$ the opposite diameters to A,B,C in the circle O and let's consider $\{A_1\}=A^{'}P\bigcap OP_1$, $\{B_1\}=B^{'}P\bigcap OP_2,\{C_1\}=C^{'}P\bigcap OP_3$, then the points A_1,B_1,C_1 are collinear.

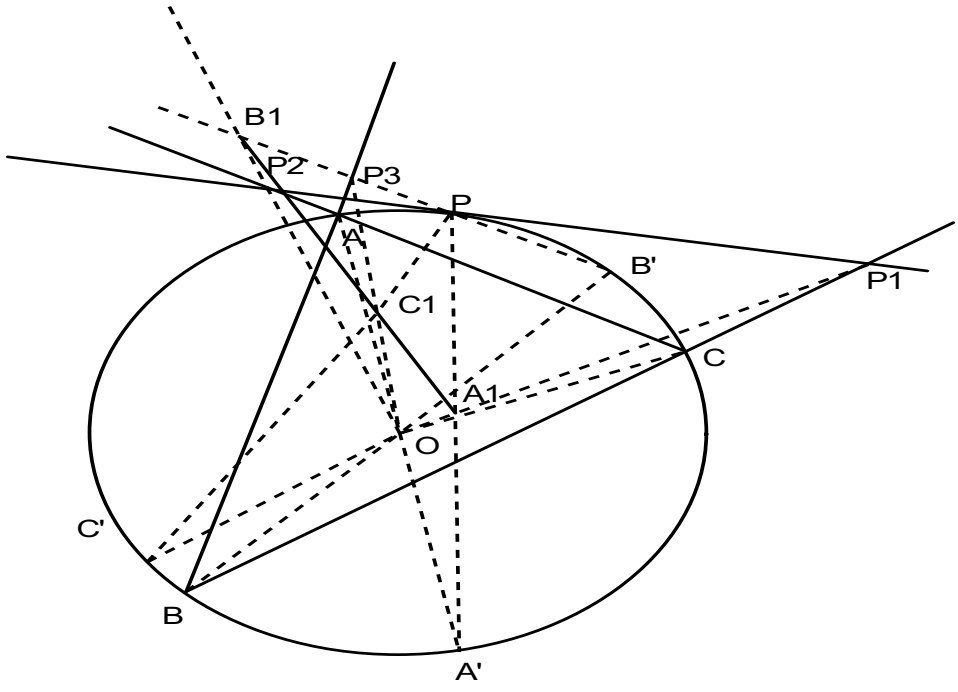

Fig. 4

Proof

The tangent in P to the circumscribed circle is perpendicular on the ray OP, therefore the points P_1,P_2,P_3 are constructed as in the hypothesis of the dual theorem of the orthopole. The point $A^{'}$ being diametric – opposite to A (see Fig. 4), we have $\mathrm{m}(APA')=90^{\circ}$, therefore A_1 is the intersection of the perpendicular in P on AP with OP_1, similarly there are constructed B_1 and C_1, and from the dual theorem of the orthopole it results their colinearity.

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SUPER-MATHEMATICS FUNCTIONS

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[translated from Romanian by Marian Nițu and Florentin Smarandache]

In this paper we talk about the so-called **Super-Mathematics Functions (SMF),** which often constitute the base for generating technical, neo-geometrical, therefore less artistic objects.

These functions are the results of 38 years of research, which began at University of Stuttgart in 1969. Since then, 42 related works have been published, written by over 19 authors, as shown in the References.

The name was given by the regretted mathematician Professor Emeritus Doctor Engineer **Gheorghe Silas** who, at the presentation of the very first work in this domain, during the First National Conference of Vibrations in Machine Constructions, Timişoara, Romania, 1978, named CIRCULAR EX-CENTRIC FUNCTIONS, declared: "Young man, you just discovered not only "some functions, but a new mathematics, a **supermathematics**!" I was glad, at my age of 40, like a teenager. And I proudly found that he might be right!

The prefix **super** is justified today, to point out the birth of the new complements in mathematics, joined together under the name of Ex-centric Mathematics (EM), with much more important and infinitely more numerous entities than the existing ones in the **actual mathematics,** which we are obliged to call it **Centric Mathematics (CM.)**

To each entity from CM corresponds an infinity of similar entities in EM, therefore the Supermathematics (SM) is the reunion of the two domains: **SM = CM ∪ EM,** where **CM** is a particular case of null ex-centricity of **EM**. Namely, **CM = SM(e = 0).** To each known function in **CM** corresponds an infinite family of functions in **EM**, and in addition, a series of new functions appear, with a wide range of applications in mathematics and technology.

In this way, to **x = cos α** corresponds the family of functions **x = cex θ = cex (θ, s, ε)** where **s = e/R** and **ε** are the polar coordinates of the **ex-center S(s,ε),** which corresponds to the unity/trigonometric circle or **E(e, ε),** which corresponds to a certain circle of radius R, considered as **pole** of a straight line **d,** which rotates around **E** or **S** with the position angle **θ,** generating in this way the ex-centric trigonometric functions, or ex-centric circular supermathematics functions (**EC-SMF**), by intersecting **d** with the unity circle (see.Fig.**1**). Amongst them the **ex-centric cosine** of **θ,** denoted **cex θ = x**, where **x** is the projection of the point **W**, which is the intersection of the straight line with the trigonometric circle **C(1,O),** or the Cartesian coordinates of the point **W**. Because a straight line, passing through **S**, interior to the circle (**s ≤ 1 → e < R**), intersects the circle in two points $\mathbf{W_1}$ and $\mathbf{W_2}$, which can be denoted $\mathbf{W_{1,2}}$, it results that there are **two determinations** of the ex-centric circular supermathematics functions (EC-SMF): a principal one of index **1** $\mathbf{cex_1}$ **θ,** and a secondary one $\mathbf{cex_2}$ **θ**, of index **2**, denoted $\mathbf{cex_{1,2}}$ **θ. E** and **S** were named **ex-centre** because they were excluded from the center **O(0,0).** This exclusion leads to the apparition of EM and implicitly of SM. By this, the number of mathematical objects grew from one to infinity: to a **unique** function from CM, for example **cos α,** corresponds an **infinity** of functions **cex θ**, due to the possibilities of placing the **ex-center S** and/or **E** in the plane.

S(e, ε) can take an infinite number of positions in the plane containing the unity or trigonometric circle. For each position of **S** and **E** we obtain a function **cex θ.** If **S** is a fixed point,

then we obtain the ex-centric circular SM functions (EC-SMF), with fixed **ex-center**, or with constant **s** and **ε**. But **S** or **E** can take different positions, in the plane, by various rules or laws, while the straight line which generates the functions by its intersection with the circle, rotates with the angle **θ** around **S** and **E.**

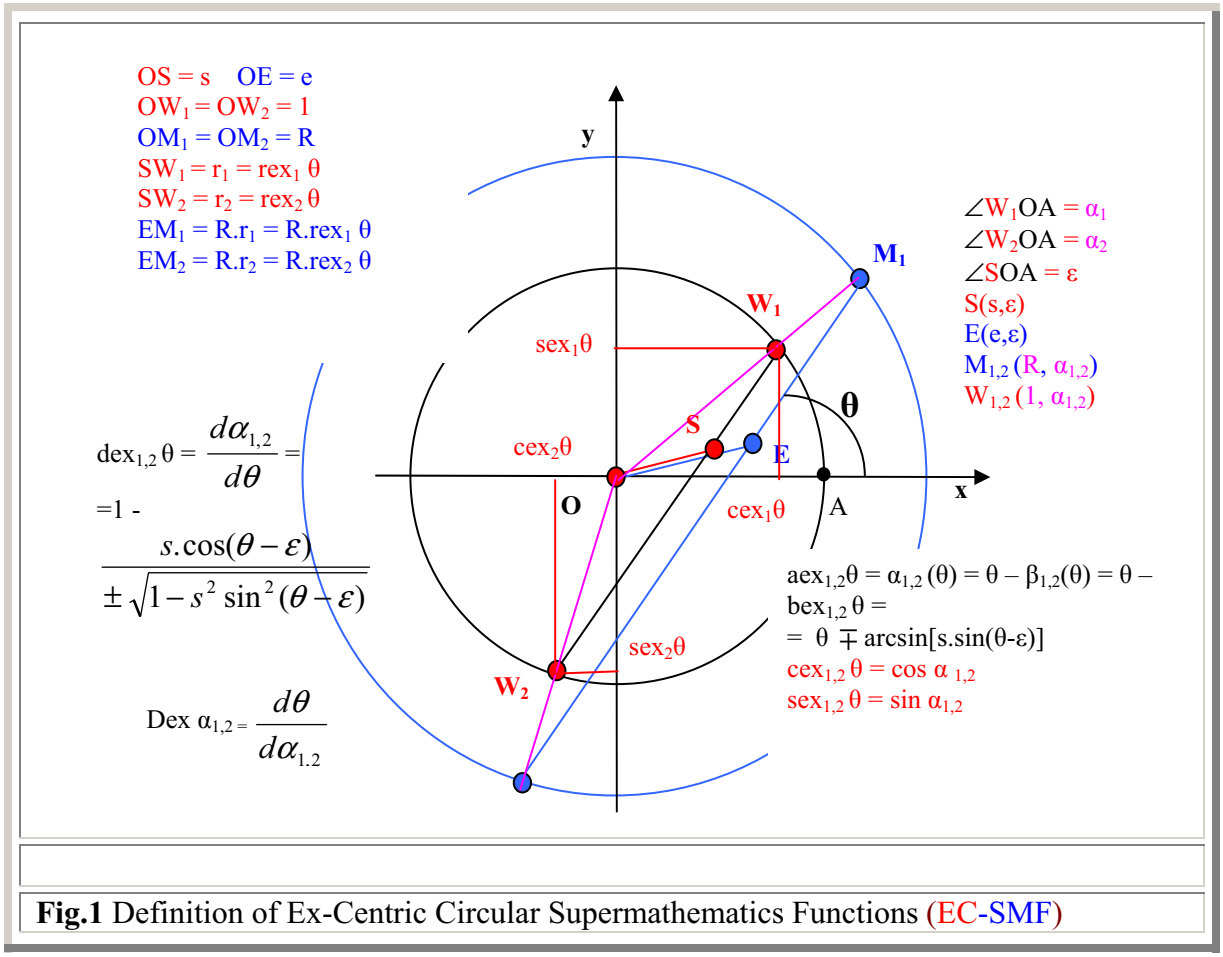


Fig.1 Definition of Ex-Centric Circular Supermathematics Functions (EC-SMF)

In the last case, we have an EC-SMF of ex-center variable point S/E, which means **s = s (θ)** and/or **ε = ε (θ).** If the variable position of **S**/**E** is represented also by EC-SMF of the same ex-center **S(s, ε)** or by another ex-center **S1**[$s_1 = s_1(\theta)$, $\varepsilon_1 = \varepsilon_1(\theta)$], then we obtain functions of double ex-centricity. By extrapolation, we'll obtain functions of triple, and multiple ex-centricity. Therefore, **EC**-SMF are functions of as many variables as we want or as many as we need**.**

If the distances from **O** to the points $\mathbf{W_{1,2}}$ on the circle **C(1,O)** are constant and equal to the radius **R = 1** of the trigonometric circle **C**, distances that will be named **ex-centric radiuses**, the distances from **S** to $\mathbf{W}_{1,2}$ denoted by $\mathbf{r_{1,2}}$ are variable and are named ex-centric radiuses of the unity circle **C**(1,O) and represent, in the same time, new ex-centric circular supermathematics functions (EC-SMF), which were named **ex-centric radial functions,** denoted $\mathbf{rex_{1,2}}$ **θ**, if are expressed in function of the **variable** named **ex-centric θ** and **motor**, which is the angle from the ex-center **E**. Or, denoted Rex1,2 **α,** if it is expressed in function of the angle **α** or the **centric variable**, the angle at **O(0,0).** The $\mathbf{W_{1,2}}$ are seen under the angles $\boldsymbol{\alpha_{1,2}}$ from **O(0,0)** and under the angles **θ** and **θ + π** from **S(e, ε)** and **E**. The straight line **d** is divided by **S ⊂ d** in the two semi-straight lines, one positive $\mathbf{d}^+$ and the other negative $\mathbf{d}^-$. For this reason, we can consider $\mathbf{r_1} = \mathrm{rex}_1\ \boldsymbol{\theta}$ a positive oriented segment

on **d** (→ $\mathbf{r_1} > 0$) and $\mathbf{r_2} = \text{rex}_2\ \boldsymbol{\theta}$ a negative oriented segment on **d** (→ $\mathbf{r_2} < 0$) in the negative sense of the semi-straight line $\mathbf{d}^-$.

Using simple trigonometric relations, in certain triangles $\mathbf{OEW_{1,2}}$, or, more precisely, writing the sine theorem (as function of **θ**) and Pitagora's generalized theorem (for the variables $\boldsymbol{\alpha_{1,2}}$) in these triangles, it immediately results the **invariant expressions** of the ex-centric radial functions:

$$\mathbf{r}_{1,2}(\boldsymbol{\theta}) = \mathbf{rex}_{1,2}\,\boldsymbol{\theta} = -\,s.\cos(\boldsymbol{\theta} - \varepsilon) \pm \sqrt{1 - s^2 \sin^2(\theta - \varepsilon)}$$

and

$$\mathbf{r}_{1,2}(\alpha_{1,2}) = \mathbf{Rex}\,\alpha_{1,2} = \pm \sqrt{1 + s^2 - 2.s.\cos(\theta - \varepsilon)}\ .$$

All **EC-SMF** have **invariant** expressions, and because of that they don't need to be tabulated, tabulated being only the centric functions from **CM**, which are used to express them. In all of their expressions, we will always find one of the square roots of the previous expressions, of ex-centric radial functions.

Finding these two determinations is simple: for **+ (plus)** in front of the square roots we always obtain the first determination ($\mathbf{r_1} > 0$) and for the **– (minus)** sign we obtain the second determination ($\mathbf{r_2} < 0$). The rule remains true for all EC-SMF**.** By convention, the first determination, of index **1**, can be used or written without index.

Some remarks about these REX ("King") functions:

- The ex-centric radial functions are the expression of the distance between two points, in the plane, in polar coordinates: **S**(s,ε) and $\mathbf{W_{1,2}}$ (**R** =1, $\alpha_{1,2}$), on the direction of the straight line **d,** skewed at an angle **θ** in relation to **Ox** axis;
- Therefore, using exclusively these functions, we can express the equations of all known **plane curves,** as well as of other new ones, which surfaced with the introduction of **EM.** An example is represented by **Booth's lemniscates** (see Fig. **2, a, b, c**), expressed, in polar coordinates, by the equation:
  $\rho(\boldsymbol{\theta}) = \mathbf{R}(\mathbf{rex}_1\theta + \text{rex}_2\theta) = -2\ s.R\cos(\theta - \varepsilon)$ for **R=1**, **ε** = 0 and **s** ∈ [0, 3].

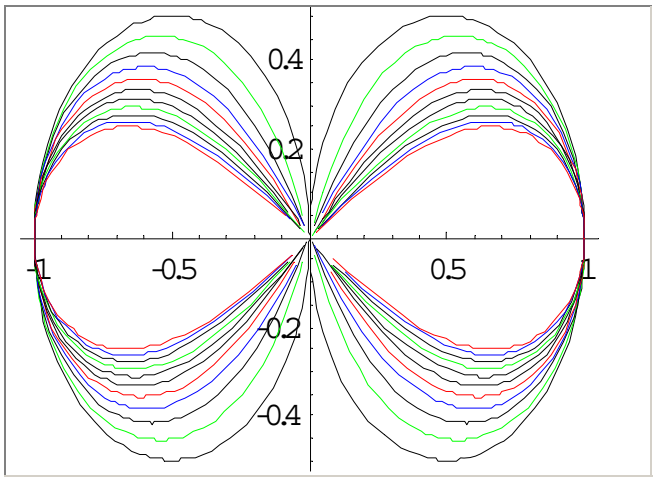


Fig.**2,a Booth's Lemniscates** for **R = 1** and numerical ex-centricity **e** ∈ [1.1, 2]

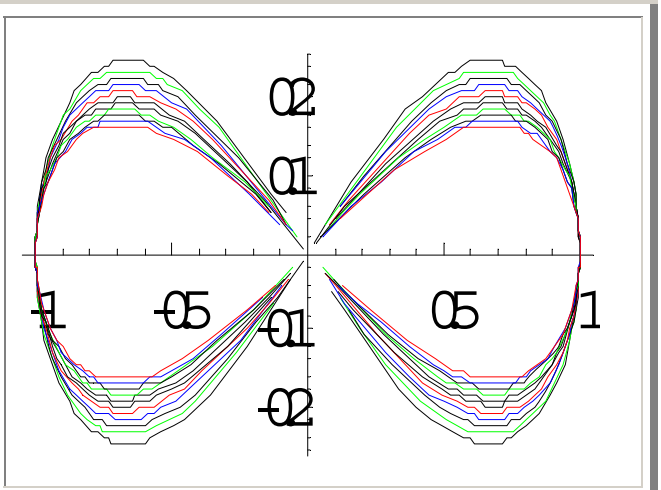
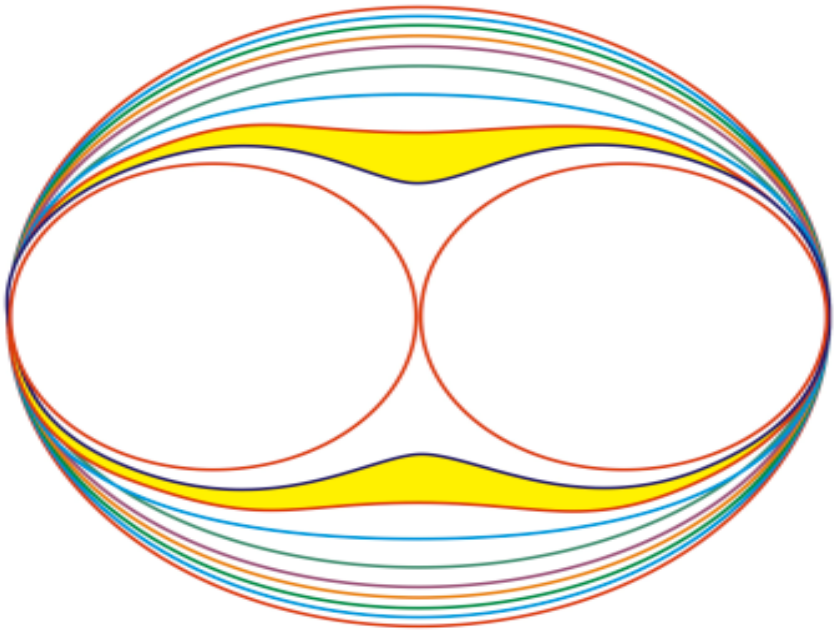


Fig. **2,b Booth's Lemniscates** for **R = 1** and numerical ex-centricity **e** ∈ [2.1, 3]

- Another consequence is the generalization of the definition of a circle:

"**The Circle** is the plane curve whose points M are at the distances **r(θ) = R**.rex **θ = R**.rex **[θ, E(e, ε)]** in relation to a **certain** point from the circle's plane **E**(e, ε)".
If **S ≡ O**(0,0), then **s = 0** and rex θ = 1 = constant, and **r(θ) = R** = constant, we obtain the circle's **classical definition**: the points situated at the same distance **R** from a point, the center of the circle.

Booth Lemniscate Functions



Polar coordinate equation with
supermathematics circle functions rex $_{1,2}$ θ :
$\rho = R\,(rex_1\,\theta + rex_2\,\theta)$
for
circle radius R = 1
and
the numerical ex-centricity $s \in [0, 1]$

Fig. **2,c**

- The functions **rex θ** and **Rex α** expresses the transfer functions of zero degree, or of the position of transfer, from the mechanism theory, and it is the ratio between the parameter $\mathbf{R}(\alpha_{1,2})$, which positions the conducted element $\mathbf{OM}_{1,2}$ and parameter $\mathbf{R}.r_{1,2}(\theta)$, which positions the leader element $\mathbf{EM}_{1,2}$.
  Between these two parameters, there are the following relations, which can be deduced similarly easy from Fig. 1 that defines EC-SMF.
  Between the position angles of the two elements, leaded and leader, there are the following relations:

$$\alpha_{1,2} = \theta \curlyvee \arcsin[e.\sin(\theta - \varepsilon)] = \theta \curlyvee \beta_{1,2}(\theta) = \mathbf{aex}_{1,2}\ \theta$$

  and

$$\theta = \alpha_{1,2} \pm \beta_{1,2}(\alpha_{1,2}) = \alpha_{1,2} \pm \arcsin\left[\pm \frac{s.\sin(\alpha_{1,2} - \varepsilon)}{\sqrt{1 + s^2 - 2.s.\cos(\alpha_{1,2} - \varepsilon)}} \right] = \mathbf{Aex}\ (\alpha_{1,2}).$$

  The functions **aex** $_{1,2}\ \theta$ and **Aex** $\alpha_{1,2}$ are EC-SMF, called **ex-centric amplitude,** because of their usage in defining the ex-centric cosine and sine from EC-SMF, in the same manner as the amplitude function or amplitudinus am(k,u) is used for defining the elliptical Jacobi functions:

  sn(k,u) = sn[am(k,u)], cn(k,u) = cos[am(k,u)],

  or:

$$\mathbf{cex}_{1,2}\ \theta = \mathbf{cos}(\mathbf{aex}_{1,2}\ \theta), \qquad \mathbf{Cex}\ \alpha_{1,2} = \mathbf{cos}(\mathbf{Aex}\ \alpha_{1,2})$$

  and

$$\mathbf{sex}_{1,2}\ \theta = \mathbf{sin}\ (\mathbf{aex}_{1,2}\ \theta), \qquad \mathbf{Sex}\ \alpha_{1,2} = \cos\ (\mathbf{Aex}\ \alpha_{1,2})$$

- The radial ex-centric functions can be considered as modules of the position vectors $\vec{r}_{1,2}$ for the $\mathbf{W}_{1,2}$ on the unity circle **C (1,O)**. These vectors are expressed by the following relations:

$$\vec{r}_{1,2} = rex_{1,2}\theta.rad\theta\ ,$$

  where **rad θ** is the unity vector of variable direction, or the versor/**phasor** of the straight line direction $\mathbf{d}^+$, whose derivative is the phasor **der θ = d(rad θ)/d θ** and represents normal vectors on the straight lines $\mathbf{OW}_{1,2}$, directions, tangent to the circle in the $\mathbf{W}_{1,2}$. They are named the **centric derivative** phasors. In the same time, the modulus **rad θ** function is the corresponding, in **CM,** of the function **rex θ** for **s = 0 → θ = α** when **rex θ = 1** and **der** $\alpha_{1,2}$ are the tangent versors to the unity circle in $\mathbf{W}_{1,2}$.
- The derivative of the $\vec{r}_{1,2}$ vectors are the velocity vectors:

$$\vec{v}_{1,2} = \frac{d\,\vec{r}_{1,2}}{d\theta} = dex_{1,2}\theta.der\alpha_{1,2}$$

  of the $\mathbf{W}_{1,2} \subset \mathbf{C}$ points in their rotating motion on the circle, with velocities of variable modulus $\mathbf{v}_{1,2} = \mathbf{dex}_{1,2}\ \theta$, when the generating straight line **d** rotates around the ex-center **S** with a constant angular speed and equal to the unity, namely **Ω = 1.** The velocity

vectors have the expressions presented above, where **der $\alpha_{1,2}$** are the phasors of centric radiuses $\mathbf{R_{1,2}}$ of module **1** and of $\boldsymbol{\alpha_{1,2}}$ directions. The expressions of the functions **EC-SM $\mathbf{dex_{1,2}\,\theta}$, ex-centric derivative** of **θ,** are, in the same time, also the $\boldsymbol{\alpha_{1,2}}$ (**θ**) angles derivatives, as function of the motor or independent variable **θ,** namely

$$\mathbf{dex_{1,2}\,\theta = d\alpha_{1,2}\,(\theta)/d\,\theta} = 1 - \frac{s.\cos(\theta-\varepsilon)}{\pm\sqrt{1-s^2.\sin^2(\theta-\varepsilon)}}$$

as function of **θ,** and

$$\mathbf{Dex\ \alpha_{1,2} = d(\theta)/d\alpha_{1,2}} = \frac{1-s.\cos(\alpha_{1,2}-\varepsilon)}{1+s^2-2.s.\cos(\alpha_{1,2}-\varepsilon)} = \frac{1-s.\cos(\alpha_{1,2}-\varepsilon)}{\mathrm{Re}\,x^2\alpha_{1,2}},$$

as functions of $\boldsymbol{\alpha_{1,2}}$ **.**

It has been demonstrated that the **ex-centric derivative** functions **EC-SM** express the transfer functions of the first order, or of the angular velocity, from the Mechanisms Theory, for **all (!)** known plane mechanisms.

- The radial ex-centric function **rex θ** expresses exactly the movement of push-pull mechanism **S = R. rex θ,** whose motor connecting rod has the length **r**, equal with **e** the real ex-centricity, and the length of the crank is equal to **R,** the radius of the circle, a very well-known mechanism, because it is a component of all automobiles, except those with Wankel engine.

The applications of radial ex-centric functions could continue, but we will concentrate now on the more general applications of **EC-SMF**.

Concretely, to the unique forms as those of the circle, square, parabola, ellipse, hyperbola, different spirals, etc. from **CM**, which are now grouped under the name of **centrics**, correspond an infinity of **ex-centrics** of the same type**:** circular, square (quadrilobe), parabolic, elliptic, hyperbolic, various spirals **ex-centrics**, etc. Any **ex-centric function**, with null ex-centricity (e = 0), degenerates into a **centric function**, which represents, at the same time its **generating** curve. Therefore, the CM itself belongs to **EM,** for the unique case (**s = e** = 0), which is one case from an infinity of possible cases, in which a point named eccenter **E(e, ε)** can be placed in plane. In this case, **E** is overleaping on one or two points named **center**: the **origin O(0,0)** of a frame, considered the origin **O(0,0)** of the referential system, and/or the **center C(0,0)** of the unity circle for circular functions, respectively, the symmetry center of the two arms of the equilateral hyperbola, for hyperbolic functions.

It was enough that a point **E** be eliminated from the center (**O** and/or **C**) to generate from the old CM a new world of **EM**. The reunion of these two worlds gave birth to the **SM** world.

This discovery occurred in the city of the Romanian Revolution from 1989, **Timişoara,** which is the same city where on November 3rd, 1823 **Janos Bolyai** wrote: "From **nothing** I've created a new world". With these words, he announced the discovery of the fundamental formula of the first **non-Euclidean geometry**.

He – from nothing, I – in a joint effort, proliferated the periodical functions which are so helpful to engineers to describe some periodical phenomena. In this way, I have enriched the mathematics with new objects.

When **Euler** defined the trigonometric functions, as direct circular functions, if he wouldn't have chosen **three superposed points**: the origin **O**, the center of the circle **C** and **S** as a pole of a semi straight line, with which he intersected the trigonometric/unity circle, the **EC-SMF** would have been discovered much earlier, eventually under another name.

Depending on the way of the "**split**" (we isolate one point at the time from the superposed ones, or all of them at once), we obtain the following types of **SMF**:

O ≡ C ≡ S → Centric functions belonging to CM;
and those which belong to **EM** are:

O ≡ C ≠ S → Ex-centric Circular Supermathematics Functions (EC-SMF);
O ≠ C ≡ S → Elevated Circular Supermathematics Functions (ELC-SMF);
O ≠ C ≠ S → Exotic Circular Supermathematics Functions (EXC-SMF).

These **new mathematics complements,** joined under the **temporary** name of **SM**, are extremely useful tools or instruments, long awaited for. The proof is in the large number and the diversity of periodical functions introduced in mathematics, and, sometimes, the complex way of reaching them, by trying the substitution of the circle with other curves, most of them closed.

To obtain new special, periodical functions, it has been attempted the replacement of the trigonometric circle with the **square** or the **diamond**. This was the proceeding of Prof. Dr. Math. **Valeriu Alaci,** the former head of the Mathematics Department of Mechanics College from Timişoara, who discovered the **square** and **diamond** trigonometric functions. Hereafter, the mathematics teacher **Eugen Visa** introduced the **pseudo-hyperbolic** functions, and the mathematics teacher **M. O. Enculescu** defined the **polygonal** functions, replacing the circle with an n-sides polygon; for n = 4 he obtained the square **Alaci** trigonometric functions. Recently, the mathematician, Prof. **Malvina Baica,** (of Romanian origin) from the University of Wisconsin together with Prof**. Mircea Cârdu,** have completed the gap between the **Euler** circular functions and **Alaci** square functions, with the so-called **Periodic Transtrigonometric functios.**

The Russian mathematician **Marcusevici** describes, in his work "**Remarcable sine functions**" the **generalized trigonometric functions** and the trigonometric functions **lemniscates**.

Even since 1877, the German mathematician Dr. **Biehringer**, substituting the right triangle with an oblique triangle, has defined the **inclined** trigonometric functions. The British scientist of Romanian origin Engineer **George** (Gogu) **Constantinescu** replaced the **circle** with the **evolvent** and defined the **Romanian** trigonometric functions: **Romanian cosine** and **Romanian sine,** expressed by Cor α and Sir α functions, which helped him to resolve some non-linear differential equations of the Sonicity Theory, which he created. And how little known are all these functions even in Romania!

Also the elliptical functions are defined on an ellipse. A rotated one, with its main axis along Oy axis.

How simple the complicated things can become, and as a matter of fact they are! This **paradox**(ism) suggests that by a simple displacement/expulsion of a **point** from a **center** and by the apparition of the notion of the **ex-center**, a new world appeared, the world of EM and, at the same time, a new Universe, the **SM** Universe.

Notions like "**Supermathematics Functions**" and "**Circular Ex-centric Functions**" appeared on most search engines like Google, Yahoo, AltaVista etc., from the beginning of the Internet. The new notions, like **quadrilobe** "quadrilobas", how the **ex-centrics** are named, and which continuously fill the space between a **square** circumscribed to a circle and the **circle** itself were included in the Mathematics Dictionary. The intersection of the **quadriloba** with the straight line **d** generates the new functions called **cosine quadrilobe-**ic and **sine quadrilobe**-ic**.**

The benefits of **SM** in science and technology are too numerous to list them all here. But we are pleased to remark that **SM** removes the boundaries between **linear** and **non-linear**; the linear belongs to **CM**, and the non-linear is the appanage of **EM**, as between **ideal** and **real**, or as between perfection and **imperfection**.

It is known that the **Topology** does not differentiate between a pretzel and a cup of tea. Well, **SM** does not differentiate between a **circle** (e = 0) and a **perfect square** (s = ± 1), between a **circle** and a **perfect triangle**, between an **ellipse** and a **perfect rectangle**, between a **sphere** and a perfect **cube,** etc. With the same parametric equations we can obtain, besides the **ideal** forms of **CM** (circle, ellipse, sphere etc.), also the **real** ones (square, oblong, cube, etc.). For **s** ∈ [-1,1], in the case of ex-centric functions of variable **θ**, as in the case of centric functions of variable **α,** for **s**∈[-∞,+∞], it can be obtained an infinity of intermediate forms, for example, square, oblong or cube with rounded corners and slightly curved sides or, respectively, faces. All of these facilitate the utilization of the new SM functions for drawing and representing of some technical parts, with rounded or splayed edges, in the **CAD/ CAM-SM** programs, which don't use the computer as drawing boards any more, but create the technical object instantly, by using the parametric equations, that speed up the processing, because only the equations are memorized, not the vast number of pixels which define the technical piece.

The numerous functions presented here, are introduced in mathematics for the first time, therefore, for a better understanding, the author considered that it was necessary to have a short presentation of their equations, such that the readers, who wish to use them in their application's development, be able to do it.

SM is not a finished work; it's merely an **introduction** in this vast domain, a first step, the author's small step, and a giant leap for mathematics.

The **elevated circular SM** functions (**ELC-SMF**), named this way because by the modification of the numerical ex-centricity **s** the points of the curves of elevated sine functions **sel θ** as of the elevated circular function elevated cosine **cel θ** is elevating – in other words it rises on the vertical, getting out from the space {-1, +1] of the other sine and cosine functions, centric or ex-centric. The functions' **cex θ** and **sex θ** plots are shown in Fig. **3,** where it can be seen that the points of these graphs get modified on the horizontal direction, but all remaining in the space [-1,+1], named the existence domain of these functions.

The functions' cel **θ** and sel **θ** plots can be simply represented by the products:

$$\mathbf{cel}_{1,2}\ \theta = \mathbf{rex}_{1,2}\ \theta \cdot \cos\theta \quad \text{and} \quad \mathbf{Cel}\ \alpha_{1,2} = \mathbf{Rex}\ \alpha_{1,2} \cdot \cos\theta$$

$$\mathbf{sel}_{1,2}\ \theta = \mathbf{rex}_{1,2}\ \theta \cdot \sin\theta \quad \text{and} \quad \mathbf{Sel}\ \alpha_{1,2} = \mathbf{Rex}\ \alpha_{1,2} \cdot \sin\theta$$

and are shown Fig. **4.**

The **exotic circular functions** are the most general **SM,** and are defined on the unity circle which is not centered in the origin of the x**O**y axis system, neither in the eccenter **S,** but in a certain point **C (c,γ)** from the plane of the unity circle, of polar coordinates **(c, γ)** in the x**O**y coordinate system.

Many of the drawings from this album are done with **EC-SMF** of ex-center variable and with arcs that are multiples of **θ (n.θ).** The used relations for each particular case are explicitly shown, in most cases using the **centric** mathematical functions, with which, as we saw, we could express all **SM** functions, especially when the image programs cannot use SMF. This doesn't mean that, in the future, the new math complements will not be implemented in computers, to facilitate their vast utilization.

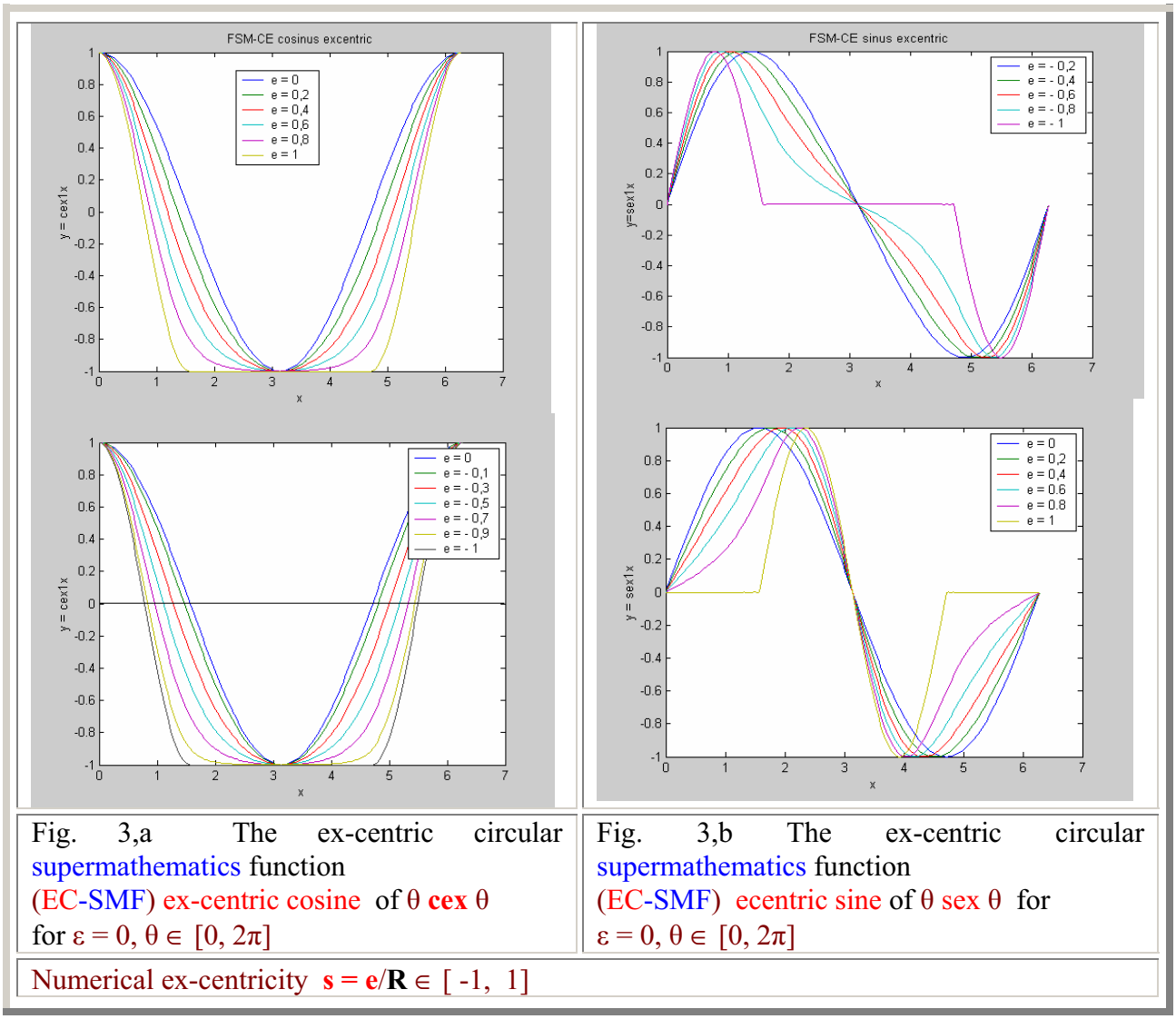


Fig. 3,a The ex-centric circular supermathematics function (EC-SMF) ex-centric cosine of θ **cex** θ for ε = 0, θ ∈ [0, 2π]

Fig. 3,b The ex-centric circular supermathematics function (EC-SMF) ecentric sine of θ sex θ for ε = 0, θ ∈ [0, 2π]

Numerical ex-centricity **s = e/R** ∈ [-1, 1]

The computer specialists working in programming the computer assisted design software **CAD/CAM/CAE,** are on their way to develop these new programs fundamentally different, because the technical objects are created with parametric **circular** or **hyperbolic** SMFs, as it has been exemplified already with some achievements such as airplanes, buildings, etc. in http://www.eng.upt.ro/~mselariu and how a washer can be represented as a toroid ex-centricity (or as an "ex-centric torus"), square or oblong in an axial section, and, respectively, a square plate with a central square hole can be a "square torus of square section". And all of these, because **SM** doesn't make distinction between a circle and a square or between an ellipse and a rectangle, as we mentioned before.

But the most important achievements in science can be obtained by solving some non-linear problems, because **SM** reunites these two domains, so different in the past, in a single entity. Among these differences we mention that the non-linear domain asks for ingenious approaches for each problem. For example, in the domain of vibrations, static elastic characteristics (SEC) soft non-linear (regressive) or hard non-linear (progressive) can be obtained simply by writing y = m. x, where **m** is

not anymore m = tan α as in the linear case (**s** = 0), but **m** = $tex_{1,2}$ **θ** and depending on the numerical ex-centricity **s** sign, positive or negative, or for **S** placed on the negative x axis (**ε** = π) or on the positive x axis (ε = 0), we obtain the two nonlinear elastic characteristics, and obviously for s=0 we'll obtain the linear SEC.

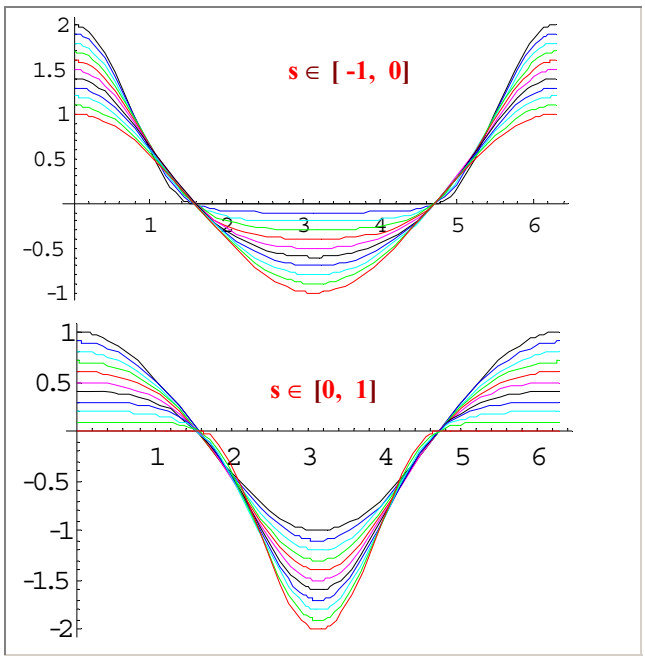


Fig. 4,a ELC-SMF elevated cosine of **θ - cel θ**, for **s** ∈ [-1, +1], **ε** = 0, **θ** ∈ [0, 2π].

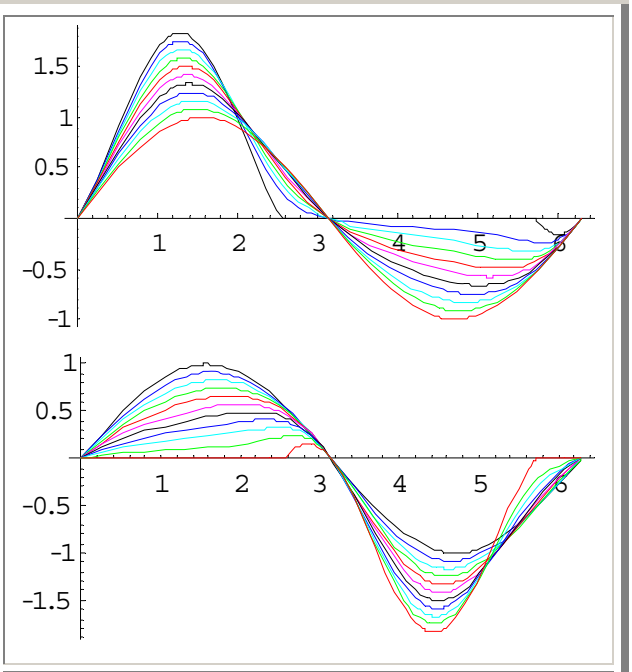


Fig. 4,b ELC-SMF elevated sine of **θ - sel θ**, for **s** ∈ [-1, +1], **ε** = **1**, **θ** ∈ [0, 2π].

Due to the fact that the functions **cex θ** and **sex θ**, as well **Cex** α and **Sex α** and their combinations, are solutions of some differential equations of second degree with variable coefficients, it has been stated that the linear systems (Tchebychev) are obtained also for **s** = ± **1,** and not only for **s** = **0.** In these equations, the mass (the point **M**) rotates on the circle with a double angular speed **ω** = 2.**Ω** (reported to the linear system where **s** = 0 and **ω** = **Ω** = **constant**) in a half of a period, and in the other half of period stops in the point **A**(**R**,0) for **e** = **s**R = **R** or ε = 0 and in **A'**(—**R**, 0) for **e** = — **s**.**R** = —1, or ε = π. Therefore, the oscillation period **T** of the **three linear systems** is the same and equal with **T** = **Ω** / **2π**. The nonlinear **SEC** systems are obtained for the others values, intermediates, of **s** and **e**. The projection, on any direction, of the rotating motion of **M** on the circle with radius **R**, equal to the oscillation amplitude, of a variable angular speed **ω** = **Ω**.**dex θ** (after **dex θ** function) is an **non-linear** oscillating motion**.**

The discovery of "**king**" function **rex θ,** with its properties, facilitated the apparition of a **hybrid** method (analytic-numerical), by which a simple **relation** was obtained, with only two terms, to **calculate** the first degree elliptic complete integral **K**(**k**), with an unbelievable precision, with a minimum of **15 accurate decimals,** after only 5 steps. Continuing with the next steps, can lead us to a new relation to compute **K** (**k**), with a considerable higher precision and with possibilities to expand the method to other elliptic integrals, and not only to those. After 6 steps, the relation of **E** (**k**) has the same precision of computation.

The discovery of **SMF** facilitated the apparition of a new integration method, named **integration through the differential dividing**.

We will stop here, letting to the readers the pleasure to delight themselves by viewing the drawings from this album.

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GRAPH THEORY

Vectored Route-Length Minimization – a Heuristic and an Open Conjecture

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Abstract

We have posed a simple but interesting graph theoretic problem and posited a heuristic solution procedure, which we have christened as *Vectored Route-length Minimization Search* (VeRMinS). Basically, it constitutes of a re-casting of the classical "shortest route" problem within a strictly Euclidean space. We have only presented a heuristic solution process with the hope that a formal proof will eventually emerge as the problem receives wider exposure within mathematical circles.

Key words: graph theory, Euclidean space, network connectivity matrix

A short historical background of similar class of problems

The classical "shortest route" (or shortest path) problem is properly associated with the branch of mathematics formally known as *graph theory* (or *network theory*). History has it that this theory originated in an attempt to solve a famous 18th century routing problem concerning the Prussian city of Konigsberg (Kaliningrad in modern Russia). The city is located along the two banks and on two islands formed by the river Pregel, which effectively divides the city into four separate landmasses. Seven bridges connected the various regions of the city and the resulting "Konigsberg bridge problem" had to do with finding an optimal route around the city that would require a traveler to cross each of the seven bridges only once in the whole trip (Alexanderson, 2006). That it is impossible to make such a trip was originally proved by the Swiss mathematical genius Leonhard Euler (Euler, 1766) thus formally giving birth to the mathematics of networks.

The classical "shortest route" problem born out of the Konigsberg bridge problem subsequently branched into a number of well-known variants popularly grouped as "traveling salesman"

problems. The shortest route problem is one of the many practical adaptations of Eulerian graph theory. The basic problem is concerned with finding the shortest distance between a "source" and a "sink" node in any sufficiently generalized network consisting of a finite number of nodes.

The usual practical applications of similar class of problems in modern times are in the configuration of telecommunications networks e.g. connecting one transmission tower to another in a network so that the total network up-linking time is minimized. There are also interesting application possibilities in the realm of social sciences especially in *social network analysis* that has provided valuable insights into the governance and behavior of organized groups in society and social capital generation (Nan Lin, 1999). In business and finance applications, network data mining is being applied to detect fraud and money laundering activities (Yue et. al., 2007) and in following terrorist money trails by identifying the likely "shortest paths" through social networks (Keefe, 2006)

Many alternative algorithms to solving the shortest route problem have been devised e.g. *Djikstra's algorithm* (Djikstra, 1959), and *Ford-Fulkerson's algorithm* (Ford and Fulkerson, 1962), which have many applications in the fields of telecommunications and internetworking. However, in positing our problem, we have been concerned with the most simplistic version of the classical shortest route problem in strictly Euclidean space of unrestricted dimensionality, which we proceed to define as follows:

"*Given a partially connected network of N nodes in a strictly Euclidean space of any dimension, find a route through the network from a pre-specified source node S_0 to a pre-specified sink node S_N such that the overall route length (in terms of the total Euclidean distance) is minimum*"

Mathematical basis of VeRMinS: a proposed heuristic solution procedure

The Vectored Route-length Minimization Search (VeRMinS) is a *heuristic search* that aims to find the shortest route from a source node to a sink node in a network in Euclidean space of any dimension by identifying the *linear-most connectivity* between the source and sink nodes.

With every route in a network, we associate a corresponding *weight* factor, which is the sum of the Euclidean distance between the nodes on that route. Then the best (i.e. linear-most) route through the network is the one having the minimum weight (Rote, 1990). For any network consisting of $N = m + 1$ nodes, we can set up a *network connectivity matrix* **M** as follows:

1	R_{01}	R_{02}	…	R_{0m}
R_{10}	1	R_{12}	…	R_{1m}
R_{20}	R_{21}	1	…	R_{2m}
…	…	…	…	…
R_{m0}	R_{m1}	R_{m2}	…	1

In the network connectivity matrix, when $i \neq j$, $R_{ij} = 1$ if and only if a connectivity exists between nodes i and j and $R_{ij} = 0$ otherwise. Since a node is necessarily 'self-connected', $R_{ij} = 1$ when $i = j$ i.e. for all the diagonal elements of **M**.

A finite number, say q, of *route vectors* $\mathbf{P}_t$ (with $t = 1, 2 \ldots q$) can then be extricated from **M** such that $\mathbf{P}_1 = [k_{10}\ k_{11}\ k_{12} \ldots k_{1m}]$, $\mathbf{P}_2 = [k_{20}\ k_{21}\ k_{22} \ldots k_{2m}] \ldots \mathbf{P}_t = [k_{t0}\ k_{t1}\ k_{t2} \ldots k_{tm}] \ldots \mathbf{P}_q = [k_{q0}\ k_{q1}\ k_{q2} \ldots k_{qm}]$, where $k_{qj} = 1$ if node j lies on the q-th route and $k_{qj} = 0$ otherwise.

A (m x 1) weight vector **W** is defined as follows:

$$\mathbf{W} = [0\ w_1\ w_2 \ldots w_i, w_{m-1}\ 0]^{\mathbf{T}}$$

where w_i is the vertical Euclidean distance of the i-th node from the *ideal route* (which is simply a hypothetical straight line connecting the source and sink nodes), as determined by its position vector with respect to the *ideal route*. Since both the source and sink nodes must necessarily lie on the shortest route (i.e. a route must be *effective* before it can be *efficient*), $w_0 = w_m = 0$.

Then, $\mathbf{P}_q \cdot \mathbf{W} = \sum\sum (k_{qj} w_i)$ would yield the deciding criterion for the q-th route in terms of the vertical Euclidean distances of each of the nodes along the q-th route from the *ideal* route.

Introducing the property of *Euclidean dominance*

The route vector $\mathbf{P}_a$ *exhibits Euclidean dominance over the route vector* $\mathbf{P}_b$ *(written henceforth as* $\mathbf{P}_a \supset \mathbf{P}_b$*) when at least one element of* $\mathbf{P}_a$ *is 0 with the corresponding element in* $\mathbf{P}_b$ *being 1 and all other elements being same for* $\mathbf{P}_a$ *and* $\mathbf{P}_b$.

Proof: This property follows from the principle of triangular inequality in Euclidean geometry whereby the sum of two sides of a triangle is always greater in magnitude than the third side.

Each of the nodes in a network corresponds to a particular position vector in Euclidean space. Therefore, it implies that if node A is connected to both nodes B and C while node B is also connected to node C, then the route that goes directly from node A to node C will always be more preferable than one which goes from node A to node B to node C. This of course assumes that the remaining segments of the two routes coincide with each other.

So the property of Euclidean dominance may be used to effectively eliminate some of the q route vectors extricated from **M**. Assuming h route vectors are eliminated after applying Euclidean dominance, then the *linear-most route* is obtainable as $\text{Min}_t [\mathbf{P}_1.\mathbf{W}, \mathbf{P}_2.\mathbf{W}, \ldots, \mathbf{P}_t.\mathbf{W}, \ldots, \mathbf{P}_{(q-h)}.\mathbf{W}]$.

Applying the VeRMinS – a numerical illustration

Let a simple network in 2*D*-Euclidean space consisting of ten nodes 0, 1, 2 … 9 be as follows:

Preceding node	**Succeeding node**	$\mathbf{w_i}$
0	1, 2, 3	0
1	4, 7	3
2	4, 5, 6	0

3	6, 8	3
4	7	2
5	7, 8	0
6	8	1
7	9	5
8	9	6
9	-	0

We wish to find the best (i.e. linear-most) route from node 0 to node 9.

We wish to make the readers aware that here we only present an illustrative exercise outlining a numerical solution procedure. However, we supply no formal proof that the outlined procedure is necessary and sufficient in obtaining the shortest route through a network in any Euclidean space.

The network connectivity matrix $\mathbf{M}_{10x10}$ for the above network is obtained as follows:

1	1	1	1	0	0	0	0	0	0
1	1	0	0	1	0	0	1	0	0
1	0	1	0	1	1	1	0	0	0
1	0	0	1	0	0	1	0	1	0
0	1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	1	1	0
0	0	1	1	0	0	1	0	1	0

0	1	0	0	1	1	0	1	0	1
0	0	0	1	0	1	1	0	1	0
0	0	0	0	0	0	0	1	1	1

The following route vectors may be extricated from **M**:

$\mathbf{P}_1$ = [1 1 0 0 1 0 0 1 0 1]

$\mathbf{P}_2$ = [1 1 0 0 0 0 0 1 0 1]

$\mathbf{P}_3$ = [1 0 1 0 1 0 0 1 0 1]

$\mathbf{P}_4$ = [1 0 1 0 0 1 0 1 0 1]

$\mathbf{P}_5$ = [1 0 1 0 0 1 0 0 1 1]

$\mathbf{P}_6$ = [1 0 1 0 0 0 1 0 1 1]

$\mathbf{P}_7$ = [1 0 0 1 0 0 1 0 1 1], and

$\mathbf{P}_8$ = [1 0 0 1 0 0 0 0 1 1].

It may be easily observed that $\mathbf{P}_2 \supset \mathbf{P}_1$ and $\mathbf{P}_8 \supset \mathbf{P}_7$, so, using the property of Euclidean dominance one can eliminate $\mathbf{P}_1$ and $\mathbf{P}_7$ straightaway.

The weight vector is obtained as: $\mathbf{W} = [0\ 3\ 0\ 3\ 2\ 0\ 1\ 5\ 6\ 0]^{\mathbf{T}}$

Therefore $\mathbf{P}_2.\mathbf{W} = 8$, $\mathbf{P}_3.\mathbf{W} = 7$, $\mathbf{P}_4.\mathbf{W} = 5$, $\mathbf{P}_5.\mathbf{W} = 6$, $\mathbf{P}_6.\mathbf{W} = 7$ and $\mathbf{P}_8.\mathbf{W} = 9$.

So $\mathbf{W}^* = \mathrm{Min}_t\,[\mathbf{P}_t.\mathbf{W}] = 5$, which corresponds to the route vector $\mathbf{P}_4$ thereby identifying it as the linear-most route from source to sink.

An open conjecture

VeRMinS is proposed at this stage as no more than a heuristic search procedure. We have not supplied a formal proof that the outlined search procedure is necessary and sufficient in

obtaining the shortest route through a network of a finite number of nodes in any Euclidean space of unrestricted dimensionality. This problem is left open at this stage that may either be proved by showing that all other possible search procedures will always yield less optimal (i.e. longer) routes or disproved via a counter-example that shows that a shorter route exists through a network in any strictly Euclidean space that is not picked by the outlined VeRMinS procedure.

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Graph Distance, Optimal Communication and Group Stability: A Preliminary Conjecture

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Introduction

In recent years, there has been a rapid increase in the literature which discusses new phenomenon associated to social network. One of the well-known phenomenon in this regards is known as 'six degrees of separation' [1], which implies that one can always keep a communication with each other anywhere within a six-step. A number of experiments has verified this hypothesis, either in the context of offline communication (postal mail), or online communication (email, etc.).

In this article, we argue that by introducing this known 'six degrees of separation' into the context of group instability problem, one can find a new type of wisdom in organization. Therefore, we offer a new conjecture, which may be called 'Group stability conjectures based on Graph/Network distance."

To our knowledge this conjecture has not been discussed elsewhere, and therefore may be useful for further research, in particular in the area of organization development and group stability studies. The purpose of this article was of course not to draw a conclusive theory, but to suggest further study of this proposed conjecture.

Graph Distance

Let $G(V,E)$ be a graph, where V is a set of vertices, and E a set of edges:

$$V=\{v_1,v_2,...\}, E=\{e_1,e_2,...\}.$$

We say that there is a route between vertices v_i and v_j. We define the distance between vertices v_i and v_j, noted by $d(v_i,v_j)$ as the shortest chain of edges that connects v_i with v_j.

In the graph $G(V,E)$ let's consider

$$d(v_i,v_j)=n\geq 1$$

where n is the number of edges connecting v_i with v_j, and for each such edge an equiprobability $\frac{1}{n}$.

Using Shannon's entropy

$$H(x) = -\sum_{i=1}^{n} P_i \log_2 (P_i)$$

In order to find the entropy of the distance between two vertices we get

$$H\left(d\left(v_i, v_j\right)\right) = -\sum_{i=1}^{n} P_i \log_2 (P_i) = -\sum_{i=1}^{n} \frac{1}{n} \log_2 (P_i) = -\log_2 \left(\frac{1}{n}\right) = \log_2 n ,$$

since all $P_i = \frac{1}{n}$.

The longer is the distance between two vertices, the bigger is the entropy, since $\log_2 n_1 > \log_2 n_2$ when $n_1 > n_2 \geq 1$, therefore the more degree of disorder, of loss of information, as both ambiguity and imprecision increase.

A Conjecture of Group Stability Based on Graph Distance

A hierarchical structure is a widespread organization form in many areas.

The hidden assumption behind Small-world hypothesis is that everyone is around six-steps away from any other person on Earth, which is known as 'six degrees of separation' principle. A number of experiments have been conducted in order to prove this hypothesis. [2]

In this regards, apparently we can draw analogy from this 'six degrees of separation' to the concept of graph distance. In this context, graph distance can be viewed as the number of 'nodes' that one should reach to come to a destination. This study of graph distance and group stability is quite new, and only a number of published articles have appeared in journals, see for example [3], [4].

Once this analogy is set, it becomes apparent that the 'six degrees of separation' may be interpreted as an optimum graph distance, where any given organization can function in its best, provided we can consider an organization as an actual social-network which functions better if and only if communication can be preserved in optimal way.

In other words, any given organization which expands rapidly beyond these six-degrees of node separation (let say, between the CEO and its factory workers) will be more prone to instability. This phenomenon may also be viewed as another example of '*self-organized criticality'* process in any given organization/structure.

At this point, now we will write down our new conjecture of Group stability based on Graph distance:

(a) For a given organization in any industry, there is an optimal graph distance which will keep communication in organization in its optimum. We can call this as 'optimal graph distance number'.
(b) This optimal graph distance number is inversely proportional to the innovation cycle time in any given organization. This optimal graph distance corresponds to both the hierarchy of organization and also to the degrees of separation.
(c) There is tendency that any organization will increase its size such that the graph distance number always grows such that it exceeds its own level of incompetence (similar to Lawrence's principle).

(d) In order to keep internal and healthy communication for its own survival preservation, a good organization will keep its graph distance number at optimal level.
(e) If an organization has graph distance number which exceeds its optimal number (let say 5 or 6 degrees of separation), then it will be prone to instability.
(f) Group instability can take the form of de-formation of organization in order to meet the communication works again, in other words an organization has tendency to keep the graph distance number at optimal, in accordance with self-organized criticality phenomena.
(g) This is what can be called as Conjecture of Group Stability based on Graph Distance.

While the above conjecture may appear quite simple and obvious at first sight, it covers the phenomena corresponding to group stability in uniquer way, i.e. from the viewpoint of preservation of optimum communication. Therefore any organization has its own tendency to keep the size of its hierarchy such that the graph distance is kept optimal.

Let us mention a simple example here: Toyota has a unique management way, which is well-known in management literature. What is not quite well-studied is perhaps the fact that it has less hierarchical structure compared to other large automobile companies in USA. As a result, the innovation cycle time tends to be faster. For instance, Toyota has released its first generation of hybrid car (Prius) in 2007, while GM only expects to release a first version of hybrid cars by 2010. The simple lesson here (see point b) is that keep graph distance at minimum in order to reach faster innovation cycle.

The same lesson we often hear when an organization performs excellently in the past when its hierarchy remains small, but during the course of its history it tends to increase in 'graph distance' and then gradually it loses its 'agility'. In this regards one can observe that group stability has deep link with cooperation level in any given organization, because in large organization there is strong tendency that coordination becomes very difficult [4]. From the viewpoint of game theory, it becomes very difficult to maintain the condition such that all of its members have optimal return [3].

In turn, a good communication in organization can be viewed as part of 'social capital', which can play significant role to keep its stability. This viewpoint has been discussed in [5].

Similarly, a city which becomes too large and exceeds its capacity to maintain good communication tends to form smaller-cities (just like sub-urbans areas) in order to keep its optimal size.

In our opinion this conjecture has not been discussed elsewhere.

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INFORMATION FUSION

An Algorithm for the Unification of Fusion Theories (UFT)

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Abstract: Since no fusion theory neither rule fully satisfy all needed applications, the author proposes an algorithm for the Unification of Fusion Theories and a combination of fusion rules in solving problems/applications. For each particular application, one selects the most appropriate model, rule(s), and algorithm of implementation.
We are working in the unification of the fusion theories and rules, which looks like a cooking recipe, better we'd say like a logical chart for a computer programmer, but we don't see another method to comprise/unify all things.
The unification scenario presented herein, which is now in an incipient form, should periodically be updated incorporating new discoveries from the fusion and engineering research.

Keywords: Distributive lattice, Boolean algebra, Conjunctive rule, Disjunctive rule, Partial and Total conflicts, Weighted Operator (WO), Proportional Conflict Redistribution (PCR) rules, Murphy's average rule, Dempster-Shafer Theory (DST), Yager's rule, Transferable Belief Model (TBM), Dubois-Prade's rule (DP), Dezert-Smarandache Theory (DSmT), static and dynamic fusion

ACM Classification: Artificial Intelligence (I.2.3).

1. Introduction.

Each theory works well for some applications, but not for all.
We extend the power and hyper-power sets from previous theories to a Boolean algebra obtained by the closure of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers as complement the fuzzy or neutrosophic complement). All bba's and rules are redefined on this Boolean algebra.
A similar generalization has been previously used by Guan-Bell (1993) for the Dempster-Shafer rule using propositions in sequential logic, herein we reconsider the Boolean algebra for all fusion rules and theories but using sets instead of propositions, because generally it is harder to work in sequential logic with summations and inclusions than in the set theory.

2. Fusion Space.

For $n \geq 2$ let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the frame of discernment of the fusion problem/application under consideration. Then $(\Theta, \cup, \cap, \complement)$, Θ closed under these three operations: union, intersection, and complementation of sets respectively, forms a Boolean

algebra. With respect to the partial ordering relation, the inclusion $\subseteq$, the minimum element is the empty set ϕ, and the maximal element is the total ignorance $I = \bigcup_{i=1}^{n} \theta i$.

Similarly one can define: (Θ, $\cup$, $\cap$, \) for sets, Θ closed with respect to each of these operations: union, intersection, and difference of sets respectively.
(Θ, $\cup$, $\cap$, $\mathcal{C}$) and (Θ, $\cup$, $\cap$, \) generate the same super-power set S^{Θ} closed under $\cup$, $\cap$, $\mathcal{C}$, and \ because for any $A, B \in S^{\Theta}$ one has $\mathcal{C}A = I \setminus A$ and reciprocally $A \setminus B = A \cap \mathcal{C}B$.

If one considers propositions, then (Θ, $\vee$, $\wedge$, $\neg$) forms a Lindenbaum algebra in sequential logic, which is isomorphic with the above (Θ, $\cup$, $\cap$, $\mathcal{C}$) Boolean algebra.

By choosing the frame of discernment Θ closed under $\cup$ only one gets DST, Yager's, TBM, DP theories. Then making Θ closed under both $\cup$, $\cap$ one gets DSm theory. While, extending Θ for closure under $\cup$, $\cap$, and $\mathcal{C}$ one also includes the complement of set (or negation of proposition if working in sequential logic); in the case of non-exclusive elements in the frame of discernment one considers a fuzzy or neutrosophic complement. Therefore the super-power set (Θ, $\cup$, $\cap$, $\mathcal{C}$) includes all the previous fusion theories.

The power set 2^{Θ}, used in DST, Yager's, TBM, DP, which is the set of all subsets of Θ, is also a Boolean algebra, closed under $\cup$, $\cap$, and $\mathcal{C}$, but does not contain intersections of elements from Θ.
The Dedekind distributive lattice D^{Θ}, used in DSmT, is closed under $\cup$, $\cap$, and if negations/complements arise they are directly introduced in the frame of discernment, say Θ', which is then closed under $\cup$, $\cap$. Unlike others, DSmT allows intersections, generalizing the previous theories.
The Unifying Theory contains intersections and complements as well.

Let's consider a frame of discernment Θ with exclusive or non-exclusive hypotheses, exhaustive or non-exhaustive, closed or open world (all possible cases).

We need to make the remark that in case when these $n \geq 2$ elementary hypotheses $\theta_1, \theta_2, \ldots, \theta_n$ are *exhaustive and exclusive* one gets the Dempster-Shafer Theory, Yager's, Dubois-Prade Theory, Dezert-Smarandache Theory, while for the case when the hypotheses are *non-exclusive* one gets Dezert-Smarandache Theory, but for *non-exhaustivity* one gets TBM. An exhaustive frame of discernment is called *close world*, and a non-exhaustive frame of discernment is called *open world* (meaning that new hypotheses might exist in the frame of discernment that we are not aware of). Θ may be finite or infinite.

Let $m_j: S^{\Theta} \rightarrow [0, 1]$, $1 \leq j \leq s$, be $s \geq 2$ basic belief assignments,
(when bbas are working with crisp numbers),
or with subunitary subsets, $m_j: S^{\Theta} \rightarrow \rho([0. 1])$, where $\rho([0. 1])$ is the set of all subsets of the interval [0,1] (when dealing with very imprecise information).

Normally the sum of crisp masses of a bba, m(.), is 1, i.e. $\sum_{X \in S^{\wedge} T} m(X) = 1$.

3. Incomplete and Paraconsistent Information.

For incomplete information the sum of a bba components can be less than 1 (not enough information known), while in paraconsistent information the sum can exceed 1 (overlapping contradictory information).

The masses can be normalized (i.e. getting the sum of their components =1), or not (sum of components < 1 in incomplete information; or > 1 in paraconsistent information).

For a bba valued on subunitary subsets one can consider, as normalization of m(.),

either $\sum_{X \in S^{\wedge}T} \sup\{m(X)\} = 1$,

or that there exist crisp numbers $x \in X$ for each $X \in S^{\Theta}$ such that $\sum_{\substack{X \in S^{\wedge}T \\ x \in X}} m(x) = 1$.

Similarly, for a bba m(.) valued on subunitary subsets dealing with paraconsistent and incomplete information respectively:

a) for incomplete information, one has $\sum_{X \in S^{\wedge}T} \sup\{m(X)\} < 1$,

b) while for paraconsistent information one has $\sum_{X \in S^{\wedge}T} \sup\{m(X)\} > 1$ and there do not exist crisp numbers $x \in X$ for each $X \in S^{\Theta}$ such that $\sum_{\substack{X \in S^{\wedge}T \\ x \in X}} m(x) = 1$.

4. Specificity Chains.

We use the min principle and the precocious/prudent way of computing and transferring the conflicting mass.

Normally by transferring the conflicting mass and by normalization we diminish the specificity.

If $A \cap B$ is empty, then the mass is moved to a less specific element A (also to B), but if we have a pessimistic view on A and B we move the mass $m(A \cap B)$ to $A \cup B$ (entropy increases, imprecision increases), and even more if we are very pessimistic about A and B: we move the conflicting mass to the total ignorance in a closed world, or to the empty set in an open world.

Specificity Chains:

a) From specific to less and less specific (in a closed world):

$(A \cap B) \subset A \subset (A \cup B) \subset I$ or $(A \cap B) \subset B \subset (A \cup B) \subset I$.

Also from specific to unknown (in an open world):

$A \cap B \rightarrow \phi$.

b) And similarly for intersections of more elements: $A \cap B \cap C$, etc.

$A \cap B \cap C \subset A \cap B \subset A \subset (A \cup B) \subset (A \cup B \cup C) \subset I$

or $(A \cap B \cap C) \subset (B \cap C) \subset B \subset (A \cup B) \subset (A \cup B \cup C) \subset I$, etc. in a closed world.

Or $A \cap B \cap C \rightarrow \phi$ in an open world.

c) Also in a closed world:

$A \cap (B \cup C) \subset B \cup C \subset (B \cup C) \subset (A \cup B \cup C) \subset I$ or $A \cap (B \cup C) \subset A \subset (A \cup B) \subset (A \cup B \cup C) \subset I$.

Or $A \cap (B \cup C) \to \phi$ in an open world.

5. Static and Dynamic Fusion.
According to Wu Li we have the following classification and definitions:
Static fusion means to combine all belief functions simultaneously.
Dynamic fusion means that the belief functions become available one after another sequentially, and the current belief function is updated by combining itself with a newly available belief function.

6. An Algorithm (or Scenario) for the Unification of Fusion Theories.
Since everything depends on the application/problem to solve, this scenario looks like a logical chart designed by the programmer in order to write and implement a computer program, or like a cooking recipe.

Here it is the scenario attempting for a unification and reconciliation of the fusion theories and rules:

1) If all sources of information are reliable, then apply the conjunctive rule, which means consensus between them (or their common part):
2) If some sources are reliable and others are not, but we don't know which ones are unreliable, apply the disjunctive rule as a cautious method (and no transfer or normalization is needed).
3) If only one source of information is reliable, but we don't know which one, then use the exclusive disjunctive rule based on the fact that $X_1 \veebar X_2 \veebar \ldots \veebar X_n$ means either X_1 is reliable, or X_2, or and so on or X_n, but not two or more in the same time.
4) If a mixture of the previous three cases, in any possible way, use the mixed conjunctive-disjunctive rule.
As an example, suppose we have four sources of information and we know that: either the first two are telling the truth or the third, or the fourth is telling the truth.
The mixed formula becomes:
$m_{\cap\cup}(\phi) = 0$, and $\forall A \in S^{\Theta} \backslash \phi$, one has $m_{\cap\cup}(A) = \sum_{\substack{X_1, X_2, X_3, X_4 \in S^{\wedge}\Theta \\ ((X_1 \cap X_2) \cup X_3) e \cup X_4 = A}} m_1(X_1) m_2(X_2) m_3(X_3) m_4(X_4)$.

5) If we know the sources which are unreliable, we discount them. But if all sources are fully unreliable (100%), then the fusion result becomes vacuum bba (i.e. $m(\Theta) = 1$, and the problem is indeterminate. We need to get new sources which are reliable or at least they are not fully unreliable.
6) If all sources are reliable, or the unreliable sources have been discounted (in the default case), then use the DSm classic rule (which is commutative, associative, Markovian) on Boolean algebra $(\Theta, \cup, \cap, \complement)$, no matter what contradictions (or model) the problem has. I emphasize that the super-power set S^{Θ} generated by this Boolean algebra contains singletons, unions, intersections, and complements of sets.
7) If the sources are considered from a statistical point of view, use *Murphy's average rule* (and no transfer or normalization is needed).
8) In the case the model is not known (the default case), it is prudent/cautious to use the free model (i.e. all intersections between the elements of the frame of discernment are non-empty) and DSm classic rule on S^{Θ}, and later if the model is found out (i.e. the constraints of

empty intersections become known), one can adjust the conflicting mass at any time/moment using the DSm hybrid rule.
9) Now suppose the model becomes known [i.e. we find out about the contradictions (= empty intersections) or consensus (= non-empty intersections) of the problem/application]. Then :

9.1) If an intersection A∩B is not empty, we keep the mass m(A∩B) on A∩B, which means consensus (common part) between the two hypotheses A and B (i.e. both hypotheses A and B are right) [here one gets *DSmT*].

9.2) If the intersection $A \cap B = \phi$ is empty, meaning contradiction, we do the following :

9.2.1) if one knows that between these two hypotheses A and B one is right and the other is false, but we don't know which one, then one transfers the mass m(A∩B) to m(A∪B), since A∪B means at least one is right [here one gets *Yager's* if n=2, or *Dubois-Prade,* or *DSmT*];

9.2.2) if one knows that between these two hypotheses A and B one is right and the other is false, and we know which one is right, say hypothesis A is right and B is false, then one transfers the whole mass m(A∩B) to hypothesis A (nothing is transferred to B);

9.2.3) if we don't know much about them, but one has an optimistic view on hypotheses A and B, then one transfers the conflicting mass m(A∩B) to A and B (the nearest specific sets in the Specificity Chains) [using *Dempster's*, *PCR2-5*]

9.2.4) if we don't know much about them, but one has a pessimistic view on hypotheses A and B, then one transfers the conflicting mass m(A∩B) to A∪B (the more pessimistic the further one gets in the Specificity Chains: (A∩B) ⊂ A ⊂ (A∪B) ⊂ I); this is also the default case [using *DP's*, *DSm hybrid rule*, *Yager's*];
if one has a very pessimistic view on hypotheses A and B then one transfers the conflicting mass m(A∩B) to the total ignorance in a closed world [*Yager's*, *DSmT*], or to the empty set in an open world [*TBM*];

9.2.5.1) if one considers that no hypothesis between A and B is right, then one transfers the mass m(A∩B) to other non-empty sets (in the case more hypotheses do exist in the frame of discernment) - different from A, B, A∪B - for the reason that: if A and B are not right then there is a bigger chance that other hypotheses in the frame of discernment have a higher subjective probability to occur; we do this transfer in a **closed world** [DSm hybrid rule]; but, if it is an **open world**, we can transfer the mass m(A∩B) to the empty set leaving room for new possible hypotheses [here one gets *TBM*];

9.2.5.2) if one considers that none of the hypotheses A, B is right and no other hypothesis exists in the frame of discernment (i.e. n = 2 is the size of the frame of discernment), then one considers the **open world** and one transfers the mass to the empty set [here *DSmT* and *TBM* converge to each other].

Of course, this procedure is extended for any intersections of two or more sets: A∩B∩C, etc. and even for mixed sets: A∩ (B∪C), etc.

If it is a dynamic fusion in a real time and associativity and/or Markovian process are needed, use an algorithm which transforms a rule (which is based on the conjunctive rule and the transfer of the conflicting mass) into an associative and Markovian rule by storing

the previous result of the conjunctive rule and, depending of the rule, other data. Such rules are called quasi-associative and quasi-Markovian.

Some applications require the necessity of **decaying the old sources** because their information is considered to be worn out.

If some bba is not normalized (i.e. the sum of its components is < 1 as in incomplete information, or > 1 as in paraconsistent information) we can easily divide each component by the sum of the components and normalize it. But also it is possible to fusion incomplete and paraconsistent masses, and then normalize them after fusion. Or leave them unnormalized since they are incomplete or paraconsistent.

PCR5 does the most mathematically exact (in the fusion literature) redistribution of the conflicting mass to the elements involved in the conflict, redistribution which exactly

follows the tracks of the conjunctive rule.

7. Examples:

7.1. Bayesian Example:

Let $\Theta = \{A, B, C, D, E\}$ be the frame of discernment.

	A	B	C	D	E	A∩B	A∩C	A∩D	A∩E	B∩C	B∩D
						≠ ϕ	= ϕ	= ϕ	= ϕ	Not known if = or ≠ ϕ	= ϕ
						Consensus between A and B	Contradiction between A and C, but optimistic in both of them	One right, one wrong, but don't know which one	A is right, E is wrong	Don't know the exact model	Unknown any relation between B and D.
m_1	0.2	0	0.3	0.4	0.1						
m_2	0.5	0.2	0.1	0	0.2						
m_{12}	0.10	0	0.03	0	0.02	0.04	0.17	0.20	0.09	0.06	0.08
						↓ A∩B	↓ A, C	↓ A∪B	↓ A	↓ B∩C We keep the mass 0.06 on B∩C till we find out more information on the model.	↓ B∪D
m_r						0.04	0.107, 0.063	0.20	0.09	0.06	0.08
m_{UFT}	**0.324**	**0.040**	**0.119**	**0**	**0.027**	**0.04**	**0**	**0**	**0**	**0.06**	**0**
m_{lower}	0.10	0	0.03	0	0.02						

(closed world)											
m_{lower} (open world)	0.10	0	0.03	0	0.02						
m_{middle} (default)	0.10	0	0.03	0	0.02						
m_{upper}	0.400	0.084	0.178	0.227	0.111						

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 1).

	$B \cap E$	$C \cap D$	$C \cap E$	$D \cap E$	$A \cup B$	$A \cup C$	$A \cup D$	$A \cup E$	$B \cup C$
	$\neq \phi$	$= \phi$	$= \phi$	$= \phi$					
	The intersection is not empty, but neither $B \cap E$ nor $B \cup E$ interest us	Pessimistic in both C and D	Very pessimistic in both C and E	Both D and E are wrong					
m_1									
m_2									
m_{12}	0.02	0.04	0.07	0.08					
	$\downarrow$ B, E	$\downarrow$ $C \cup D$	$\downarrow$ $A \cup B \cup C \cup D \cup E$	$\downarrow$ A,B,C					
m_r	0.013, 0.007	0.04	0.07	0.027, 0.027, 0.027					
$\mathbf{m_{UFT}}$	**0**	**0**	**0**	**0**	**0**	**0**	**0.20**	**0**	**0**
m_{lower} (closed world)									
m_{lower} (open world)									
m_{middle} (default)					0.04	0.17	0.20	0.09	0.06
m_{upper}									

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 2).

	$B \cup D$	$B \cup E$	$C \cup D$	$C \cup E$	$D \cup E$	$A \cup B \cup C \cup D \cup E$	ϕ
m_1							
m_2							
m_{12}							
m_r							
$\mathbf{m_{UFT}}$	**0.08**	**0**	**0.04**	**0**	**0**	**0.07**	**0**

m_{lower} (closed world)						0.85	
m_{lower} (open world)							0.85
m_{middle} (default)	0.08	0.02	0.04	0.07	0.08		
m_{upper}							

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 3).

We keep the mass $m_{12}(B\cap C) = 0.06$ on $B\cap C$ (eleventh column in Table 1, part 1) although we don't know if the intersection $B\cap C$ is empty or not (this is considered the default model), since in the case when it is empty one considers an open world because $m_{12}(\phi)=0.06$ meaning that there might be new possible hypotheses in the frame of discernment, but if $B\cap C \neq \phi$ one considers a consensus between B and C.
Later, when finding out more information about the relation between B and C, one can transfer the mass 0.06 to $B\cup C$, or to the total ignorance I, or split it between the elements B, C, or even keep it on $B\cap C$.

$m_{12}(A\cap C)=0.17$ is redistributed to A and C using the PCR5:
a1/0.2 = c1/0.1 = 0.02(0.2+0.1),
whence a1 = 0.2(0.02/0.3) = 0.013,
c1 = 0.1(0.02/0.3) = 0.007.
a2/0.5 = c2/0.3 = 0.15(0.5+0.3),
whence a2 = 0.5(0.15/0.8) = 0.094,
c2 = 0.3(0.15/0.8) = 0.056.
Thus A gains a1+a2 = 0.013+0.0.094 = 0.107 and C gains c1+c2 = 0.007+0.056 = 0.063.

$m_{12}(B\cap E)=0.02$ is redistributed to B and E using the PCR5:
b/0.2 = e/0.1 = 0.02/(0.2+0.1),
whence b = 0.2(0.02/0.3) = 0.013,
e = 0.1(0.02/0.3) = 0.007.
Thus B gains 0.013 and E gains 0.007.

Then one sums the masses of the conjunctive rule $\mathbf{m_{12}}$ and the redistribution of conflicting masses $\mathbf{m_r}$ (according to the information we have on each intersection, model, and relationship between conflicting hypotheses) in order to get the mass of the Unification of Fusion Theories rule m_{UFT}.

$\mathbf{m_{UFT}}$, the Unification of Fusion Theories rule, is a combination of many rules and gives the optimal redistribution of the conflicting mass for each particular problem, following the given model and relationships between hypotheses; this extra-information allows the choice of the combination rule to be used for each intersection. The algorithm is presented above. $\mathbf{m_{lower}}$, the lower bound believe assignment, the most pessimistic/prudent belief, is obtained by transferring the whole conflicting mass to the total ignorance (Yager's rule) in a closed

world, or to the empty set (Smets' TBM) in an open world herein meaning that other hypotheses might belong to the frame of discernment.
$\mathbf{m_{middle}}$, the middle believe assignment, half optimistic and half pessimistic, is obtained by transferring the partial conflicting masses $m_{12}(X \cap Y)$ to the partial ignorance $X \cup Y$ (as in Dubois-Prade theory or more general as in Dezert-Smarandache theory).
Another way to compute a middle believe assignment would be to average the $\mathbf{m_{lower}}$ and $\mathbf{m_{upper}}$.
$\mathbf{m_{upper}}$, the upper bound believe assignment, the most optimistic (less prudent) belief, is obtained by transferring the masses of intersections (empty or non-empty) to the elements in the frame of discernment using the PCR5 rule of combination, i.e. $m_{12}(X \cap Y)$ is split to the elements X, Y (see Table 2). We use PCR5 because it is more exact mathematically (following backwards the tracks of the conjunctive rule) than Dempster's rule, minC, and PCR1-4.

X	$m_{12}(X)$	A	B	C	D	E
A∩B	0.040	0.020	0.020			
A∩C	0.170	0.107		0.063		
A∩D	0.200	0.111			0.089	
A∩E	0.090	0.020 0.042				0.020 0.008
B∩C	0.060		0.024	0.036		
B∩D	0.080		0.027		0.053	
B∩E	0.020		0.013			0.007
C∩D	0.040			0.008	0.032	
C∩E	0.070			0.036 0.005		0.024 0.005
D∩E	0.080				0.053	0.027
Total	0.850	0.300	0.084	0.148	0.227	0.091

Table 2. Redistribution of the intersection masses to the singletons A, B, C, D, E using the PCR5 rule only, needed to compute the upper bound belief assignment m_{upper}.

7.2. Negation/Complement Example:

Let $\Theta = \{A, B, C, D\}$ be the frame of discernment. Since $(\Theta, \cup, \cap, \mathcal{C})$ is Boolean algebra, the super-power set S^{Θ} includes complements/negations, intersections and unions. Let's note by $\mathcal{C}(B)$ the complement of B.

	A	B	D	$\mathcal{C}(B)$	A∩C {Later in the dynamic fusion process we find out that A∩C is empty }	B∪C= B	A∩B	A∩$\mathcal{C}$(B)=A
					= ϕ		= ϕ	≠ ϕ
					Unknown		Optimistic in	Consensus between

					relationship between A and C		both A and B.	A and $\mathcal{C}$(B), but A$\subset \mathcal{C}$(B)
m_1	0.2	0.3	0	0.1	0.1	0.3		
m_2	0.4	0.1	0	0.2	0.2	0.1		
m_{12}	0.08	0.09	0	0.02	0.17	0.03	0.14	0.08
					↓ A∪C	↓ B	↓ A,B	↓ A
m_r					0.17	0.03	0.082, 0.058	0.08
m_{UFT}	**0.277**	**0.318**	**0.035**	**0.020**	**0**	**0**	**0**	**0**
m_{lower} (closed world)	0.16	0.26	0	0.02	0	0		
m_{lower} (open world)	0.16	0.26	0	0.02	0	0		
m_{middle} (default)	0.16	0.26	0	0.02	0	0		
m_{upper}	0.296	0.230	0	0.126	0.219	0.129		

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 1).

	A∩ (B∪C)	B∩ $\mathcal{C}$(B)	B∩(A∩C)	$\mathcal{C}$(B)∩(A∩C)	$\mathcal{C}$(B)∩(B∪C) = $\mathcal{C}$(B)∩C	B∪(A∩C) =B
	= ϕ	= ϕ	= ϕ	= ϕ	= ϕ	
	At least one is right between A and B∪C	B is right, $\mathcal{C}$(B) is wrong	No relationship known between B and A∩C (default case)	Very pessimistic on $\mathcal{C}$(B) and A∩C	Neither $\mathcal{C}$(B) nor B∪C are right	
m_1						
m_2						
m_{12}	0.14	0.07	0.07	0.04	0.07	
	↓ A∪(B∪C)	↓ B	↓ B∪(A∩C)=B	↓ A∪B∪C∪D	↓ A, D	↓ B
m_r	0.14	0.07	0.07	0.04	0.035, 0.035	
m_{UFT}	**0**	**0**	**0**	**0**	**0**	**0**
m_{lower} (closed world)						
m_{lower} (open world)						
m_{middle} (default)						
m_{upper}						

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 2).

	A∪B	A∪C	A∪D	B∪C	B∪D	C∪D	A∪B∪C	A∪B∪C∪D	ϕ
m_1									
m_2									
m_{12}									
m_r									
m_{UFT}	**0**	**0.170**	**0**	**0**	**0**	**0**	**0.140**	**0.040**	**0**
m_{lower} (closed world)								0.56	
m_{lower} (open world)									0.56
m_{middle} (default)	0.14	0.17		0.03			0.14	0.11	
m_{upper}									

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 3).

Model of Negation/Complement Example:

$A \cap B = \phi$, $C \subset B$, $A \subset \mathcal{C}(B)$.

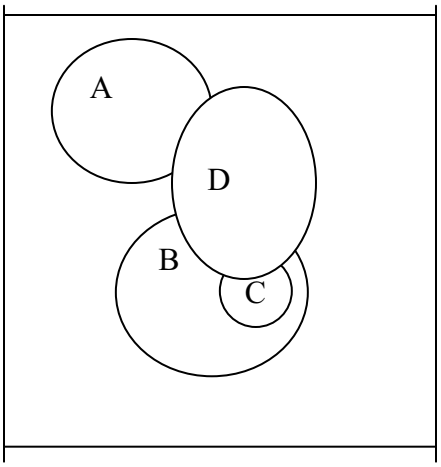

Fig. 1

$m_{12}(A \cap B) = 0.14$.
x1/0.2 = y1/0.1 = 0.02/0.3, whence x1 = 0.2(0.02/0.3) = 0.013, y1 = 0.1(0.02/0.3) = 0.007;

x2/0.4 = y2/0.3 = 0.12/0.7, whence x2 = 0.4(0.12/0.7) = 0.069, y2 = 0.3(0.12/0.7) = 0.051.
Thus, A gains 0.013+0.069 = 0.082 and B gains 0.007+0.051 = 0.058.

For the upper belief assignment m_{upper} one considered all resulted intersections from results of the conjunctive rule as empty and one transferred the partial conflicting masses to the elements involved in the conflict using PCR5.
All elements in the frame of discernment were considered non-empty.

7.3. Example with Intersection:

Look at this:

Suppose A={x<0.4}, B={0.3<x<0.6}, C={x>0.8}. The frame of discernment T={A, B, C} represents the possible cross section of a target, and there are two sensors giving the following bbas:
m1(A)=0.5, m1(B)=0.2, m1(C)=0.3.
m2(A)=0.4, m2(B)=0.4, m2(C)=0.2.

	A	B	C	A∩B= {.3<x<.4}	A∪C	B∪C
m_1	.5	.2	.3			
m_2	.4	.4	.2			
m_1&m_2 DSmT	.20	.08	.06	.28	.22	.16

We have a DSm hybrid model (one intersection A&B=nonempty).
This example proves the necessity of allowing intersections of elements in the frame of discernment. [Shafer's model doesn't apply here.]
Dezert-Smarandache Theory of Uncertain and Paradoxist Reasoning (DSmT) is the only theory which accepts intersections of elements.

7.4. Another Multi-Example of UFT:

Cases:
1. Both sources reliable: use conjunctive rule [default case]:
  1.1. A∩B≠ϕ:
    1.1.1. Consensus between A and B; mass → A∩B;
    1.1.2. Neither A∩B nor A∪B interest us; mass → A, B;
  1.2. A∩B=ϕ:
      1.2.1. Contradiction between A and B, but optimistic in both of them; mass → A, B;
      1.2.2. One right, one wrong, but don't know which one; mass → A∪B;
    1.2.3. Unknown any relation between A and B [default case]; mass → A∪B;
    1.2.4. Pessimistic in both A and B; mass → A∪B;
    1.2.5. Very pessimistic in both A and B;

1.2.5.1. Total ignorance $\supset$ A∪B; mass → A∪B∪C∪D (total ignorance);
1.2.5.2. Total ignorance = A∪B; mass → ϕ (open world);
1.2.6. A is right, B is wrong; mass → A;
1.2.7. Both A and B are wrong; mass → C, D;
1.3. Don't know if A∩B = or ≠ ϕ (don't know the exact model); mass → A∩B (keep the mass on intersection till we find out more info) [default case];
2. One source reliable, other not, but not known which one: use disjunctive rule; no normalization needed.
3. S1 reliable, S2 not reliable 20%: discount S2 for 20% and use conjunctive rule.

	A	B	A∪B	A∩B	ϕ (open world)	A∪B∪C∪D	C	D
S1	.2	.5	.3					
S2	.4	.4	.2					
S1&S2	.24	.42	.06	.28				
S1 or S2	.08	.20	.72	0				
UFT 1.1.1	.24	.42	.06	.28				
UFT 1.1.2 (PCR5)	.356	.584	.060	0				
UFT 1.2.1	.356	.584	.060	0				
UFT 1.2.2	.24	.42	.34	0				
UFT 1.2.3	.24	.42	.34	0				
UFT 1.2.4	.24	.42	.34	0				
UFT 1.2.5.1	.24	.42	.06	0	0	.28		
UFT 1.2.5.2	.24	.42	.06	0	.28			
80% S2	.32	.32	.16			.20		
UFT 1.2.6	.52	.42	.06					
UFT 1.2.7	.24	.42	.06	0			.14	.14
UFT 1.3	.24	.42	.06	.28				
UFT 2	.08	.20	.72	0				
UFT 3	.232	.436	.108	.224		0		

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Unification of Fusion Rules (UFR)

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In this short note we give a formula for the unification of a class of fusion rules based on the conjunctive and/or disjunctive rule at the first step, and afterwards the redistribution of the conflicting and/or non-conflicting mass to the non-empty sets at the second step.

Fusion of masses $m_1(.)$ and $m_2(.)$ is done directly proportional with some parameters and inversely proportional with other parameters (parameters that the hypotheses depend upon). The resulting mass is noted by $m_{UFR}(.)$.

a) If variable y is <u>directly proportional</u> with variable p, then $y=k_1 \cdot p$, where $k_1 \neq 0$ is a constant.
b) If variable y is <u>inversely proportional</u> with variable q, then $y=k_2 \cdot (1/q)$, where $k_2 \neq 0$ is a constant; we can also say herein that y is directly proportional with variable $1/q$.

In a general way, we say that if y is directly proportional with variables $p_1, p_2, \ldots, p_m$ and inversely proportionally with variables $q_1, q_2, \ldots, q_n$, then:

$y = k \cdot (p_1 \cdot p_2 \cdot \ldots \cdot p_m)/(q_1 \cdot q_2 \cdot \ldots \cdot q_n) = k \cdot P/Q$, where $P=\prod_{i=1}^{m} p_i$, $Q=\prod_{j=1}^{n} q_j$, and $k \neq 0$ is a constant.

With such notations we have a general formula for a **UFR rule**:

$$m_{UFR}(\phi) = 0, \text{ and } \forall A \in S^{\Theta} \setminus \phi \text{ one has:}$$

$$m_{UFR}(A) = \sum_{\substack{X_1, X_2 \in S^{\wedge}\Theta \\ X_1 * X_2 = A}} d(X_1 * X_2) T(X_1, X_2)$$

$$+ \frac{P(A)}{Q(A)} \sum_{\substack{X \in S^{\wedge}\Theta \setminus A \\ X * A \in Tr}} d(X * A) \frac{T(A, X)}{P(A)/Q(A) + P(X)/Q(X)}$$

where * can be an intersection or a union of sets,
$d(X*Y)$ is the degree of intersection or union,
$T(X,Y)$ is a *T-norm* fusion combination rule (extension of conjunctive or disjunctive rules),
Tr is the ensemble of sets (in majority cases they are empty sets) whose masses must be transferred,
$P(A)$ is the product of all parameters directly proportional with A,
while $Q(A)$ the product of all parameters inversely proportional with A,
$S^{\wedge}\Theta$ is the fusion space (i.e. the frame of discernment closed under union, intersection, and complement of the sets).
At the end we normalize the result.

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Unification/Combination of Image Fusion Methods

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Ming Zhang, Ling Zhang, H. D. Cheng use a novel approach, i.e. neutrosophic logic which is a generalization of fuzzy logic and especially of intuitionistic fuzzy logic, to image segmentation - following one of the authors (H. D. Cheng) together with his co-author Y. Guo previous published paper on neutrosophic approach to image thresholding.

The authors improved the watershed algorithms using a neutrosophic approach (i.e. they consider the objects as the T set, the background as the F set, and the edges as the I set); their method is less sensitive to noise and performs better on non-uniform images since it uses the indeterminacy (I) from neutrosophic logic and set, while this indeterminacy is not featured in fuzzy logic.

Since using neutrosophic logic/set/probability/statistics is a new trend in image processing and the authors prove that the neutrosophic approach is better than other methods (such as: histogram-based, edge-based, region-based, model-based, watershed/topographic in MatLab or using Toboggan-Based) I recommend the publication of this paper.

Next step for these authors would be to use the neutrosophic approach to image registration and similarly compare the result with those obtained from other methods.

Interesting also is to use the neutrosophic approach to the control theory.

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An Algorithm for Quasi-Associative and Quasi-Markovian Rules of Combination in Information Fusion

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Abstract: *In this paper one proposes a simple algorithm of combining the fusion rules, those rules which first use the conjunctive rule and then the transfer of conflicting mass to the non-empty sets, in such a way that they gain the property of associativity and fulfill the Markovian requirement for dynamic fusion.*
Also, a new fusion rule, SDL-improved, is presented.

Keywords: Conjunctive rule, partial and total conflicts, Dempster's rule, Yager's rule, TBM, Dubois-Prade's rule, Dezert-Smarandache classic and hybrid rules, SDL-improved rule, quasi-associative, quasi-Markovian, fusion algorithm

ACM Classification: I.2.4.

1. Introduction.

We first present the formulas for the conjunctive rule and total conflict, then try to unify some theories using an adequate notation. Afterwards, we propose an easy fusion algorithm in order to transform a quasi-associative rule into an associative rule, and a quasi-Markovian rule into a Markovian rule. One gives examples using the DSm classic and hybrid rules and SDL-improved rule within DSmT. One studies the impact of the VBF on SDLi and one makes a short discussion on the degree of the fusion rules' ad-hoc-ity

2. The Conjunctive Rule:

For $n \geq 2$ let $T = \{t_1, t_2, \ldots, t_n\}$ be the frame of discernment of the fusion problem under consideration.
We need to make the remark that in the case when these n elementary hypotheses $t_1, t_2, \ldots, t_n$ are *exhaustive and exclusive* one can use the Dempster-Shafer Theory, Yager's, TBM,

Dubois-Prade Theory, while for the case when the hypotheses are *not exclusive* one can use Dezert-Smarandache Theory, while for non-exhaustivity one uses TBM.

Let m: $2^T \rightarrow [0, 1]$ be a basic belief assignment or mass.
The conjunctive rule works in any of these theories, and it is the following in the first theories:

$$\text{for } A \in 2^T,\ m_c(A) = \sum_{\substack{X,Y \in 2^{\wedge}T \\ X \cap Y = A}} m_1(X) m_2(Y) \qquad (1)$$

while in DSmT the formula is similar, but instead of the power set 2^T one uses the hyper-power set D^T, and similarly m: $D^T \rightarrow [0, 1]$ be a basic belief assignment or mass:

$$\text{for } A \in D^T,\ m_c(A) = \sum_{\substack{X,Y \in D^{\wedge}T \\ X \cap Y = A}} m_1(X) m_2(Y). \qquad (2)$$

The power set is closed under $\cup$, while the hyper-power set is closed under both $\cup$ and $\cap$.
Formula (2) allows the use of intersection of sets (for the non-exclusive hypotheses) and it is called DSm classic rule.

The conjunctive rule (1) and its extension (2) to DSmT are associative, which is a nice property needed in fusion combination that we need to extend to other rules derived from it.
Unfortunately, only three fusion rules derived from the conjunctive rule are known as associative, i.e. Dempster's rule, Smets's TBM's rule, and Dezert-Smarandache classic rule, the others are not.
For unification of theories let's note by G either 2^T or D^T depending on theories.

The conflicting mass k_{12} is computed similarly:

$$k_{12} = m_c(\emptyset) = \sum_{\substack{X,Y \in G \\ X \cap Y = \phi}} m_1(X) m_2(Y). \qquad (3)$$

Formulas (1), (2), (3) can be generalized for any number of masses $s \geqslant 2$.

3. Associativity.
The propose of this article is to show a simple method to combine the masses in order to keep the associativity and the Markovian requirement, important properties for information fusion.

Let $m_1, m_2, m_3 : G \rightarrow [0, 1]$ be any three masses, and a fusion rule denoted by $\oplus$ operating on these masses. One says that this *fusion rule is associative* if:
$((m_1 \oplus m_2) \oplus m_3)(A) = (m_1 \oplus (m_2 \oplus m_3))(A)$ for all $A \in G$, (4)
which is also equal to $(m_1 \oplus m_2 \oplus m_3)(A)$ for all $A \in G$. (5)

4. Markovian Requirement.

Let $m_1, m_2, \ldots, m_k : G \rightarrow [0, 1]$ be any $k \geq 2$ masses, and a fusion rule denoted by $\oplus$ operating on these masses. One says that this *fusion rule satisfies Markovian requirement* if: $(m_1 \oplus m_2 \oplus \ldots \oplus m_n)(A) = ((m_1 \oplus m_2 \oplus \ldots \oplus m_{n-1}) \oplus m_n)(A)$ for all $A \in G$. (6)

Similarly, only three fusion rules derived from the conjunctive rule are known satisfying the Markovian Requirement, i.e. Dempster's rule, Smets's TBM's rule, and Dezert-Smarandache classic rule.

The below algorithm will help transform a rule into a Markovian rule.

5. Fusion Algorithm.

A trivial algorithm is proposed below in order to restore the associativity and Markovian properties to any rule derived from the conjunctive rule.

Let's consider a rule ® formed by using: first the conjunctive rule, noted by ©, and second the transfer of the conflicting mass to non-empty sets, noted by operator "O" (no matter how the transfer is done, either proportionally with some parameters, or transferred to partial or total ignorances and/or to the empty set; if all conflicting mass is transferred to the empty set, as in Smets's rule, there is no need for transformation into an associative or Markovian rule since Smets's rule has already these properties).

Clearly ® = O(©).

The idea is simple, we store the conjunctive rule's result (before doing the transfer) and, when a new mass arises, one combines this new mass with the conjunctive rule's result, not with the result after the transfer of conflicting mass.

Let's have two masses m_1, m_2 defined as above.

a) One applies the conjunctive rule to m_1 and m_2 and *one stores* the result:

$m_1 © m_2 = m_{c(1,2)}$ (by notation).

b) One applies the operator O of transferring conflicting mass to the non-empty sets, i.e. $O(m_{c(12)})$.

This calculation completely does the work of our fusion rule, i.e. $m_1 ® m_2 = O(m_{c(12)})$ that we compute for decision-making proposes.

c) When a new mass, m_3, arises, we combine using the conjunctive rule this mass m_3 with the previous conjunctive rule's result $m_{c(12)}$, not with $O(m_{c(12)})$. Therefore:

$m_{c(1,2)} © m_3 = m_{c(c(1,2),3)}$ (by notation).

One stores this results, while deleting the previous one stored.

d) Now again we apply the operator O to transfer the conflicting mass, i.e. compute $ô(m_{c(c(1,2),3)})$ needed for decision-making.

e) …And so one the algorithm is continued for any number $n \geqslant 3$ of masses.

The properties of the conjunctive rule, i.e. associativity and satisfaction of the Markovian requirement, are transmitted to the fusion rule ® too.

This is the algorithm we use in DSmT in order to conserve the associativity and Markovian requirement for DSm hybrid rule and SDL improved rule for $n \geqslant 3$.

Depending on the type of problem to be solved we can use in DSmT either the hybrid rule, or the SDL rule, or a combination of both (i.e., partial conflicting mass is transferred using DSm hybrid, other conflicting mass is transferred using SDL improved rule).

Yet, this easy fusion algorithm can be extended to any rule which is composed from a conjunctive rule first and a transfer of conflicting mass second, returning the associativity and Markovian properties to that rule.

One can remark that the algorithm gives the same result if one applies the rule ® to $n \geqslant 3$ masses together, and then one does the transfer of conflicting mass.
Within DSmT we designed *fusion rules that can transfer a part of the conflicting mass to partial or total ignorance and the other part of the conflicting mass to non-empty initial sets*, depending on the type of application.

A non-associative rule that can be transformed through this algorithm into an associative rule is called *quasi-associative rule*. And similarly, a non-Markovian rule than can be transformed through this algorithm into a Markovian rule is called *quasi-Markovian rule*.

6. SDL-improved Rule.
Let $T = \{t_1, t_2, \ldots, t_n\}$ be the frame of discernment and two masses $m_1, m_2 : G \rightarrow [0, 1]$. One applies the conjunctive rule (1) or (2) depending on theory, then one calculates the conflicting mass (3). In SDL improved rule one transfers partial conflicting masses, instead of the total conflicting mass. If an intersection is empty, say $A \cap B = \emptyset$, then the mass $m(A \cap B)$ is transferred to A and B proportionally with respect to the non-zero sum of masses assigned to A and respectively B by the masses m_1, m_2. Similarly, if another intersection, say $A \cap C \cap D = \emptyset$, then again the mass $m(A \cap C \cap D)$ is transferred to A, C, and D proportionally with respect to the non-zero sum of masses assigned to A, C and respectively D by the masses m_1, m_2. And so on 'til all conflicting mass is distributed. Then one cumulates the corresponding masses to each non-empty set.

For two masses one has the formula: (7)

$$\text{for } \emptyset \neq A \in D^T,\ m_{SDLi}(A) = \sum_{\substack{X,Y \in G \\ X \cap Y = A}} m_1(X) m_2(Y) + c_{12}(A). \sum_{\substack{X \in G \\ X \cap A = \phi}} \frac{m_1(X) m_2(A) + m_1(A) m_2(X)}{c_{12}(A) + c_{12}(X)}$$

where $c_{12}(A)$ is the non-zero sum of the mass matrix column corresponding to the set A, i.e.
$c_{12}(A) = m_1(A) + m_2(A) \neq 0$. (8)

For more masses one applies the algorithm to formulas (7) and (8).

7. Ad-Hoc-ity of Fusion Rules.
Each fusion rule is more or less ad-hoc. Same thing for SDL improved. There is up to the present no rule that fully satisfies everybody. Let's analyze some of them.
Dempster's rule transfers the conflicting mass to non-empty sets proportionally with their resulting masses. What is the reasoning for doing this? Just to swallow the masses of non-empty sets in order to sum up to 1?

Smets's rule transfers the conflicting mass to the empty set. Why? Because, he says, we consider on open world where unknown hypotheses might be. Not convincing.
Yager's rule transfers the conflicting mass to the total ignorance. Should the conflicting mass be ignored?
Dubois-Prade's rule and DSm hybrid rule transfers the conflicting mass to the partial and total ignorances. Not completely justified either.
SDL improved rule is based on partial conflicting masses, transferred to the corresponding sets proportionally with respect to the non-zero sums of their assigned masses. But other weighting coefficients can be found. Inagaki (1991), Lefevre-Colot-Vannoorenberghe (2002) proved that there are infinitely many fusion rules based on the conjunctive rule and then on the transfer of the conflicting mass, all of them depending on the weighting coefficients that transfer that conflicting mass. How to choose them, what parameters should they rely on – that's the question! There is not a measure for this.
In my opinion, neither DSm hybrid rule nor SDLi rule are not more ad-hoc than other fusion rules.
"No matter how you do, people will have objections" (Wu Li).

8. Numerical Examples.
We show how it is possible to use the above fusion algorithm in order to transform a quasi-associative and quasi-Markovian rule into an associative and Markovian one.
Let T = {A, B, C}, all hypotheses exclusive, and two masses m_1, m_2 that form the corresponding mass matrix:

	A	B	A∪C
m_1	0.4	0.5	0.1
m_2	0.6	0.2	0.2

8.1 Let's take the DSm hybride rule:
8.1.1. Let's check the associativity:
a) First we use the DSm classic rule and we get at time t1:
$m_{DSmC12}(A)=0.38$, $m_{DSmC12}(B)=0.10$, $m_{DSmC12}(A\cup C)=0.02$, $m_{DSmC12}(A\cap B)=0.38$, $m_{DSmC12}(B\cap (A\cup C))=0.12$, and one stores this result. (S1)
b) One uses the DSm hybrid rule and we get:
$m_{DSmH12}(A)=0.38$, $m_{DSmH12}(B)=0.10$, $m_{DSmH12}(A\cup C)=0.02$, $m_{DSmH12}(A\cup B)=0.38$, $m_{DSmH12}(A\cup B\cup C)=0.12$. This result was computed because it is needed for decision making on two sources/masses only. (R1)
c) A new masses, m_3, arise at time t2, and has to be taken into consideration, where $m_3(A)=0.7$, $m_3(B)=0.2$, $m_3(A\cup C)=0.1$.
Now one combines the result stored at (S1) with m_3, using DSm classic rule, and we get:
$m_{DSmC(12)3}(A)=0.318$, $m_{DSmC(12)3}(B)=0.020$, $m_{DSmC(12)3}(A\cup C)=0.002$, $m_{DSmC(12)3}(A\cap B)=0.610$, $m_{DSmC(12)3}(B\cap (A\cup C))=0.050$, and one stores this result, (S2) while deleting (S1).
d) One uses the DSm hybrid rule and we get:
$m_{DSmH(12)3}(A)=0.318$, $m_{DSmH(12)3}(B)=0.020$, $m_{DSmH(12)3}(A\cup C)=0.002$, $m_{DSmH(12)3}(A\cup B)=0.610$, $m_{DSmH(12)3}(A\cup B\cup C)=0.050$. This result was also computed because it is needed for decision making on three sources/masses only. (R2)
e) And so on for as many masses as needed.

First combining the last masses, m_2, m_3, one gets:
$m_{DSmC23}(A)=0.62$, $m_{DSmC23}(B)=0.04$, $m_{DSmC23}(A\cup C)=0.02$, $m_{DSmC23}(A\cap B)=0.26$, $m_{DSmC23}(B\cap (A\cup C))=0.06$, and one stores this result. (S3)
Using DSm hybrid one gets:
$m_{DSmH23}(A)=0.62$, $m_{DSmH23}(B)=0.04$, $m_{DSmH23}(A\cup C)=0.02$, $m_{DSmH23}(A\cup B)=0.26$, $m_{DSmH23}(A\cup B\cup C)=0.06$.
Then, combining m_1 with m_{DSmC23} {stored at (S3)} using DSm classic and then using DSm hybrid one obtain the same result (R2).

If one applies the DSm hybride rule to all three masses together one gets the same result (R2).

We showed on this example that DSm hybrid applied within the algorithm is associative (i.e. using the notation DSmHa one has):
$DSmHa((m_1, m_2), m_3) = DSmHa(m_1, (m_2, m_3)) = DSmHa(m_1, m_2, m_3)$.

8.1.2. Let's check the Markov requirement:
a) Combining three masses together using DSm classic:

	A	B	A∪C	
m_1	0.4	0.5	0.1	(M1)
m_2	0.6	0.2	0.2	
m_3	0.7	0.2	0.1	

one gets as before:
$m_{DSmC123}(A)=0.318$, $m_{DSmC123}(B)=0.020$, $m_{DSmC123}(A\cup C)=0.002$, $m_{DSmC123}(A\cap B)=0.610$, $m_{DSmC123}(B\cap (A\cup C))=0.050$, and one stores this result in (S2).
b) One uses the DSm hybrid rule to transfer the conflicting mass and we get:
$m_{DSmH123}(A)=0.318$, $m_{DSmH123}(B)=0.020$, $m_{DSmH123}(A\cup C)=0.002$, $m_{DSmH123}(A\cup B)=0.610$, $m_{DSmH123}(A\cup B\cup C)=0.050$.
c) Suppose a new mass m_4 arises, $m_4(A)=0.5$, $m_4(B)=0.5$, $m_4(A\cup C)=0$.
Use DSm classic to combine m_4 with $m_{DSmC123}$ and one gets:
$m_{DSmC(123)4}(A)=0.160$, $m_{DSmC(123)4}(B)=0.010$, $m_{DSmC(123)4}(A\cup C)=0$, $m_{DSmC(123)4}(A\cap B)=0.804$, $m_{DSmC(123)4}(B\cap (A\cup C))=0.026$, and one stores this result in (S3).
d) Use DSm hybrid rule:
$m_{DSmH(123)4}(A)=0.160$, $m_{DSmH(123)4}(B)=0.010$, $m_{DSmH(123)4}(A\cup C)=0$, $m_{DSmH(123)4}(A\cup B)=0.804$, $m_{DSmH(123)4}(A\cup B\cup C)=0.026$. (R4)

Now, if one combines all previous four masses, m_1, m_2, m_3, m_4, together using first the DSm classic then the DSm hybrid one still get (R4). Whence the Markovian requirement.
We didn't take into account any discounting of masses.

8.2. Let's use the SDL improved rule on the same example.
a) One considers the above mass matrix (M1) and one combines m_1 and m_2 using DSm classic and one gets as before:
$m_{DSmC12}(A)=0.38$, $m_{DSmC12}(B)=0.10$, $m_{DSmC12}(A\cup C)=0.02$, $m_{DSmC12}(A\cap B)=0.38$, $m_{DSmC12}(B\cap (A\cup C))=0.12$, and one stores this result in (S1).

b) One transfers the partial conflicting mass 0.38 to A and B respectively:
x/1 = y/0.7 = 0.38/1.8; whence x=0.223529, y=0.156471.
One transfers the other conflicting mass 0.12 to B and A∪C respectively:
z/0.7 = w/0.3 = 0.12/1; whence z=0.084, w=0.036.
One cumulates them to the corresponding sets and one gets:
$m_{SDLi12}(A) = 0.38+0.223529 = 0.603529$;
$m_{SDLi12}(B) = 0.10+0.156471 + 0.084 = 0.340471$;
$m_{SDLi12}(A\cup C) = 0.02+0.036 = 0.056000$.
c) One uses the DSm classic rule to combine the above m_3 and the result in (S1) and one gets again:
$m_{DSmC(12)3}(A)=0.318$, $m_{DSmC(12)3}(B)=0.020$, $m_{DSmC(12)3}(A\cup C)=0.002$,
$m_{DSmC(12)3}(A\cap B)=0.610$, $m_{DSmC(12)3}(B\cap (A\cup C))=0.050$, and one stores this result in (S2) while deleting (S1).
d) One transfers the partial conflicting masses 0.610 to A and B respectively, and 0.050 to B and A∪C respectively. Then one cumulates the corresponding masses and one gets:
$m_{SDLi(12)3}(A) = 0.716846$;
$m_{SDLi(12)3}(B) = 0.265769$;
$m_{SDLi(12)3}(A\cup C) = 0.017385$.

Same result we obtain if one combine first m_2 and m_3, and the result combine with m_1, or if we combine all three masses m_1, m_2, m_3 together.

9. Vacuous Belief Function.
SDLi seems to satisfy Smets's impact of VBF (Vacuum Belief Function. i.e. m(T)=1), because there is no partial conflict ever between the total ignorance T and any of the sets of G. Since in SDLi the transfer is done after each partial conflict, T will receive no mass, not being involved in any partial conflict. Thus VBF acts as a neutral elements with respect with the composition of masses using SDLi. The end combination does not depend on the number of VBF's included in the combination.
Let's check this on the previous example. Considering the first two masses m_1 and m_2 in (M1) and using SDLi one got: $m_{SDLi12}(A) = 0.603529$, $m_{SDLi12}(B) = 0.340471$, $m_{SDLi12}(A\cup C) = 0.056000$.
Now let's combine the VBF too:

	A	B	A∪C	A∪B∪C	
VBF	0	0	0	1	(M2)
m_1	0.4	0.5	0.1	0	
m_2	0.6	0.2	0.2	0	

a) One uses the DSm classic rule to combine all three of them and one gets again:
$m_{DSmC(VBF12)}(A)=0.38$, $m_{DSmC(VBF12)}(B)=0.10$, $m_{DSmC(VBF12)}(A\cup C)=0.02$,
$m_{DSmC(VBF12)}(A\cap B)=0.38$, $m_{DSmC(VBF12)}(B\cap (A\cup C))=0.12$, $m_{DSmC(VBF12)}(A\cup B\cup C)=0$ and one stores this result in (S1).
b) One transfers the partial conflicting mass 0.38 to A and B respectively:
x/1 = y/0.7 = 0.38/1.8; whence x=0.223529, y=0.156471.
One transfers the other conflicting mass 0.12 to B and A∪C respectively:
z/0.7 = w/0.3 = 0.12/1; whence z=0.084, w=0.036.

Therefore nothing is transferred to the mass of A∪B∪C, then the results is the same as above: $m_{SDLi12}(A) = 0.603529$, $m_{SDLi12}(B) = 0.340471$, $m_{SDLi12}(A \cup C) = 0.056000$.

10. Conclusion.

We propose an elementary fusion algorithm that transforms any fusion rule (which first uses the conjunctive rule and then the transfer of conflicting masses to non-empty sets, except for Smets's rule) to an associative and Markovian rule. This is very important in information fusion since the order of combination of masses should not matter, and for the Markovian requirement the algorithm allows the storage of information of all previous masses into the last result (therefore not necessarily to store all the masses), which later will be combined with the new mass.

In DSmT, using this fusion algorithm for $n \geqslant 3$ sources, the DSm hybrid rule and SDLi are commutative, associative, Markovian, and SDLi also satisfies the impact of vacuous belief function.

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Degree of Uncertainty of a Set and of a Mass

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Abstract.
In this paper we use extend Harley's measure of uncertainty of a set and of mass to the degree of uncertainty of a set and of a mass (bba).

Measure of Uncertainty of a Set.

In *DST* (Dempster-Shafer's Theory), Hartley defined the measure of uncertainty of a set A by:

$$I(A) = \log_2 |A|, \text{ for } A \in 2^\theta \setminus \{\Phi\},$$

where $|A|$ is the cardinal of the set A .

We can extend it to *DSmT* in the same way:

$$I(A) = \log_2 |A|, \text{ for } A \in G^\theta \setminus \{\Phi\}$$

where G^θ is the super-power set, and $|A|$ means the *DSm* cardinal of the set A .

We even improve it to:

$$\cup_d^s : G^\theta \setminus \{\Phi\} \to [0,1]$$

If A is a singleton, i.e. $|A| = 1$, then $\cup_d^s(A) = 0$ (minimum degree of uncertainty of a set),

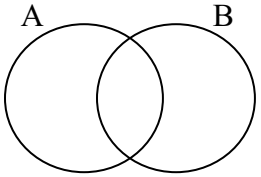
For the total ignorance I_t , since $|I_t|$ is the maximum cardinal, we get $\cup_d^s(I_t) = 1$ (maximum degree of uncertainty of a set).

For all other sets X from $G^\theta \setminus \{\Phi\}$, whose cardinal is in between 1 and $|I_t|$, we have $0 < \cup_d^s(X) < 1$.

We consider our degree of uncertainty of a set work better than Hartley Measure since it is referred to the frame of discernment.

Let's see an **Example 1**.

If $\theta = \{A, B\}$ and $A \cap B \neq \Phi$, we have the model

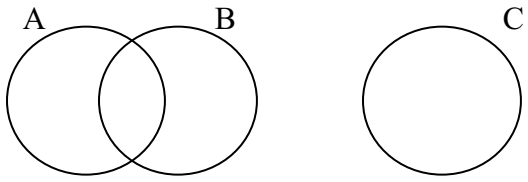


$$I(A) = \log_2 |A| = \log_2 2 = 1$$

While $\bigcup_d^s (A) = \dfrac{\log_2 |A|}{\log_2 |A \cup B|} = \dfrac{\log_2 2}{\log_2 3} = 0.63093$

Example 2.

If $\theta = \{A, B, C\}$, and $A \cap B \neq \Phi$, but $A \cap C = \Phi$, $B \cap C = \Phi$, we have the model



$I(A) = \log_2 |A| = 1$ as in Example 1.

While $\bigcup_d^s (A) = \dfrac{\log_2 |A|}{\log_2 |A \cup B \cup C|} = \dfrac{\log_2 2}{\log_2 4} = \dfrac{1}{2} = 0.5 < 0.63093$

It is normal to have a smaller degree of uncertainty of set A when the frame of discernment is larger, since herein the total ignorance has a bigger cardinal.

Generalized Hartley Measure of uncertainty for masses is defined as:

$$GH(m) = \sum_{A \in 2^\theta \setminus \{\Phi\}} m(A) \log_2 |A|$$

In *DST* we simply extend it in *DSmT* as:

$$GH(m) = \sum_{A \in G^\theta \setminus \{\Phi\}} m(A) \log_2 |A|$$

Degree of Uncertainty of a mass.

We go further and define a degree of uncertainty of a mass m as

$$\bigcup_d^M (m) = \sum_{A \in G^\theta \setminus \{\Phi\}} m(A) \cdot \frac{\log_2 |A|}{\log_2 |I_t|}$$

where I_t is the total ignorance.

If $m(\cdot)$ is a mass whose focal elements are only singletons then $\bigcup_d^M (m) = 0$ (minimum uncertainty degree of a mass).

If $m(I_t) = 1$, then $\bigcup_d^M (m) = 1$ (maximum uncertainty degree of a mass).

For all other masses $m(\cdot)$ we have $0 < \bigcup_d^M (m) < 1$.

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Fusion of Masses Defined on Infinite Countable Frames of Discernment

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Abstract.
In this paper we introduce for the first time the fusion of information on infinite discrete frames of discernment and we give general results of the fusion of two such masses using the Dempster's rule and the PCR5 rule for Bayesian and non-Bayesian cases.

Introduction.

Let $\theta = \{x_1, x_2, ..., x_i, ...x_\infty\}$ be an infinite countable frame of discernment, with $x_i \cap x_j = \Phi$ for $i \neq j$, and $m_1(\cdot)$, $m_2(\cdot)$ two masses, defined as follows:

$$m_1(x_i) = a_i \in [0,1] \text{ and } m_2(x_i) = b_i \in [0,1] \text{ for all } i \in \{1,2,...,i,...\infty\},$$

such that

$$\sum_{i=1}^{\infty} m_1(x_i) = 1 \text{ and } \sum_{i=1}^{\infty} m_2(x_i) = 1,$$

therefore $m_1(\cdot)$ and $m_2(\cdot)$ are normalized.

Bayesian masses.

1. Let's fusion $m_1(\cdot)$ and $m_2(\cdot)$, two Bayesian masses:

	x_1	x_2	...	x_i	...	x_j	...	x_∞	Φ(*conflicting mass*)
m_1	a_1	a_2	...	a_i	...	a_j	.....		
m_2	b_1	b_2	...	b_i	...	b_j	.....		
m_{12}	a_1b_1	a_2b_2	...	a_ib_i	...	a_jb_j	.....		$1 - \sum_{i=1}^{\infty} a_ib_i$

where $m_{12}(\cdot)$ represents the conjunctive rule fusion of $m_1(\cdot)$ and $m_2(\cdot)$.

a) If we use Dempster's rule to normalize $m_{12}(\cdot)$, we need to divide each $m_{12}(x_i)$ by the sum of masses of all non-null elements, and we get:

$$m_{12DS}(x_i) = \frac{a_ib_i}{\sum_{i=1}^{\infty} a_ib_i},$$

for all i.

b) Using PCR_5 the redistribution of the conflicting mass $a_i b_j + b_i a_j$ between x_i and x_j (for all $j \neq i$) is done in the following way:

$$\frac{\alpha_i}{a_i} = \frac{\alpha_j}{b_j} = \frac{a_i b_j}{a_i + b_j}, \text{ whence } \alpha_i = \frac{a_i^2 b_j}{a_i + b_j}$$

and

$$\frac{\beta_i}{b_i} = \frac{\beta_j}{a_j} = \frac{b_i a_j}{b_i + a_j}, \text{ whence } \beta_i = \frac{b_i^2 a_j}{b_i + a_j}.$$

Therefore

$$m_{12\,PCR_5}(x_i) = a_i b_i + \sum_{\substack{j=1 \\ j \neq i}}^{\infty} \left(\frac{a_i^2 b_j}{a_i + b_j} + \frac{a_j b_i^2}{a_j + b_i} \right),$$

for all i.

Non-Bayesian masses.

2. Let's consider two <u>non-Bayesian masses</u> $m_3(\cdot)$ and $m_4(\cdot)$:

	x_1	x_2	...	x_i	...	x_j	...	x_∞	θ	Φ(*conflicting mass*)
m_3	c_1	c_2	...	c_i	...	c_j	.....		C	
m_4	d_1	d_2	...	d_i	...	d_j	.....		D	
m_{34}	.....			$c_i d_i + c_i D + C d_i$			.....		CD	$1 - CD - \sum_{i=1}^{\infty} (c_i d_i + c_i D + C d_i)$

where $m_3(x_i) = c_i \in [0,1]$ for all i, and $m_3(\theta) = C \in [0,1]$,

and $m_4(x_i) = d_i \in [0,1]$ for all i, and $m_4(\theta) = D \in [0,1]$,

such that $m_3(\cdot)$ and $m_4(\cdot)$ are normalized:

$$C + \sum_{i=1}^{\infty} c_i = 1 \text{ and } D + \sum_{i=1}^{\infty} d_i = 1.$$

$m_{34}(x_i) = c_i d_i + c_i D + C d_i$ for all $i \in \{1, 2, ..., \infty\}$, and $m_{34}(\theta) = C \cdot D$, where $m_{34}(\cdot)$ represents the conjunctive combination rule.

a) If we use the Dempster's rule to normalize, we get:

$$m_{34\,DS}(x_i) = \frac{c_i d_i + c_i D + C d_i}{CD + \sum_{i=1}^{\infty} (c_i d_i + c_i D + C d_i)}$$

for all i,

and

$$m_{34DS}(\theta) = \frac{CD}{CD + \sum_{i=1}^{\infty}\left(c_i d_i + c_i D + C d_i\right)}.$$

b) If we use PCR_5, we similarly transfer the conflicting mass as in the previous 1.b) case, and we get:

$$m_{34PCR_5}(x_i) = c_i d_i + c_i D + C d_{ii} + \sum_{\substack{j=1 \\ j\neq i}}^{\infty}\left(\frac{c_i^2 d_j}{c_i + d_j} + \frac{c_j d_i^2}{c_j + d_i}\right)$$

for all i,

and $\quad m_{34PCR_5}(\theta) = C \cdot D$

Numerical Examples.

We consider infinite positive geometrical series whose ratio $0 < r < 1$ as masses for the sets $x_1, x_2, ..., x_\infty$, so the series are congruent:

If $P_1, P_2, ..., P_n ...$ is an infinite positive geometrical series whose ratio $0 < r < 1$, then

$$\sum_{i=1}^{\infty} P_i = \frac{P_1}{1-r}$$

Example 1 (Bayesian).

Let $m_1(x_i) = \frac{1}{2^i}$ for all $i \in \{1, 2, ..., \infty\}$.

$$\sum_{i=1}^{\infty} m_1(x_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

since the ratio of this infinite positive geometric series is $\frac{1}{2}$.

And $m_2(x_i) = \frac{2}{3^i}$ for all $i \in \{1, 2, ..., \infty\}$

$$\sum_{i=1}^{\infty} m_2(x_i) = \sum_{i=1}^{\infty} \frac{2}{3^i} = \frac{\frac{2}{3}}{1-\frac{1}{3}} = 1$$

since the ratio of this infinite positive geometric series is $\frac{1}{3}$.

	x_1	x_2	...	x_i	...	x_j	...	x_∞	Φ
m_1	$\frac{1}{2}$	$\frac{1}{2^2}$	...	$\frac{1}{2^i}$	...	$\frac{1}{2^j}$	.....		
m_2	$\frac{2}{3}$	$\frac{2}{3^2}$	...	$\frac{2}{3^i}$	...	$\frac{2}{3^j}$	.....		
m_{12}	$\frac{2}{6}$	$\frac{2}{6^2}$	...	$\frac{2}{6^i}$	..	$\frac{2}{6^j}$	.....		$1-\sum_{i=1}^{\infty}\frac{2}{6^i}=1-\frac{\frac{2}{6}}{1-\frac{1}{6}}=\frac{3}{5}$

$m_{12}(\cdot)$ is the conjunctive rule.

a) Normalizing with the Dempster's we get:

$$m_{12DS}(x_i)=\frac{\frac{2}{6^i}}{\sum_{i=1}^{\infty}\frac{2}{6^i}}=\frac{\frac{2}{6^i}}{\frac{\frac{2}{6}}{1-\frac{1}{6}}}=\frac{2}{6^i}\cdot\frac{5}{2}=\frac{5}{6^i}$$

for all i.

b) Normalizing with PCR_5 we get:

$$m_{12PCR_5}(x_i)=\frac{2}{6^i}+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left(\frac{\frac{1}{2^{2i}}\cdot\frac{2}{3^j}}{\frac{1}{2^i}+\frac{2}{3^j}}+\frac{\frac{1}{2^j}\cdot\frac{4}{6^{2i}}}{\frac{1}{2^j}+\frac{2}{3^i}}\right)$$

Example 2 (non-Bayesian).

Let $m_3(x_i)=\frac{1}{3^i}$ for all $i\in\{1,2,....,\infty\}$, and $m_3(\theta)=\frac{1}{2}$.

$$m_3(\theta)+\sum_{i=1}^{\infty}m_3(x_i)=\frac{1}{2}+\sum_{i=1}^{\infty}\frac{1}{3^i}=\frac{1}{2}+\frac{\frac{1}{3}}{1-\frac{1}{3}}=1,$$

so $m_3(\cdot)$ is normalized.

And $m_4(x_i)=\frac{1}{4^i}$ for all i, and $m_4(\theta)=\frac{2}{3}$.

$$m_4(\theta)+\sum_{i=1}^{\infty}m_4(x_i)=\frac{2}{3}+\sum_{i=1}^{\infty}\frac{1}{4^i}=\frac{2}{3}+\frac{\frac{1}{4}}{1-\frac{1}{4}}=1,$$

so $m_4(\cdot)$ is normalized.

	x_1	x_2	...	x_i	...	x_j	...	x_∞	θ	Φ
m_3	$\frac{1}{3}$	$\frac{1}{3^2}$	...	$\frac{1}{3^i}$	...	$\frac{1}{3^j}$	.....		$\frac{1}{2}$	
m_4	$\frac{1}{4}$	$\frac{1}{4^2}$	...	$\frac{1}{4^i}$	...	$\frac{1}{4^j}$	.....		$\frac{2}{3}$	
m_{34}	.....			$\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}$	.....				$\frac{1}{6}$	$1-\frac{1}{6}-\sum_{i=1}^{\infty}\left(\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}\right)=$

$$=\frac{5}{6}-\frac{\frac{1}{12}}{1-\frac{1}{12}}-\frac{\frac{2}{3^2}}{1-\frac{1}{3}}-\frac{\frac{1}{2\cdot 4}}{1-\frac{1}{4}}=$$

$$=\frac{5}{6}-\frac{1}{11}-\frac{1}{3}-\frac{1}{6}=\frac{8}{33}\ conflicting\ mass$$

a) Normalizing with Dempster's rule we get:

$$m_{34DS}(x_i)=\frac{33}{25}\left(\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}\right)$$

for all i,
and

$$m_{34DS}(\theta)=\frac{33}{25}\cdot\frac{1}{6}=\frac{33}{150}.$$

b) Normalizing with PCR_5 we get

$$m_{34PCR_5}(x_i)=\frac{1}{12^i}+\frac{2}{3^{i+1}}+\frac{1}{2\cdot 4^i}+\sum_{\substack{j=1\\ j\neq i}}^{\infty}\left(\frac{\frac{1}{3^{2i}}\cdot\frac{1}{4^j}}{\frac{1}{3^i}+\frac{1}{4^j}}+\frac{\frac{1}{3^j}\cdot\frac{1}{4^{2i}}}{\frac{1}{3^j}+\frac{1}{4^i}}\right)$$

for all i,
and

$$m_{34PCR_5}(\theta)=\frac{1}{6}.$$

References:

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A Simple Proportional Conflict Redistribution Rule

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Abstract – *One proposes a first alternative rule of combination to WAO (Weighted Average Operator) proposed recently by Josang, Daniel and Vannoorenberghe, called Proportional Conflict Redistribution rule (denoted PCR1). PCR1 and WAO are particular cases of WO (the Weighted Operator) because the conflicting mass is redistributed with respect to some weighting factors. In this first PCR rule, the proportionalization is done for each non-empty set with respect to the non-zero sum of its corresponding mass matrix - instead of its mass column average as in WAO, but the results are the same as Ph. Smets has pointed out. Also, we extend WAO (which herein gives no solution) for the degenerate case when all column sums of all non-empty sets are zero, and then the conflicting mass is transferred to the non-empty disjunctive form of all non-empty sets together; but if this disjunctive form happens to be empty, then one considers an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set. In addition to WAO, we propose a general formula for PCR1 (WAO for non-degenerate cases). Several numerical examples and comparisons with other rules for combination of evidence published in literature are presented too. Another distinction between these alternative rules is that WAO is defined on the power set, while PCR1 is on the hyper-power set (Dedekind's lattice). A nice feature of PCR1, is that it works not only on non-degenerate cases but also on degenerate cases as well appearing in dynamic fusion, while WAO gives the sum of masses in this cases less than 1 (WAO does not work in these cases). Meanwhile we show that PCR1 and WAO do not preserve unfortunately the neutrality property of the vacuous belief assignment though the fusion process. This severe drawback can however be easily circumvented by new PCR rules presented in a companion paper.*

Keywords: WO, WAO, PCR rules, Dezert-Smarandache theory (DSmT), Data fusion, DSm hybrid rule of combination, TBM, Smets' rule, Murphy's rule, Yager's rule, Dubois-Prade's rule, conjunctive rule, disjunctive rule.

1 Introduction

Due to the fact that Dempster's rule is not mathematically defined for conflict 1 or gives counter-intuitive results for high conflict (see Zadeh's example [22], Dezert-Smarandache-Khoshnevisan's examples [11]), we looked for another rule, similar to Dempster's, easy to implement due to its simple formula, and working in any case no matter the conflict. We present this PCR1 rule of combination, which is an alternative of WAO for non-degenerate cases, in many examples comparing it with other existing rules mainly: Smets', Yager's, Dubois-Prade's, DSm hybride rule, Murphy's, and of course Dempster's. PCR1 rule is commutative, but not associative nor Markovian (it is however quasi-associative and quasi-Markovian). More versions of PCR rules are proposed in a companion paper [12] to overcome the limitations of PCR1 presented in the sequel.

2 Existing rules for combining evidence

We briefly present here the main rules proposed in the literature for combining/aggregating several independent and equi-reliable sources of evidence expressing their belief on a given finite set of exhaustive and exclusive hypotheses (Shafer's model). We assume the reader familiar with the Dempster-Shafer theory of evidence [10] and the recent theory of plausible and paradoxical reasoning (DSmT) [11]. A detailed presentation of these rules can be found in [11] and [9]. In the sequel, we consider the Shafer's model as the valid model for the fusion problem under consideration, unless specified.

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ be the *frame of discernment* of the fusion problem under consideration having n *exhaustive* and *exclusive elementary* hypotheses θ_i. The set of all subsets of Θ is called the *power set* of Θ and is denoted 2^Θ. Within Shafer's model, a *basic belief assignment* (bba) $m(.) : 2^\Theta \to [0,1]$ associated to a given body of evidence $\mathcal{B}$ is defined by [10]

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{X \in 2^\Theta} m(X) = 1 \tag{1}$$

The belief (credibility) and plausibility functions of $X \subseteq \Theta$ are defined as

$$\text{Bel}(X) = \sum_{Y \in 2^\Theta, Y \subseteq X} m(Y) \tag{2}$$

$$\text{Pl}(X) = \sum_{Y \in 2^\Theta, Y \cap X \neq \emptyset} m(Y) = 1 - \text{Bel}(\bar{X}) \tag{3}$$

where $\bar{X}$ denotes the complement of X in Θ.

The belief functions $m(.)$, Bel$(.)$ and Pl$(.)$ are in one-to-one correspondence. The set of elements $X \in 2^\Theta$ having a positive basic belief assignment is called the *core/kernel* of the source of evidence under consideration.

The main problem is now how to combine several belief assignments provided by a set of independent sources of evidence. This problem is fundamental to pool correctly uncertain and imprecise information and help the decision-making. Unfortunately, no clear/unique and satisfactory answer to this problem exists since there is potentially an infinite number of possible rules of combination [5, 7, 9]. Our contribution here is to propose an alternative to existing rules which is very easy to implement and have a legitimate behavior (not necessary the optimal one - if such optimality exists ...) for practical applications.

2.1 The Dempster's rule

The Dempster's rule of combination is the most widely used rule of combination so far in many expert systems based on belief functions since historically it was proposed in the seminal book of Shafer in [10]. This rule, although presenting interesting advantages (mainly the commutativity and associativity properties) fails however to provide coherent results due to the normalization procedure it involves. Discussions on the justification of the Dempster's rule and its well-known limitations can be found by example in [21, 22, 23, 17]. The Dempster's rule is defined as follows: let $\text{Bel}_1(.)$ and $\text{Bel}_2(.)$ be two belief functions provided by two independent equally reliable sources of evidence $\mathcal{B}_1$ and $\mathcal{B}_2$ over the same frame Θ with corresponding belief assignments $m_1(.)$ and $m_2(.)$. Then the combined global belief function denoted $\text{Bel}(.) = \text{Bel}_1(.) \oplus \text{Bel}_2(.)$ is obtained by combining $m_1(.)$ and $m_2(.)$ according to $m(\emptyset) = 0$ and $\forall (X \neq \emptyset) \in 2^\Theta$ by

$$m(X) = \frac{\sum\limits_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)}{1 - \sum\limits_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)} \tag{4}$$

$m(.)$ is a proper basic belief assignment if and only if the denominator in equation (4) is non-zero. The *degree of conflict* between the sources $\mathcal{B}_1$ and $\mathcal{B}_2$ is defined by

$$k_{12} \triangleq \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \tag{5}$$

2.2 The Murphy's rule

The Murphy's rule of combination [8] is a commutative but not associative trade-off rule, denoted here with index M, drawn from [19, 3]. It is a special case of convex combination of bbas $m_1(.)$ and $m_2(.)$ and consists actually in a simple arithmetic average of belief functions associated with $m_1(.)$ and $m_2(.)$. $\text{Bel}_M(.)$ is then given $\forall X \in 2^\Theta$ by:

$$\text{Bel}_M(X) = \frac{1}{2}[\text{Bel}_1(X) + \text{Bel}_2(X)]$$

2.3 The Smets' rule

The Smets' rule of combination [15, 16] is the non-normalized version of the conjunctive consensus (equivalent to the non-normalized version of Dempster's rule). It is commutative and associative and allows positive mass on the null/empty set $\emptyset$ (i.e. open-world assumption). Smets' rule of combination of two independent (equally reliable) sources of evidence (denoted here by index S) is then trivially given by:

$$m_S(\emptyset) \equiv k_{12} = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)$$

and $\forall(X \neq \emptyset) \in 2^\Theta$, by

$$m_S(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)$$

2.4 The Yager's rule

The Yager's rule of combination [18, 19, 20] admits that in case of conflict the result is not reliable, so that k_{12} plays the role of an absolute discounting term added to the weight of ignorance. This commutative but not associative rule, denoted here by index Y is given[1] by $m_Y(\emptyset) = 0$ and $\forall X \in 2^\Theta, X \neq \emptyset, X \neq \Theta$ by

$$m_Y(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2)$$

and when $X = \Theta$ by

$$m_Y(\Theta) = m_1(\Theta) m_2(\Theta) + \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)$$

2.5 The Dubois & Prade's rule

The Dubois & Prade's rule of combination [3] admits that the two sources are reliable when they are not in conflict, but one of them is right when a conflict occurs. Then if one observes a value in set X_1 while the other observes this value in a set X_2, the truth lies in $X_1 \cap X_2$ as long $X_1 \cap X_2 \neq \emptyset$. If $X_1 \cap X_2 = \emptyset$, then the truth lies in $X_1 \cup X_2$ [3]. According to this principle, the commutative (but not associative) Dubois & Prade hybrid rule of combination, denoted here by index DP, which is a reasonable trade-off between precision and reliability, is defined by $m_{DP}(\emptyset) = 0$ and $\forall X \in 2^\Theta, X \neq \emptyset$ by

$$m_{DP}(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = X \\ X_1 \cap X_2 \neq \emptyset}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cup X_2 = X \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \quad (6)$$

2.6 The disjunctive rule

The disjunctive rule of combination [2, 3, 14] is a commutative and associative rule proposed by Dubois & Prade in 1986 and denoted here by the index $\cup$. $m_\cup(.)$ is defined $\forall X \in 2^\Theta$ by $m_\cup(\emptyset) = 0$ and $\forall(X \neq \emptyset) \in 2^\Theta$ by

$$m_\cup(X) = \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cup X_2 = X}} m_1(X_1) m_2(X_2)$$

The core of the belief function given by $m_\cup$ equals the union of the cores of Bel_1 and Bel_2. This rule reflects the disjunctive consensus and is usually preferred when one knows that one of the sources $\mathcal{B}_1$ or $\mathcal{B}_2$ is mistaken but without knowing which one among $\mathcal{B}_1$ and $\mathcal{B}_2$. Because we assume equi-reliability of sources in this paper, this rule will not be discussed in the sequel.

2.7 Unification of the rules (weighted operator)

In the framework of Dempster-Shafer Theory (DST), an unified formula has been proposed recently by Lefèvre, Colot and Vanoorenberghe in [7] to embed all the existing (and potentially forthcoming) combination rules (including the PCR1 combination rule presented in the next section) involving conjunctive consensus in the same general mechanism of construction. We recently discovered that actually such unification formula had been already proposed 10 years before by Inagaki [5] as reported in [9]. This formulation is known as *the Weighted Operator* (WO) in literature [6], but since these two approaches have been developed independently by Inagaki and Lefèvre et al., it seems more judicious to denote it as ILCV formula instead to refer to its authors when necessary (ILCV beeing the acronym standing for Inagaki-Lefèvre-Colot-Vannoorenberghe). The WO (ILCV unified fusion rule) is based on two steps.

- **Step 1**: Computation of the total conflicting mass based on the conjunctive consensus

$$k_{12} \triangleq \sum_{\substack{X_1, X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \quad (7)$$

[1] Θ represents here the full ignorance $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ on the frame of discernment according the notation used in [10].

- **Step 2**: This step consists in the reallocation (convex combination) of the conflicting masses on $(X \neq \emptyset) \subseteq \Theta$ with some given coefficients $w_m(X) \in [0,1]$ such that $\sum_{X\subseteq\Theta} w_m(X) = 1$ according to

$$m(\emptyset) = w_m(\emptyset) \cdot k_{12}$$

and $\forall (X \neq \emptyset) \in 2^\Theta$

$$m(X) = [\sum_{\substack{X_1,X_2\in 2^\Theta \\ X_1\cap X_2 = X}} m_1(X_1)m_2(X_2)] + w_m(X)k_{12} \tag{8}$$

This WO can be easily generalized for the combination of $N \geq 2$ independent and equi-reliable sources of information as well for step 2 by substituting k_{12} by

$$k_{12\ldots N} \triangleq \sum_{\substack{X_1,\ldots,X_N\in 2^\Theta \\ X_1\cap\ldots\cap X_N = \emptyset}} \prod_{i=1,N} m_i(X_i)$$

and for step 2 by deriving for all $(X \neq \emptyset) \in 2^\Theta$ the mass $m(X)$ by

$$m(X) = [\sum_{\substack{X_1,\ldots,X_N\in 2^\Theta \\ X_1\cap\ldots\cap X_N = X}} \prod_{i=1,N} m_i(X_i)] + w_m(X)k_{12\ldots N}$$

The particular choice of the set of coefficients $w_m(.)$ provides a particular rule of combination. Actually this nice and important general formulation shows there exists an infinite number of possible rules of combination. Some rules are then justified or criticized with respect to the other ones mainly on their ability to, or not to, preserve the associativity and commutativity properties of the combination. It can be easily shown in [7] that such general procedure provides all existing rules involving conjunctive consensus developed in the literature based on Shafer's model. We will show later how the PCR1 rule of combination can also be expressed as a special case of the WO.

2.8 The weighted average operator (WAO)

This operator has been recently proposed by Josang, Daniel and Vannoorenberghe in [6]. It is a particular case of WO where the weighting coefficients $w_m(A)$ are chosen as follows: $w_m(\emptyset) = 0$ and $\forall A \in 2^\Theta \setminus \{\emptyset\}$,

$$w_m(A) = \frac{1}{N}\sum_{i=1}^{N} m_i(A)$$

where N is the number of independent sources to combine.

2.9 The hybrid DSm rule

The hybrid DSm rule of combination is a new powerful rule of combination emerged from the recent theory of plausible and paradoxist reasoning developed by Dezert and Smarandache, known as DSmT in literature. The foundations of DSmT are different from the DST foundations and DSmT covers potentially a wider class of applications than DST especially for dealing with highly conflicting static or dynamic fusion problems. Due to space limitations, we will not go further into a detailed presentation of DSmT here. A deep presentation of DSmT can be found in [11]. The DSmT deals properly with the granularity of information and intrinsic vague/fuzzy nature of elements of the frame Θ to manipulate. The basic idea of DSmT is to define belief assignments on hyper-power set D^Θ (i.e. free Dedekind's lattice) and to integrate all integrity constraints (exclusivity and/or non-existential constraints) of the model, say $\mathcal{M}(\Theta)$, fitting with the problem into the rule of combination. This rule, known as hybrid DSm rule works for any model (including the Shafer's model) and for any level of conflicting information. Mathematically, the hybrid DSm rule of combination of N independent sources of evidence is defined as follows (see chap. 4 in [11]) for all $X \in D^\Theta$

$$m_{\mathcal{M}(\Theta)}(X) \triangleq \phi(X)\Big[S_1(X) + S_2(X) + S_3(X)\Big] \tag{9}$$

where $\phi(X)$ is the *characteristic non-emptiness function* of a set X, i.e. $\phi(X) = 1$ if $X \notin \boldsymbol{\emptyset}$ and $\phi(X) = 0$ otherwise, where $\boldsymbol{\emptyset} \triangleq \{\emptyset_\mathcal{M}, \emptyset\}$. $\emptyset_\mathcal{M}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $S_1(X)$, $S_2(X)$ and $S_3(X)$ are defined by

$$S_1(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_N\in D^\Theta \\ (X_1\cap X_2\cap\ldots\cap X_N)=X}} \prod_{i=1}^{N} m_i(X_i) \tag{10}$$

$$S_2(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_N \in \emptyset \\ [\mathcal{U}=X] \vee [(\mathcal{U} \in \emptyset) \wedge (X=I_t)]}} \prod_{i=1}^{N} m_i(X_i) \tag{11}$$

$$S_3(X) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_N \in D^\Theta \\ (X_1 \cup X_2 \cup \ldots \cup X_N)=X \\ (X_1 \cap X_2 \cap \ldots \cap X_N) \in \emptyset}} \prod_{i=1}^{N} m_i(X_i) \tag{12}$$

with $\mathcal{U} \triangleq u(X_1) \cup u(X_2) \cup \ldots \cup u(X_N)$ where $u(X_i)$, $i = 1, \ldots, N$, is the union of all singletons θ_k, $k \in \{1, \ldots, |\Theta|\}$, that compose X_i and $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ is the total ignorance. $S_1(X)$ corresponds to the conjunctive consensus on free Dedekind's lattice for N independent sources; $S_2(X)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances; $S_3(X)$ transfers the sum of relatively empty sets to the non-empty sets.

In the case of a dynamic fusion problem, when all elements become empty because one gets new evidence on integrity constraints (which corresponds to a specific hybrid model $\mathcal{M}$), then the conflicting mass is transferred to the total ignorance, which also turns to be empty, therefore the empty set gets now mass which means open-world, i.e, new hypotheses might be in the frame of discernment. For example, Let's consider the frame $\Theta = \{A, B\}$ with the 2 following bbas $m_1(A) = 0.5$, $m_1(B) = 0.3$, $m_1(A \cup B) = 0.2$ and $m_2(A) = 0.4$, $m_2(B) = 0.5$, $m_2(A \cup B) = 0.1$, but one finds out with new evidence that A and B are truly empty, then $A \cup B \equiv \Theta \overset{\mathcal{M}}{\equiv} \emptyset$. Then $m(\emptyset) = 1$.

The hybrid DSm rule of combination is not equivalent to Dempter's rule even working on the Shafer's model. DSmT is actually a natural extension of the DST. An extension of this rule for the combination of *imprecise* generalized (or eventually classical) basic belief functions is possible and is presented in [11].

3 The PCR1 combination rule

3.1 The PCR1 rule for 2 sources

Let $\Theta = \{\theta_1, \theta_2\}$ be the frame of discernment and its hyper-power set $D^\Theta = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\, \theta_1 \cap \theta_2\}$. Two basic belief assignments / masses $m_1(.)$ and $m_2(.)$ are defined over this hyper-power set. We assume that $m_1(.)$ and $m_2(.)$ are normalized belief masses following definition given by (1). The PCR1 combination rule consists in two steps:

- **Step 1**: Computation of the conjunctive consensus[2] $m_\cap(.) = [m_1 \oplus m_2](.)$ and the conflicting mass according to

$$m_\cap(X) = \sum_{\substack{X_1,X_2 \in D^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2) \tag{13}$$

  and

$$k_{12} \triangleq \sum_{\substack{X_1,X_2 \in D^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \tag{14}$$

  This step coincides with the Smets' rule of combination when accepting the open-world assumption. In the Smets' open-world TBM framework [13], k_{12} is interpreted as the mass $m(\emptyset)$ committed to the empty set. $\emptyset$ corresponds then to all missing unknown hypotheses and the absolute impossible event.

- **Step 2** (normalization): Distribution of the conflicting mass k_{12} onto $m_\cap(X)$ proportionally with the non-zero sums of their corresponding columns of non-empty sets of the effective mass matrix $\mathbf{M}_{12}[m_{ij}]$ (index 12 denotes the list of sources entering into the mass matrix). If all sets are empty, then the conflicting mass is redistributed to the disjunctive form of all these empty sets (which is many cases coincides with the total ignorance).

  More precisely, the original mass matrix $\mathbf{M}_{12}$ is a $(N = 2) \times (2^{|\Theta|} - 1)$ matrix constructed by stacking the row vectors

$$\begin{cases} \mathbf{m}_1 = [m_1(\theta_1) \quad m_1(\theta_2) \quad m_1(\theta_1 \cup \theta_2)] \\ \mathbf{m}_2 = [m_2(\theta_1) \quad m_2(\theta_2) \quad m_2(\theta_1 \cup \theta_2)] \end{cases}$$

[2] $\oplus$ denotes here the generic symbol for the fusion.

associated with the beliefs assignments $m_1(.)$ and $m_2(.)$. For convenience and by convention, the row index i follows the index of sources and the index j for columns follows the enumeration of elements of power set 2^Θ (excluding the empty set because by definition its committed mass is zero). Any permutation of rows and columns can be arbitrarily chosen as well and it doesn't not make any difference in the PCR1 fusion result. Thus, one has for the 2 sources and 2D fusion problem:

$$\mathbf{M}_{12} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix} = \begin{bmatrix} m_1(\theta_1) & m_1(\theta_2) & m_1(\theta_1 \cup \theta_2) \\ m_2(\theta_1) & m_2(\theta_2) & m_2(\theta_1 \cup \theta_2) \end{bmatrix}$$

We denote by $c_{12}(X)$ the sum of the elements of the column of the mass matrix associated with element X of the power set, i.e

$$\begin{cases} c_{12}(X = \theta_1) = m_1(\theta_1) + m_2(\theta_1) \\ c_{12}(X = \theta_2) = m_1(\theta_2) + m_2(\theta_2) \\ c_{12}(X = \theta_1 \cup \theta_2) = m_1(\theta_1 \cup \theta_2) + m_2(\theta_1 \cup \theta_2) \end{cases}$$

The conflicting mass k_{12} is distributed proportionally with all non-zero coefficients $c_{12}(X)$. For elements $X \in D^\Theta$ with zero coefficients $c_{12}(X)$, no conflicting mass will be distributed to them. Let's note by $w(\theta_1)$, $w(\theta_2)$ and $w(\theta_1 \cup \theta_2)$ the part of the conflicting mass that is respectively distributed to θ_1, θ_2 and $\theta_1 \cup \theta_2$ (assuming $c_{12}(\theta_1) > 0$, $c_{12}(\theta_2) > 0$ and $c_{12}(\theta_1 \cup \theta_2) > 0$. Then:

$$\frac{w(\theta_1)}{c_{12}(\theta_1)} = \frac{w(\theta_2)}{c_{12}(\theta_2)} = \frac{w(\theta_1 \cup \theta_2)}{c_{12}(\theta_1 \cup \theta_2)} = \frac{w(\theta_1) + w(\theta_2) + w(\theta_1 \cup \theta_2)}{c_{12}(\theta_1) + c_{12}(\theta_2) + c_{12}(\theta_1 \cup \theta_2)} = \frac{k_{12}}{d_{12}} \tag{15}$$

because

$$c_{12}(\theta_1) + c_{12}(\theta_2) + c_{12}(\theta_1 \cup \theta_2) = \sum_{X_1 \in D^\Theta \setminus \{\emptyset\}} m_1(X_1) + \sum_{X_2 \in D^\Theta \setminus \{\emptyset\}} m_2(X_2) = d_{12}$$

Hence the proportionalized conflicting masses to transfer are given by

$$\begin{cases} w(\theta_1) = c_{12}(\theta_1) \cdot \frac{k_{12}}{d_{12}} \\ w(\theta_2) = c_{12}(\theta_2) \cdot \frac{k_{12}}{d_{12}} \\ w(\theta_1 \cup \theta_2) = c_{12}(\theta_1 \cup \theta_2) \cdot \frac{k_{12}}{d_{12}} \end{cases}$$

which are added respectively to $m_\cap(\theta_1)$, $m_\cap(\theta_2)$ and $m_\cap(\theta_1 \cup \theta_2)$.

Therefore, the general formula for the PCR1 rule for 2 sources, for $|\Theta| \geq 2$, is given by $m_{PCR1}(\emptyset) = 0$ and for $(X \neq \emptyset) \in D^\Theta$,

$$m_{PCR1}(X) = \sum_{\substack{X_1, X_2 \in D^\Theta \\ X_1 \cap X_2 = X}} m_1(X_1) m_2(X_2) + c_{12}(X) \cdot \frac{k_{12}}{d_{12}} \tag{16}$$

where k_{12} is the total conflicting mass and $c_{12}(X) \triangleq \sum_{i=1,2} m_i(X) \neq 0$, i.e. the non-zero sum of the column of the mass matrix $\mathbf{M}_{12}$ corresponding to the element X, and d_{12} is the sum of all non-zero column sums of all non-empty sets (in many cases $d_{12} = 2$ but in some degenerate cases it can be less).

In the degenerate case when all column sums of all non-empty sets are zero, then the conflicting mass is transferred to the non-empty disjunctive form of all sets involved in the conflict together. But if this disjunctive form happens to be empty, then one considers an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set.

As seen, the PCR1 combination rule works for any degree of conflict $k_{12} \in [0, 1]$, while Dempster's rule does not work for $k_{12} = 1$ and gives counter-intuitive results for most of high conflicting fusion problems.

3.2 Generalization for $N \geq 2$ sources

The previous PCR1 rule of combination for two sources ($N = 2$) can be directly and easily extended for the multi-source case ($N \geq 2$) as well. The general formula of the PCR1 rule is thus given by $m_{PCR1}(\emptyset) = 0$ and for $X \neq \emptyset) \in D^\Theta$

$$m_{PCR1}(X) = \Big[\sum_{\substack{X_1, \ldots, X_N \in D^\Theta \\ X_1 \cap \ldots \cap X_N = X}} \prod_{i=1,N} m_i(X_i) \Big] + c_{12\ldots N}(X) \cdot \frac{k_{12\ldots N}}{d_{12\ldots N}} \tag{17}$$

where $k_{12...N}$ is the total conflicting mass between all the N sources which is given by

$$k_{12...N} \triangleq \sum_{\substack{X_1,...,X_N \in D^\Theta \\ X_1 \cap ... \cap X_N = \emptyset}} \prod_{i=1,N} m_i(X_i) \tag{18}$$

and $c_{12...N}(X) \triangleq \sum_{i=1,N} m_i(X) \neq 0$, i.e. the non-zero sum of the column of the mass matrix $\mathbf{M}_{12...N}$ corresponding to the element X, while $d_{12...N}$ represents the sum of all non-zero column sums of all non-empty sets (in many cases $d_{12...N} = N$ but in some degenerate cases it can be less).

Similarly for N sources, in the degenerate case when all column sums of all non-empty sets are zero, then the conflicting mass is transferred to the non-empty disjunctive form of all sets involved in the conflict together. But if this disjunctive form happens to be empty, then one considers an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set.

The PCR1 rule can be seen as a cheapest, easiest implementable approximated version of the sophisticated MinC combination rule proposed by Daniel in [1] and [11] (chap. 10). Note also that the PCR1 rule works in the DSmT framework and can serve as a cheap alternative to the more sophisticated and specific DSm hybrid rule but preferentially when none of sources is totally ignorant (see discussion in section 3.6). One applies the DSm classic rule [11] (i.e. the conjunctive consensus on D^Θ), afterwards one identifies the model and its integrity constraints and one eventually employs the PCR1 rule instead of DSm hybrid rule (depending of the dimension of the problem to solve, the number of sources involved and the computing resources available). PCR1 can be used on the power set 2^Θ and within the DS Theory.

The PCR1 combination rule is commutative but not associative. It converges towards Murphy's rule (arithmetic mean of masses) when the conflict is approaching 1, and it converges towards the conjunctive consensus rule when the conflict is approaching 0.

3.3 Implementation of the PCR1 rule

For practical use and implementation of the PCR1 combination rule, it is important to save memory space and avoid useless computation as best as possible and especially when dealing with many sources and for frames of high dimension. To achieve this, it's important to note that since all zero-columns of the mass matrix do not play a role in the normalization, all zero-columns (if any) of the original mass matrix can be removed to *compress* the matrix horizontally (this can be easily done using MatLab programming language) to get an effective mass matrix of smaller dimension for computation the set of proportionalized conflicting masses to transfer. The list of elements of power set corresponding to non-empty colums must be maintained in parallel to this compression for implementation purpose. By example, let's assume $|\Theta| = 2$ and only 2 sources providing $m_1(\theta_2) = m_2(\theta_2) = 0$ and all other masses are positive, then the *effective* mass matrix will become

$$\mathbf{M}_{12} = \begin{bmatrix} m_1(\theta_1) & m_1(\theta_1 \cup \theta_2) \\ m_2(\theta_1) & m_2(\theta_1 \cup \theta_2) \end{bmatrix}$$

with now the following correspondance for column indexes: $(j = 1) \leftrightarrow \theta_1$ and $(j = 2) \leftrightarrow \theta_1 \cup \theta_2$.

The computation the set of proportionalized conflicting masses to transfer will be done using the PCR1 general formula directly from this previous effective mass matrix rather than from

$$\mathbf{M}_{12} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix} = \begin{bmatrix} m_1(\theta_1) & m_1(\theta_2) = 0 & m_1(\theta_1 \cup \theta_2) \\ m_2(\theta_1) & m_2(\theta_2) = 0 & m_2(\theta_1 \cup \theta_2) \end{bmatrix}$$

3.4 PCR1 rule as a special case of WO

The PCR1 rule can be easily expressed as a special case of the WO (8) for the combination of two sources by choosing as weighting coefficients for each $X \in 2^\Theta \setminus \{\emptyset\}$,

$$w_m(X) = c_{12}(X)/d_{12}$$

For the combination of $N \geq 2$ independent and equi-reliable sources, the weighting coefficients will be given by

$$w_m(X) = c_{12...N}(X)/d_{12...N}$$

3.5 Advantages of the PCR1 rule

- the PCR1 rule works in any cases, no matter what the conflict is (it may be 1 or less); Zadeh's example, examples with $k_{12} = 1$ or $k_{12} = 0.99$, etc. All work;
- the implementation of PCR1 rule is very easy and thus presents a great interest for engineers who look for a cheap and an easy alternative fusion rule to existing rules;
- the PCR1 formula is simple (it is not necessary to go by proportionalization each time when fusionning);
- the PCR1 rule works quite well with respect to some other rules since the specificity of information is preserved (i.e no mass is transferred onto partial or total ignorances, neither onto the empty set as in TBM);
- the PCR1 rule reflects the majority rule;
- the PCR1 rule is convergent towards idempotence for problems with no unions or intersections of sets (we know that, in fact, no combination rule is idempotent, except Murphy elementary fusion mean rule);
- the PCR1 rule is similar to the classical Dempster-Shafer's rule instead of proportionalizing with respect to the results of the conjunctive rule as is done in Dempster's, we proportionalize with respect to the non-zero sum of the columns masses, the only difference is that in the DS combination rule one eliminates the denominator (which caused problems when the degree of conflict is 1 or close to 1); PCR1 on the power set and for non-degenerate cases gives the same results as WAO [6]; yet, for the storage proposal in a dynamic fusion when the associativity is needed, for PCR1 is needed to store only the last sum of masses, besides the previous conjunctive rules result, while in WAO it is in addition needed to store the number of the steps and both rules become quasi-associative;
- the normalization, done proportionally with the corresponding non-zero sum of elements of the mass matrix, is natural - because the more mass is assigned to an hypothesis by the sources the more mass that hypothesis deserves to get after the fusion.

3.6 Disadvantages of the PCR1 rule

- the PCR1 rule requires normalization/proportionalization, but the majority of rules do; rules which do not require normalization loose information through the transfer of conflicting mass to partial and/or total ignorances or to the empty set.
- the results of PCR1 combination rule do not bring into consideration any new set: formed by unions (uncertainties); or intersections (consensus between some hypotheses); yet, in the DSmT framework the intersections show up through the hyper-power set.
- the severe drawback of PCR1 and WAO rules is that they do not preserve the neutrality property of the vacuous belief assignment $m_v(.)$ (defined by $m_v(\Theta) = 1$) as one legitimately expects since if one or more bbas $m_s(.)$, $s \geq 1$, different from the vacuous belief, are combined with the vacuous belief assignment the result is not the same as that of the combination of the bbas only (without including $m_v(.)$), i.e. $m_v(.)$ does not act as a neutral element for the fusion combination.

  In other words, for $s \geq 1$, one gets for $m_1(.) \neq m_v(.), \ldots, m_s(.) \neq m_v(.)$:

$$m_{PCR1}(.) = [m_1 \oplus \ldots m_s \oplus m_v](.) \neq [m_1 \oplus \ldots m_s](.) \tag{19}$$

$$m_{WAO}(.) = [m_1 \oplus \ldots m_s \oplus m_v](.) \neq [m_1 \oplus \ldots m_s](.) \tag{20}$$

  For the cases of the combination of only one non-vacuous belief assignment $m_1(.)$ with the vacuous belief assignment $m_v(.)$ where $m_1(.)$ has mass asigned to an empty element, say $m_1(\emptyset) > 0$ as in Smets' TBM, or as in DSmT dynamic fusion where one finds out that a previous non-empty element A, whose mass $m_1(A) > 0$, becomes empty after a certain time, then this mass of an empty set has to be transferred to other elements using PCR1, but for such case $[m_1 \oplus m_v](.)]$ is different from $m_1(.)$.

  Example: Let's have $\Theta = \{A, B\}$ and two bbas

$$m_1(A) = 0.4 \qquad m_1(B) = 0.5 \qquad m_1(A \cup B) = 0.1$$

$$m_2(A) = 0.6 \qquad m_2(B) = 0.2 \qquad m_2(A \cup B) = 0.2$$

together with the vacuous bba $m_v(\Theta = A \cup B) = 1$. If one applies the PCR1 rule to combine the 3 sources altogether, one gets

$$m_{PCR1|12v}(A) = 0.38 + 1 \cdot \frac{0.38}{3} = 0.506667$$

$$m_{PCR1|12v}(B) = 0.22 + 0.7 \cdot \frac{0.38}{3} = 0.308667$$

$$m_{PCR1|12v}(A \cup B) = 0.02 + 1.3 \cdot \frac{0.38}{3} = 0.184666$$

since the conjunctive consensus is given by $m_{12v}(A) = 0.38$, $m_{12v}(B) = 0.22$, $m_{12v}(A \cup B) = 0.02$; the conflicting mass is $k_{12v} = 0.38$ and one has

$$\frac{x}{1} = \frac{y}{0.7} = \frac{z}{1.3} = \frac{0.38}{3}$$

while the combination of only the sources 1 and 2 withe the PCR1 provides

$$m_{PCR1|12}(A) = 0.38 + 0.19 = 0.570$$

$$m_{PCR1|12}(B) = 0.22 + 0.133 = 0.353$$

$$m_{PCR1|12}(A \cup B) = 0.02 + 0.057 = 0.077$$

since the conjunctive consensus is given by $m_{12}(A) = 0.38$, $m_{12}(B) = 0.22$, $m_{12}(A \cup B) = 0.02$; the conflicting mass is $k_{12} = 0.38$ but one has now the following redistribution condition

$$\frac{x}{1} = \frac{y}{0.7} = \frac{z}{0.3} = \frac{0.38}{2} = 0.19$$

Thus clearly $m_{PCR1|12v}(.) \neq m_{PCR1|12}(.)$ although the third source brings no information in the fusion since it is fully ignorant. This behavior is abnormal and counter-rintuitive. WAO gives the same results in this example, therefore WAO also doesn't satisfy the neutrality property of the vacuous belief assignment for the fusion. That's why we have improved PCR1 to PCR2-4 rules in a companion paper [12].

3.7 Comparison of the PCR1 rule with the WAO

3.7.1 The non degenerate case

Let's compare in this section the PCR1 with the WAO for a very simple 2D general non degenerate case (none of the elements of the power set or hyper-power set of the frame Θ are known to be truly empty but the universal empty set itself) for the combination of 2 sources. Assume that the non degenerate mass matrix $\mathbf{M}_{12}$ associated with the beliefs assignments $m_1(.)$ and $m_2(.)$ is given by

$$\begin{cases} \mathbf{m}_1 = [m_1(\theta_1) & m_1(\theta_2) & m_1(\theta_1 \cup \theta_2)] \\ \mathbf{m}_2 = [m_2(\theta_1) & m_2(\theta_2) & m_2(\theta_1 \cup \theta_2)] \end{cases}$$

In this very simple case, the total conflict is given by

$$k_{12} = m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)$$

According to the WAO definition, one gets $m_{WAO}(\emptyset) = w_m(\emptyset) \cdot k_{12} = 0$ because by definition $w_m(\emptyset) = 0$. The other weighting coefficients of WAO are given by

$$w_m(\theta_1) = \frac{1}{2}[m_1(\theta_1) + m_2(\theta_1)]$$

$$w_m(\theta_2) = \frac{1}{2}[m_1(\theta_2) + m_2(\theta_2)]$$

$$w_m(\theta_1 \cup \theta_2) = \frac{1}{2}[m_1(\theta_1 \cup \theta_2) + m_2(\theta_1 \cup \theta_2)]$$

Thus, one obtains

$$m_{WAO}(\theta_1) = [m_1(\theta_1)m_2(\theta_1) + m_1(\theta_1 \cup \theta_2)m_2(\theta_1) + m_1(\theta_1)m_2(\theta_1 \cup \theta_2)] + \frac{1}{2}[m_1(\theta_1) + m_2(\theta_1)] \cdot [m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)]$$

$$m_{WAO}(\theta_2) = [m_1(\theta_2)m_2(\theta_2) + m_1(\theta_1 \cup \theta_2)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1 \cup \theta_2)] + \frac{1}{2}[m_1(\theta_2) + m_2(\theta_2)] \cdot [m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)]$$

$$m_{WAO}(\theta_1 \cup \theta_2) = [m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)] + \frac{1}{2}[m_1(\theta_1 \cup \theta_2) + m_2(\theta_1 \cup \theta_2)] \cdot [m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)]$$

It is easy to verify that $\sum_{X \in 2^\Theta} m_{WAO}(X) = 1$.

Using the PCR1 formula for 2 sources explicated in section 3.1, one has $m_{PCR1}(\emptyset) = 0$ and the weighting coefficients of the PCR1 rule are given by

$$\begin{cases} c_{12}(\theta_1) = m_1(\theta_1) + m_2(\theta_1) \\ c_{12}(\theta_2) = m_1(\theta_2) + m_2(\theta_2) \\ c_{12}(\theta_1 \cup \theta_2) = m_1(\theta_1 \cup \theta_2) + m_2(\theta_1 \cup \theta_2) \end{cases}$$

and d_{12} by $d_{12} = c_{12}(\theta_1) + c_{12}(\theta_2) + c_{12}(\theta_1 \cup \theta_2) = 2$. Therefore, one finally gets:

$$m_{PCR1}(\theta_1) = [m_1(\theta_1)m_2(\theta_1) + m_1(\theta_1 \cup \theta_2)m_2(\theta_1) + m_1(\theta_1)m_2(\theta_1 \cup \theta_2)] + \frac{c_{12}(\theta_1)}{d_{12}} \cdot [m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)]$$

$$m_{PCR1}(\theta_2) = [m_1(\theta_2)m_2(\theta_2) + m_1(\theta_1 \cup \theta_2)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1 \cup \theta_2)] + \frac{c_{12}(\theta_2)}{d_{12}} \cdot [m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)]$$

$$m_{PCR1}(\theta_1 \cup \theta_2) = [m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)] + \frac{c_{12}(\theta_1 \cup \theta_2)}{d_{12}} \cdot [m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1)]$$

Therefore for all X in 2^Θ, one has $m_{PCR1}(X) = m_{WAO}(X)$ if no singletons or unions of singletons are (or become) empty at a given time, otherwise the results are different as seen in the below three examples. This property holds for the combination of $N > 2$ sources working on a $n - D$ frame ($n > 2$) Θ as well if no singletons or unions of singletons are (or become) empty at a given time, otherwise the results become different.

3.7.2 The degenerate case

In the dynamic fusion, when one or more singletons or unions of singletons become empty at a certain time t which corresponds to a degenerate case, the WAO does not work.

Example 1: Let's consider the Shafer's model (exhaustivity and exclusivity of hypotheses) on $\Theta = \{A, B, C\}$ and the two following bbas

$$m_1(A) = 0.3 \quad m_1(B) = 0.4 \quad m_1(C) = 0.3$$
$$m_2(A) = 0.5 \quad m_2(B) = 0.1 \quad m_2(C) = 0.4$$

Then the conjunctive consensus yields

$$m_{12}(A) = 0.15 \quad m_{12}(B) = 0.04 \quad m_{12}(C) = 0.12$$

and the conflicting mass $k_{12} = 0.69$. Now assume that at time t, one finds out that $B = \emptyset$, then the new conflict mass which becomes $k'_{12} = 0.69 + 0.04 = 0.73$ is re-distributed to A and B according to the WAO formula:

$$m_{WAO}(B) = 0$$

$$m_{WAO}(A) = 0.15 + (1/2)(0.3 + 0.5)(0.73) = 0.4420$$
$$m_{WAO}(C) = 0.12 + (1/2)(0.3 + 0.4)(0.73) = 0.3755$$

From this WAO result, one sees clearly that the sum of the combined masses $m(.)$ is $0.8175 < 1$ while using PCR1, one redistributes 0.73 to A and B following the PCR1 formula:

$$m_{PCR1}(B) = 0$$

$$m_{PCR1}(A) = 0.15 + \frac{(0.3 + 0.5)(0.73)}{(0.3 + 0.5 + 0.3 + 0.4)} = 0.539333$$

$$m_{PCR1}(C) = 0.12 + \frac{(0.3+0.4)(0.73)}{(0.3+0.5+0.3+0.4)} = 0.460667$$

which clearly shows that he sum of masses $m_{PCR1}(.)$ is 1 as expected for a proper belief assignment.

Example 2 (totally degenerate case) : Let's take exactly the same previous example with exclusive hypotheses A, B and C but assume now that at time t one finds out that A, B and C are all truly empty, then $k'_{12} = 1$. In this case, the WAO is not able to redistribute the conflict to any element A, B, C or partial/total ignorances because they are empty. But PCR1 transfers the conflicting mass to the ignorance $A \cup B \cup C$, which is the total ignorance herein, but this is also empty, thus the conflicting mass is transferred to the empty set, meaning we have an open world, i.e. new hypotheses might belong to the frame of discernment.

Example 3 (Open-world): In the Smets' open-world approach (when the empty set gets some mass assigned by the sources), the WAO doesn't work either. For example, let's consider $\Theta = \{A, B\}$ and the following bbas $m_1(\emptyset) = 0.1$, $m_2(\emptyset) = 0.2$ and

$$m_1(A) = 0.4 \quad m_1(B) = 0.3 \quad m_1(A \cup B) = 0.2$$
$$m_2(A) = 0.5 \quad m_2(B) = 0.2 \quad m_2(A \cup B) = 0.1$$

Then the conjunctive consensus yields $m_{12}(\emptyset) = 0.28$ and

$$m_{12}(A) = 0.34 \quad m_{12}(B) = 0.13 \quad m_{12}(A \cup B) = 0.02$$

with the conflicting mass

$$k_{12} = m_{12}(A \cap B) + m_{12}(\emptyset) = 0.23 + 0.28 = 0.51$$

Using WAO, one gets

$$m_{WAO}(\emptyset) = 0$$
$$m_{WAO}(A) = 0.34 + (1/2)(0.4+0.5)(0.51) = 0.5695$$
$$m_{WAO}(B) = 0.13 + (1/2)(0.3+0.2)(0.51) = 0.2275$$
$$m_{WAO}(A \cup B) = 0.02 + (1/2)(0.2+0.1)(0.51) = 0.0965$$

The sum of masses$m_{WAO}(.)$ is $0.9235 < 1$ while PCR1 gives:

$$m_{PCR1}(\emptyset) = 0$$
$$m_{PCR1}(A) = 0.34 + \frac{(0.4+0.5)(0.51)}{(0.4+0.5+0.3+0.2+0.2+0.1)} = 0.61$$
$$m_{PCR1}(B) = 0.13 + \frac{(0.3+0.2)(0.51)}{(0.4+0.5+0.3+0.2+0.2+0.1)} = 0.28$$
$$m_{PCR1}(A \cup B) = 0.02 + \frac{(0.2+0.1)(0.51)}{(0.4+0.5+0.3+0.2+0.2+0.1)} = 0.11$$

which shows that the sum of masses $m_{PCR1}(.)$ is 1.

3.7.3 Comparison of memory storages

In order to keep the associativity of PCR1 one stores the previous result of combination using the conjunctive rule, and also the sums of mass columns [2 storages]. For the WAO one stores the previous result of combination using the conjunctive rule (as in PCR1), and the mass columns averages (but the second one is not enough in order to compute the next average and that's why one still needs to store the number of masses combined so far) [3 storages].

For example, let's $\Theta = \{A, B, C\}$ and let's suppose first that only five bbas available, $m_1(.)$, $m_2(.)$, $m_3(.)$, $m_4(.)$, $m_5(.)$, have been combined with WAO, where for example $m_1(A) = 0.4$, $m_2(A) = 0.2$, $m_3(A) = 0.3$, $m_4(A) = 0.6$, $m_5(A) = 0.0$. Their average $m_{12345}(A) = 0.3$ was then obtained and stored. Let's assume now that a new bba $m_6(.)$, with $m_6(A) = 0.4$ comes in as a new evidence. Then, how to compute with WAO the new average $m_{123456}(A) = [m_{12345} \oplus m_6](A)$? We need to know how many masses have been combined so far with WAO (while in PCR1 this is not necessary). Therefore $n = 5$, the number of combined bbas so far, has to be stored too when using WAO in sequential/iterative fusion. Whence, the new average is possible to be computed with WAO :

$$m_{123456}(A) = \frac{5 \cdot 0.3 + 0.4}{5+1} = 0.316667$$

but contrariwise to WAO, we don't need an extra memory storage for keep in memory $n = 5$ when using PCR1 to compute[3] $m_{PCR1|123456}(A)$ from $m_{PCR1|12345}(A)$ and $m_6(A)$ which is more interesting since PCR1 reduces the memory storage requirement versus WAO. Indeed, using PCR1 we only store the sum of previous masses: $c_{12345}(A) = 0.4 + 0.2 + 0.3 + 0.6 + 0.0 = 1.5$, and when another bba $m_6(.)$ with $m_6(A) = 0.4$ comes in as a new evidence one only adds it to the previous sum of masses: $c_{123456}(A) = 1.5 + 0.4 = 1.9$ to get the coefficient of proportionalization for the set A.

4 Some numerical examples

4.1 Example 1

Let's consider a general 2D case (i.e. $\Theta = \{\theta_1, \theta_2\}$) including epistemic uncertainties with the two following belief assignments

$$m_1(\theta_1) = 0.6, \quad m_1(\theta_2) = 0.3, \quad m_1(\theta_1 \cup \theta_2) = 0.1$$
$$m_2(\theta_1) = 0.5, \quad m_2(\theta_2) = 0.2, \quad m_2(\theta_1 \cup \theta_2) = 0.3$$

The conjunctive consensus yields:

$$m_\cap(\theta_1) = 0.53, \quad m_\cap(\theta_2) = 0.17, \quad m_\cap(\theta_1 \cup \theta_2) = 0.03$$

with the total conflicting mass $k_{12} = 0.27$.

Applying the proportionalization from the mass matrix

$$\mathbf{M}_{12} = \begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$$

one has

$$\frac{w_{12}(\theta_1)}{0.6+0.5} = \frac{w_{12}(\theta_2)}{0.3+0.2} = \frac{w_{12}(\theta_1 \cup \theta_2)}{0.1+0.3} = \frac{w_{12}(\theta_1) + w_{12}(\theta_2) + w_{12}(\theta_1 \cup \theta_2)}{2} = \frac{0.27}{2} = 0.135$$

and thus one deduces:

$$w_{12}(\theta_1) = 1.1 \cdot 0.135 = 0.1485 \qquad w_{12}(\theta_2) = 0.5 \cdot 0.135 = 0.0675 \qquad w_{12}(\theta_1 \cup \theta_2) = 0.4 \cdot 0.135 = 0.0540$$

One adds $w_{12}(\theta_1)$ to $m_\cap(\theta_1)$, $w_{12}(\theta_2)$ to $m_\cap(\theta_2)$ and $w_{12}(\theta_1 \cup \theta_2)$ to $m_\cap(\theta_1 \cup \theta_2)$. One finally gets the result of the PCR1 rule of combination:

$$m_{PCR1}(\theta_1) = 0.53 + 0.1485 = 0.6785$$
$$m_{PCR1}(\theta_2) = 0.17 + 0.0675 = 0.2375$$
$$m_{PCR1}(\theta_1 \cup \theta_2) = 0.03 + 0.0540 = 0.0840$$

4.2 Example 2

Let's consider the frame of discernment with only two exclusive elements, i.e. $\Theta = \{\theta_1, \theta_2\}$ and consider the two following *Bayesian* belief assignments

$$m_1(\theta_1) = 0.2, m_1(\theta_2) = 0.8$$
$$m_2(\theta_1) = 0.9, m_2(\theta_2) = 0.1$$

The associated (effective) mass matrix will be

$$\mathbf{M}_{12} = \begin{bmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{bmatrix}$$

The first row of $\mathbf{M}_{12}$ corresponds to basic belief assignment $m_1(.)$ and the second row of $\mathbf{M}_{12}$ corresponds to basic belief assignment $m_2(.)$. The columns of the mass matrix $\mathbf{M}_{12}$ correspond to focal elements of $m_1(.)$ and $m_2(.)$ and the choice for ordering these elements doesn't matter. any arbitrary choice is possible. In this example the first column of $\mathbf{M}_{12}$ is associated with θ_1 and the second column with θ_2.

[3] The notation $m_{PCR1|12\ldots n}(.)$ denotes explicitly the fusion of n bbas $m_1(.), m_2(.), \ldots, m_n(.)$; i.e. given the knowledge of the n bbas combined altogether.

4.2.1 Fusion with the PCR1 rule

The conjunctive consensus yields:

$$\begin{cases} m_{\cap}(\theta_1) = [m_1 \oplus m_2](\theta_1) = 0.2 \cdot 0.9 = 0.18 \\ m_{\cap}(\theta_2) = [m_1 \oplus m_2](\theta_2) = 0.8 \cdot 0.1 = 0.08 \end{cases}$$

The remaining mass corresponds to the conflict k_{12}, i.e.

$$k_{12} = 1 - m_{\cap}(\theta_1) - m_{\cap}(\theta_2) = m_1(\theta_1)m_2(\theta_2) + m_1(\theta_2)m_2(\theta_1) = (0.2 \cdot 0.1) + (0.9 \cdot 0.8) = 0.74$$

Now the conflicting mass, $k_{12} = 0.74$, is distributed between $m_{\cap}(\theta_1)$ and $m_{\cap}(\theta_2)$ proportionally with the non-zero sums of their columns. Thus, the column vector associated with θ_1 is $[0.2 \ 0.9]'$ and we add the elements $0.2 + 0.9 = 1.1$. The column vector associated with θ_2 is $[0.8 \ 0.1]'$ and we add the elements $0.8 + 0.1 = 0.9$.

Let $w_{12}(\theta_1)$, $w_{12}(\theta_2)$ be the parts from the conflicting mass to be assigned to $m_{\cap}(\theta_1)$ and $m_{\cap}(\theta_2)$ respectively. Then:

$$\frac{w_{12}(\theta_1)}{1.1} = \frac{w_{12}(\theta_2)}{0.9} = \frac{w_{12}(\theta_1) + w_{12}(\theta_2)}{1.1 + 0.9} = \frac{0.74}{2} = 0.37$$

Whence, $w_{12}(\theta_1) = 1.1 \cdot 0.37 = 0.407$, $w_{12}(\theta_2) = 0.9 \cdot 0.37 = 0.333$. One adds $w_{12}(\theta_1)$ to $m_{\cap}(\theta_1)$ and $w_{12}(\theta_2)$ to $m_{\cap}(\theta_2)$ and one finally gets the result of the PCR1 rule of combination:

$$m_{PCR1}(\theta_1) = 0.18 + 0.407 = 0.587$$
$$m_{PCR1}(\theta_2) = 0.08 + 0.333 = 0.413$$

where $m_{PCR1}(.)$ means the normalized mass resulting from the PCR1 rule of combination.

We can directly use the PCR1 formula for computing the mass, instead of doing proportionalizations all the time.

4.2.2 Fusion with the Dempster's rule

Based on the close-world Shafer's model and applying the Dempster's rule of combination, one gets (index DS standing here for Dempster-Shafer)

$$m_{DS}(\theta_1) = \frac{m_{\cap}(\theta_1)}{1 - k_{12}} = \frac{0.18}{0.26} = 0.692308$$
$$m_{DS}(\theta_2) = \frac{m_{\cap}(\theta_2)}{1 - k_{12}} = \frac{0.08}{0.26} = 0.307692$$

4.2.3 Fusion with the Smets' rule

Based on the open-world model with TBM interpretation [13] and applying the Smets' rule of combination (i.e. the non-normalized Dempster's rule of combination), one trivially gets (index S standing here for Smets)

$$m_S(\theta_1) = m_{\cap}(\theta_1) = 0.18$$
$$m_S(\theta_2) = m_{\cap}(\theta_2) = 0.08$$
$$m_S(\emptyset) = k_{12} = 0.74$$

4.2.4 Fusion with other rules

While different in their essence, the Yager's rule [18], Dubois-Prade [3] rule and the hybrid DSm rule [11] of combination provide the same result for this specific 2D example. That is

$$m(\theta_1) = 0.18 \qquad m(\theta_2) = 0.08 \qquad m(\theta_1 \cup \theta_2) = 0.74$$

4.3 Example 3 (Zadeh's example)

Let's consider the famous Zadeh's examples [21, 22, 23, 24] with the frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$, two independent sources of evidence corresponding to the following Bayesian belief assignment matrix (where columns 1, 2 and 3 correspond respectively to elements θ_1, θ_2 and θ_3 and rows 1 and 2 to belief assignments $m_1(.)$ and $m_2(.)$ respectively), i.e.

$$\mathbf{M}_{12} = \begin{bmatrix} 0.9 & 0 & 0.1 \\ 0 & 0.9 & 0.1 \end{bmatrix}$$

In this example, one has

$$\begin{cases} m_{\cap}(\theta_1) = [m_1 \oplus m_2](\theta_1) = 0 \\ m_{\cap}(\theta_2) = [m_1 \oplus m_2](\theta_2) = 0 \\ m_{\cap}(\theta_3) = [m_1 \oplus m_2](\theta_3) = 0.1 \cdot 0.1 = 0.01 \end{cases}$$

and the conflict between the sources is very high and is given by

$$k_{12} = 1 - m_{\cap}(\theta_1) - m_{\cap}(\theta_2) - m_{\cap}(\theta_3) = 0.99$$

4.3.1 Fusion with the PCR1 rule

Using the PCR1 rule of combination, the conflict $k_{12} = 0.99$ is proportionally distributed to $m_{\cap}(\theta_1)$, $m_{\cap}(\theta_2)$, $m_{\cap}(\theta_3)$ with respect to their corresponding sums of columns, i.e. 0.9, 0.9, 0.2 respectively. Thus: $w_{12}(\theta_1)/0.9 = w_{12}(\theta_2)/0.9 = w_{12}(\theta_3)/0.2 = 0.99/2 = 0.495$. Hence: $w_{12}(\theta_1) = 0.9 \cdot 0.495 = 0.4455$, $w_{12}(\theta_2) = 0.9 \cdot 0.495 = 0.4455$ and $w_{12}(\theta_3) = 0.2 \cdot 0.495 = 0.0990$. Finally the result of the PCR1 rule of combination is given by

$$m_{PCR1}(\theta_1) = 0 + 0.4455 = 0.4455$$

$$m_{PCR1}(\theta_2) = 0 + 0.4455 = 0.4455$$

$$m_{PCR1}(\theta_3) = 0.01 + 0.099 = 0.109$$

This is an acceptable result if we don't want to introduce the partial ignorances (epistemic partial uncertainties). This result is close to Murphy's arithmetic mean combination rule [8], which is the following (M index standing here for the Murphy's rule) :

$$m_M(\theta_1) = (m_1(\theta_1) + m_2(\theta_1))/2 = (0.9 + 0)/2 = 0.45$$

$$m_M(\theta_2) = (m_1(\theta_2) + m_2(\theta_2))/2 = (0 + 0.9)/2 = 0.45$$

$$m_M(\theta_3) = (m_1(\theta_3) + m_2(\theta_3))/2 = (0.1 + 0.1)/2 = 0.10$$

4.3.2 Fusion with the Dempster's rule

The use of the Dempster's rule of combination yields here to the counter-intuitive result $m_{DS}(\theta_3) = 1$. This example is discussed in details in [11] where several other infinite classes of counter-examples to the Dempster's rule are also presented.

4.3.3 Fusion with the Smets' rule

Based on the open-world model with TBM, the Smets' rule of combination gives very little information, i;e. $m_S(\theta_3) = 0.01$ and $m_S(\emptyset) = k_{12} = 0.99$.

4.3.4 Fusion with the Yager's rule

The Yager's rule of combination transfers the conflicting mass k_{12} onto the total uncertainty and thus provides little specific information since one gets $m_Y(\theta_3) = 0.01$ and $m_Y(\theta_1 \cup \theta_2 \cup \theta_3) = 0.99$.

4.3.5 Fusion with the Dubois & Prade and DSmT rule

In zadeh's example, the hybrid DSm rule and the Dubois-Prade rule give the same result: $m(\theta_3) = 0.01$, $m(\theta_1 \cup \theta_2) = 0.81$, $m(\theta_1 \cup \theta_3) = 0.09$ and $m(\theta_2 \cup \theta_3) = 0.09$. This fusion result is more informative/specific than previous rules of combination and is acceptable if one wants to take into account all aggregated partial epistemic uncertainties.

4.4 Example 4 (with total conflict)

Let's consider now the 4D case with the frame $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$ and two independent equi-reliable sources of evidence with the following Bayesian belief assignment matrix (where columns 1, 2, 3 and 4 correspond to elements θ_1, θ_2, θ_3 and θ_4 and rows 1 and 2 to belief assignments $m_1(.)$ and $m_2(.)$ respectively)

$$\mathbf{M}_{12} = \begin{bmatrix} 0.3 & 0 & 0.7 & 0 \\ 0 & 0.4 & 0 & 0.6 \end{bmatrix}$$

4.4.1 Fusion with the PCR1 rule

Using the PCR1 rule of combination, one gets $k_{12} = 1$ and

$$m_{\cap}(\theta_1) = m_{\cap}(\theta_2) = m_{\cap}(\theta_3) = m_{\cap}(\theta_4) = 0$$

We distribute the conflict among $m_{\cap}(\theta_1)$, $m_{\cap}(\theta_2)$, $m_{\cap}(\theta_3)$ and $m_{\cap}(\theta_4)$ proportionally with their sum of columns, i.e., 0.3, 0.4, 0.7 and 0.6 respectively. Thus:

$$\frac{w_{12}(\theta_1)}{0.3} = \frac{w_{12}(\theta_2)}{0.4} = \frac{w_{12}(\theta_3)}{0.7} = \frac{w_{12}(\theta_4)}{0.6} = \frac{w_{12}(\theta_1) + w_{12}(\theta_2) + w_{12}(\theta_3) + w_{12}(\theta_4)}{0.3 + 0.4 + 0.7 + 0.6} = \frac{1}{2} = 0.5$$

Then $w_{12}(\theta_1) = 0.3 \cdot 0.5 = 0.15$, $w_{12}(\theta_2) = 0.4 \cdot 0.5 = 0.20$, $w_{12}(\theta_3) = 0.7 \cdot 0.5 = 0.35$ and $w_{12}(\theta_4) = 0.6 \cdot 0.5 = 0.30$ and add them to the previous masses. One easily gets:

$$m_{PCR1}(\theta_1) = 0.15 \qquad m_{PCR1}(\theta_2) = 0.20 \qquad m_{PCR1}(\theta_3) = 0.35 \qquad m_{PCR1}(\theta_4) = 0.30$$

In this case the PCR1 combination rule gives the same result as Murphy's arithmetic mean combination rule.

4.4.2 Fusion with the Dempster's rule

In this example, the Dempster's rule can't be applied since the sources are in total contradiction because $k_{12} = 1$. Dempster's rule is mathematically not defined because of the indeterminate form 0/0.

4.4.3 Fusion with the Smets' rule

Using open-world assumption, the Smets' rule provides no specific information, only $m_S(\emptyset) = 1$.

4.4.4 Fusion with the Yager's rule

The Yager's rule gives no information either: $m_Y(\theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4) = 1$ (total ignorance).

4.4.5 Fusion with the Dubois & Prade and DSmT rule

The hybrid DSm rule and the Dubois-Prade rule give here the same result:

$$m(\theta_1 \cup \theta_2) = 0.12 \qquad m(\theta_1 \cup \theta_4) = 0.18 \qquad m(\theta_2 \cup \theta_3) = 0.28 \qquad m(\theta_3 \cup \theta_4) = 0.42$$

4.5 Example 5 (convergent to idempotence)

Let's consider now the 2D case with the frame of discernment $\Theta = \{\theta_1, \theta_2\}$ and two independent equi-reliable sources of evidence with the following Bayesian belief assignment matrix (where columns 1 and 2 correspond to elements θ_1 and θ_2 and rows 1 and 2 to belief assignments $m_1(.)$ and $m_2(.)$ respectively)

$$\mathbf{M}_{12} = \begin{bmatrix} 0.7 & 0.3 \\ 0.7 & 0.3 \end{bmatrix}$$

The conjunctive consensus yields here:

$$m_{\cap}(\theta_1) = 0.49 \qquad \text{and} \qquad m_{\cap}(\theta_2) = 0.09$$

with conflict $k_{12} = 0.42$.

4.5.1 Fusion with the PCR1 rule

Using the PCR1 rule of combination, one gets after distributing the conflict proportionally among $m_{\cap}(\theta_1)$ and $m_{\cap}(\theta_2)$ with $0.7 + 0.7 = 1.4$ and $0.3 + 0.3 = 0.6$ such that

$$\frac{w_{12}(\theta_1)}{1.4} = \frac{w_{12}(\theta_2)}{0.6} = \frac{w_{12}(\theta_1) + w_{12}(\theta_2)}{1.4 + 0.6} = \frac{0.42}{2} = 0.21$$

whence $w_{12}(\theta_1) = 0.294$ and $w_{12}(\theta_2) = 0.126$ involving the following result

$$m_{PCR1}(\theta_1) = 0.49 + 0.294 = 0.784 \qquad m_{PCR1}(\theta_2) = 0.09 + 0.126 = 0.216$$

4.5.2 Fusion with the Dempster's rule

The Dempster's rule of combination gives here:

$$m_{DS}(\theta_1) = 0.844828 \qquad \text{and} \qquad m_{DS}(\theta_2) = 0.155172$$

4.5.3 Fusion with the Smets' rule

Based on the open-world model with TBM, the Smets' rule of combination provides here:

$$m_S(\theta_1) = 0.49 \qquad m_S(\theta_2) = 0.09 \qquad m_S(\emptyset) = 0.42$$

4.5.4 Fusion with the other rules

The hybrid DSm rule, the Dubois-Prade rule and the Yager's give here:

$$m(\theta_1) = 0.49 \qquad m(\theta_2) = 0.09 \qquad m(\theta_1 \cup \theta_2) = 0.42$$

4.5.5 Behavior of the PCR1 rule with respect to idempotence

Let's combine now with the PCR1 rule four equal sources $m_1(.) = m_2(.) = m_3(.) = m_4(.)$ with $m_i(\theta_1) = 0.7$ and $m_i(\theta_2) = 0.3$ ($i = 1, \ldots, 4$). The PCR1 result[4] is now given by

$$m^{1234}_{PCR1}(\theta_1) = 0.76636 \qquad m^{1234}_{PCR1}(\theta_2) = 0.23364$$

Then repeat the fusion with the PCR1 rule for eight equal sources $m_i(\theta_1) = 0.7$ and $m_i(\theta_2) = 0.3$ ($i = 1, \ldots, 8$). One gets now:

$$m^{1...8}_{PCR1}(\theta_1) = 0.717248 \qquad m^{1...8}_{PCR1}(\theta_2) = 0.282752$$

Therefore $m_{PCR1}(\theta_1) \to 0.7$ and $m_{PCR1}(\theta_2) \to 0.3$. We can prove that the fusion using PCR1 rule converges towards idempotence, i.e. for $i = 1, 2$

$$\lim_{n\to\infty} \underbrace{[m \oplus m \oplus \ldots \oplus m](\theta_i)}_{n \text{ times}} = m(\theta_i)$$

in the 2D simple case with exclusive hypotheses, no unions, neither intersections (i.e. with Bayesian belief assignments).

Let $\Theta = \{\theta_1, \theta_2\}$ and the mass matrix

$$\mathbf{M}_{1...n} = \begin{bmatrix} a & 1-a \\ a & 1-a \\ \vdots & \vdots \\ a & 1-a \end{bmatrix}$$

Using the general PCR1 formula, one gets for any $A \neq \emptyset$,

$$\lim_{n\to\infty} m^{1...n}_{PCR1}(\theta_1) = a^n + n \cdot a \cdot \frac{k_{1...n}}{n} = a^n + a[1 - a^n - (1-a)^n] = a$$

because $\lim_{n\to\infty} a^n = \lim_{n\to\infty} (1-a)^n = 0$ when $0 < a < 1$; if $a = 0$ or $a = 1$ also $\lim_{n\to\infty} m^{1...n}_{PCR1}(\theta_1) = a$. We can prove similarly $\lim_{n\to\infty} m^{1...n}_{PCR1}(\theta_2) = 1 - a$

One similarly proves the n-D, $n \geq 2$, simple case for $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ with exclusive elements when no mass is on unions neither on intersections.

4.6 Example 6 (majority opinion)

Let's consider now the 2D case with the frame $\Theta = \{\theta_1, \theta_2\}$ and two independent equi-reliable sources of evidence with the following belief assignment matrix (where columns 1 and 2 correspond to elements θ_1 and θ_2 and rows 1 and 2 to belief assignments $m_1(.)$ and $m_2(.)$ respectively)

$$\mathbf{M}_{12} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

Then after a while, assume that a third independent source of evidence is introduces with belief assignment $m_3(\theta_1) = 0.3$ and $m_3(\theta_2) = 0.7$. The previous belief matrix is then extended/updated as follows (where the third row of matrix $\mathbf{M}$ corresponds to the new source $m_3(.)$)

$$\mathbf{M}_{123} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \\ 0.3 & 0.7 \end{bmatrix}$$

[4] The verification is left to the reader.

4.6.1 Fusion with the PCR1 rule

The conjunctive consensus for sources 1 and 2 gives (where upper index 12 denotes the fusion of source 1 and 2)

$$m_{\cap}^{12}(\theta_1) = 0.24 \qquad m_{\cap}^{12}(\theta_2) = 0.14$$

with conflict $k_{12} = 0.62$.

We distribute the conflict 0.62 proportionally with 1.1 and 0.9 respectively to $m_{\cap}^{12}(\theta_1)$ and $m_{\cap}^{12}(\theta_2)$ such that

$$\frac{w_{12}(\theta_1)}{1.1} = \frac{w_{12}(\theta_2)}{0.9} = \frac{w_{12}(\theta_1) + w_{12}(\theta_2)}{1.1 + 0.9} = \frac{0.62}{2} = 0.31$$

and thus $w_{12}(\theta_1) = 1.1 \cdot 0.31 = 0.341$ and $w_{12}(\theta_2) = 0.9 \cdot 0.31 = 0.279$.

Using the PCR1 combination rule for sources 1 and 2, one gets:

$$m_{PCR1}^{12}(\theta_1) = 0.24 + 0.341 = 0.581 \qquad m_{PCR1}^{12}(\theta_2) = 0.14 + 0.279 = 0.419$$

Let's combine again the previous result with $m_3(.)$ to check the majority rule (if the result's trend is towards $m_3 = m_2$). Consider now the following matrix (where columns 1 and 2 correspond to elements θ_1 and θ_2 and rows 1 and 2 to belief assignments $m_{PCR1}^{12}(.)$ and $m_3(.)$ respectively)

$$\mathbf{M}_{12,3} = \begin{bmatrix} 0.581 & 0.419 \\ 0.3 & 0.7 \end{bmatrix}$$

The conjunctive consensus obtained from $m_{PCR1}^{12}(.)$ and $m_3(.)$ gives

$$m_{\cap}^{12,3}(\theta_1) = 0.1743 \qquad m_{\cap}^{12,3}(\theta_2) = 0.2933$$

with conflict $k_{12,3} = 0.5324$ where the index notation 12,3 stands here for the combination of the result of the fusion of sources 1 and 2 with the new source 3. The proportionality coefficients are obtained from

$$\frac{w_{12}(\theta_1)}{0.581 + 0.3} = \frac{w_{12}(\theta_2)}{0.419 + 0.7} = \frac{w_{12}(\theta_1) + w_{12}(\theta_2)}{0.581 + 0.3 + 0.419 + 0.7} = \frac{0.5324}{2} = 0.2662$$

and thus:

$$w_{12}(\theta_1) = 0.881 \cdot 0.2662 = 0.234522 \qquad w_{12}(\theta_2) = 1.119 \cdot 0.2662 = 0.297878$$

The fusion result obtained by the PCR1 after the aggregation of sources 1 and 2 with the new source 3 is:

$$m_{PCR1}^{12,3}(\theta_1) = 0.1743 + 0.234522 = 0.408822 \qquad m_{PCR1}^{12,3}(\theta_2) = 0.2933 + 0.297878 = 0.591178$$

Thus $\mathbf{m}_{PCR1}^{12,3} = [0.408822 \ 0.591178]$ starts to reflect the majority opinion $\mathbf{m}_2(.) = \mathbf{m}_3 = [0.3 \ 0.7]$ (i.e. the mass of θ_1 becomes smaller than the mass of θ_2).

If now we apply the PCR1 rule for the 3 sources taken directly together, one gets

$$m_{\cap}^{123}(\theta_1) = 0.072 \qquad m_{\cap}^{123}(\theta_2) = 0.098$$

with the total conflicting mass $k_{123} = 0.83$.

Applying the proportionalization from $\mathbf{M}_{123}$, one has

$$\frac{w_{123}(\theta_1)}{0.8 + 0.3 + 0.3} = \frac{w_{123}(\theta_2)}{0.2 + 0.7 + 0.7} = \frac{w_{123}(\theta_1) + w_{123}(\theta_2)}{3} = \frac{0.83}{3}$$

Thus, the proportionalized conflicting masses to transfer onto $m_{\cap}^{123}(\theta_1)$ and $m_{\cap}^{123}(\theta_2)$ are respectively given by

$$w_{123}(\theta_1) = 1.4 \cdot \frac{0.83}{3} = 0.387333 \qquad w_{123}(\theta_2) = 1.6 \cdot \frac{0.83}{3} = 0.442667$$

The final result of the PCR1 rule combining all three sources together is then

$$m_{PCR1}^{123}(\theta_1) = 0.072 + 0.387333 = 0.459333 \qquad m_{PCR1}^{123}(\theta_2) = 0.098 + 0.442667 = 0.540667$$

The majority opinion is reflected since $m^{123}_{PCR1}(\theta_1) < m^{123}_{PCR1}(\theta_2)$. Note however that the PCR1 rule of combination is clearly not associative because $(m^{12,3}_{PCR1}(\theta_1) = 0.408822) \neq (m^{123}_{PCR1}(\theta_1) = 0.459333)$ and $(m^{12,3}_{PCR1}(\theta_2) = 0.591178) \neq (m^{123}_{PCR1}(\theta_2) = 0.540667)$.

If we now combine the three previous sources altogether with the fourth source providing the majority opinion, i.e. $m_4(\theta_1) = 0.3$ and $m_4(\theta_2) = 0.7$ one will get

$$m^{1234}_{\cap}(\theta_1) = 0.0216 \qquad m^{123}_{\cap}(\theta_2) = 0.0686$$

with the total conflicting mass $k_{1234} = 0.9098$.

Applying the proportionalization from mass matrix

$$\mathbf{M}_{1234} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \\ 0.3 & 0.7 \\ 0.3 & 0.7 \end{bmatrix}$$

yields

$$w_{1234}(\theta_1) = [0.8 + 0.3 + 0.3 + 0.3] \cdot \frac{0.9098}{4} \qquad w_{1234}(\theta_2) = [0.2 + 0.7 + 0.7 + 0.7] \cdot \frac{0.9098}{4}$$

and finally the followwing result

$$m^{1234}_{PCR1}(\theta_1) = 0.0216 + [0.8 + 0.3 + 0.3 + 0.3] \cdot \frac{0.9098}{4} = 0.408265$$
$$m^{1234}_{PCR1}(\theta_2) = 0.0686 + [0.2 + 0.7 + 0.7 + 0.7] \cdot \frac{0.9098}{4} = 0.591735$$

Hence $m^{1234}_{PCR1}(\theta_1)$ is decreasing more and more while $m^{1234}_{PCR1}(\theta_2)$ is increasing more and more, which reflects again the majority opinion.

4.7 Example 7 (multiple sources of information)

Let's consider now the 2D case with the frame $\Theta = \{\theta_1, \theta_2\}$ and 10 independent equi-reliable sources of evidence with the following Bayesian belief assignment matrix (where columns 1 and 2 correspond to elements θ_1 and θ_2 and rows 1 to 10 to belief assignments $m_1(.)$ to $m_{10}(.)$ respectively)

$$\mathbf{M}_{1...10} = \begin{bmatrix} 1 & 0 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \\ 0.1 & 0.9 \end{bmatrix}$$

The conjunctive consensus operator gives here

$$m_{\cap}(\theta_1) = (0.1)^9 \qquad m_{\cap}(\theta_2) = 0$$

with the conflict $k_{1...10} = 1 - (0.1)^9$.

4.7.1 Fusion with the PCR1 rule

Using the general PCR1 formula (17), one gets

$$\begin{aligned} m^{1...10}_{PCR1}(\theta_1) &= (0.1)^9 + c_{1...10}(\theta_1) \cdot \frac{k_{1...10}}{10} = (0.1)^9 + (1.9) \cdot \frac{1 - (0.1)^9}{10} = (0.1)^9 + (0.19) \cdot [1 - (0.1)^9] \\ &= (0.1)^9 + 0.19 - 0.19 \cdot (0.1)^9 = (0.1)^9 \cdot 0.81 + 0.19 \approx 0.19 \end{aligned}$$

$$\begin{aligned} m^{1...10}_{PCR1}(\theta_2) &= (0.9)^9 + c_{1...10}(\theta_2) \cdot \frac{k_{1...10}}{10} = (0.9)^9 + (8.1) \cdot \frac{1 - (0.1)^9}{10} = (0.9)^9 + (0.81) \cdot [1 - (0.1)^9] \\ &= (0.9)^9 + 0.81 - 0.81 \cdot (0.1)^9 = (0.1)^9 \cdot 0.19 + 0.81 \approx 0.81 \end{aligned}$$

The PCR1 rule's result is converging towards the Murphy's rule in this case, which is $m_M(\theta_1) = 0.19$ and $m_M(\theta_2) = 0.81$.

4.7.2 Fusion with the Dempster's rule

In this example, the Dempster's rule of combination gives $m_{DS}(\theta_1) = 1$ which looks quite surprising and certainly wrong since nine sources indicate $m_i(\theta_1) = 0.1$ ($i = 2, \ldots, 10$) and only one shows $m_1(\theta_1) = 1$.

4.7.3 Fusion with the Smets' rule

In this example when assuming open-world model, the Smets' rule provide little specific information since one gets

$$m_S(\theta_1) = (0.1)^9 \qquad m_S(\emptyset) = 1 - (0.1)^9$$

4.7.4 Fusion with the other rules

The hybrid DSm rule, the Dubois-Prade's rule and the Yager's rule give here:

$$m(\theta_1) = (0.1)^9 \qquad m(\theta_1 \cup \theta_2) = 1 - (0.1)^9$$

which is less specific than PCR1 result but seems more reasonable and cautious if one introduces/takes into account epistemic uncertainty arising from the conflicting sources if we consider that the majority opinion does not necessary reflect the reality of the solution of a problem. The answer to this philosophical question is left to the reader.

4.8 Example 8 (based on hybrid DSm model)

In this last example, we show how the PCR1 rule can be applied on a fusion problem characterized by a hybrid DSm model rather than the Shafer's model and we compare the result of the PCR1 rule with the result obtained from the hybrid DSm rule.

Let's consider a 3D case (i.e. $\Theta = \{\theta_1, \theta_2, \theta_2\}$) including epistemic uncertainties with the two following belief assignments

$$m_1(\theta_1) = 0.4 \qquad m_1(\theta_2) = 0.1 \qquad m_1(\theta_3) = 0.3 \qquad m_1(\theta_1 \cup \theta_2) = 0.2$$
$$m_2(\theta_1) = 0.6 \qquad m_2(\theta_2) = 0.2 \qquad m_2(\theta_3) = 0.2$$

We assume here a *hybrid DSm model* [11] (chap. 4) in which the following integrity constraints hold

$$\theta_1 \cap \theta_2 = \theta_1 \cap \theta_3 = \emptyset$$

but where $\theta_2 \cap \theta_3 \neq \emptyset$.

The conjunctive consensus rule extended to the hyper-power set D^Θ (i.e. the Dedekind's lattice built on Θ with union and intersection operators) becomes now the classic DSm rule and we obtain

$$m_\cap(\theta_1) = 0.36 \qquad m_\cap(\theta_2) = 0.06 \qquad m_\cap(\theta_3) = 0.06 \qquad m_\cap(\theta_2 \cap \theta_3) = 0.12$$

One works on hyper-power set (which contains, besides unions, intersections as well), not on power set as in all other theories based on the Shafer's model (because power set contains only unions, not intersections).

The conflicting mass k_{12} is thus formed together by the masses of $\theta_1 \cap \theta_2$ and $\theta_1 \cap \theta_3$ and is given by

$$\begin{aligned} k_{12} &= m(\theta_1 \cap \theta_2) + m(\theta_1 \cap \theta_3) = [0.4 \cdot 0.2 + 0.6 \cdot 0.1] + [0.4 \cdot 0.2 + 0.6 \cdot 0.2] = 0.14 + 0.26 = 0.40 \\ &= 1 - m_\cap(\theta_1) - m_\cap(\theta_2) - m_\cap(\theta_3) - m_\cap(\theta_2 \cap \theta_3) \end{aligned}$$

The classic DSm rule (denoted here with index DSmc) provides also

$$m_{DSmc}(\theta_2 \cap \theta_3) = 0.1 \cdot 0.2 + 0.2 \cdot 0.3 = 0.08 \qquad m_{DSmc}(\theta_3 \cap (\theta_1 \cup \theta_2)) = 0.04$$

but since $\theta_3 \cap (\theta_1 \cup \theta_2) = (\theta_3 \cap \theta_1) \cup (\theta_3 \cap \theta_2) = \theta_2 \cap \theta_3$ because integrity constraint $\theta_1 \cap \theta_3 = \emptyset$ of the model, the total mass committed to $\theta_2 \cap \theta_3$ is finally

$$m_{DSmc}(\theta_2 \cap \theta_3) = 0.08 + 0.04 = 0.12$$

4.8.1 Fusion with the hybrid DSm rule

If one uses the hybrid DSm rule, one gets

$$m_{DSmh}(\theta_1) = 0.36 \qquad m_{DSmh}(\theta_2) = 0.06 \qquad m_{DSmh}(\theta_3) = 0.06$$
$$m_{DSmh}(\theta_1 \cup \theta_2) = 0.14 \qquad m_{DSmh}(\theta_1 \cup \theta_3) = 0.26 \qquad m_{DSmh}(\theta_2 \cap \theta_3) = 0.12$$

4.8.2 Fusion with the PCR1 rule

If one uses the PCR1 rule, one has to distribute the conflicting mass 0.40 to the others according to

$$\frac{w_{12}(\theta_1)}{1.0} = \frac{w_{12}(\theta_2)}{0.3} = \frac{w_{12}(\theta_3)}{0.5} = \frac{w_{12}(\theta_1 \cup \theta_2)}{0.2} = \frac{0.40}{2} = 0.20$$

Thus one deduces $w_{12}(\theta_1) = 0.20$, $w_{12}(\theta_2) = 0.06$, $w_{12}(\theta_3) = 0.10$ and $w_{12}(\theta_1 \cup \theta_2) = 0.04$.

Nothing is distributed to $\theta_1 \cup \theta_2$ because its column in the mass matrix is $[0\ 0]'$, therefore its sum is zero. Finally, one gets the following results with the PCR1 rule of combination:

$$m_{PCR1}(\theta_1) = 0.36 + 0.20 = 0.56 \qquad m_{PCR1}(\theta_2) = 0.06 + 0.06 = 0.12 \qquad m_{PCR1}(\theta_3) = 0.06 + 0.10 = 0.16$$

$$m_{PCR1}(\theta_1 \cup \theta_2) = 0 + 0.0.4 = 0.04 \qquad m_{PCR1}(\theta_2 \cap \theta_3) = 0.12 + 0 = 0.12$$

5 Conclusion

In this paper a very simple alternative rule to WAO has been proposed for managing the transfer of epistemic uncertainty in any framework (Dempster-Shafer Theory, Dezert-Smarandache Theory) which overcomes limitations of the Dempster's rule yielding to counter-intuitive results for highly conflicting sources to combine. This rule is interesting both from the implementation standpoint and the coherence of the result if we don't accept the transfer of conflicting mass to partial ignorances. It appears as an interesting compromise between the Dempster's rule of combination and the more complex (but more cautious) hybrid DSm rule of combination. This first and simple Proportional Conflict Redistribution (PCR1) rule of combination works in all cases no matter how big the conflict is between sources, but when some sources become totally ignorant because in such cases, PCR1 (as WAO) does not preserve the neutrality property of the vacuous belief assignment in the combination. PCR1 corresponds to a given choice of proportionality coefficients in the infinite continuum family of possible rules of combination (i.e. weighted operator - WO) involving conjunctive consensus pointed out by Inagaki in 1991 and Lefèvre, Colot and Vannoorenberghe in 2002. The PCR1 on the power set and for non-degenerate cases gives the same results as WAO; yet, for the storage proposal in a dynamic fusion when the associativity is needed, for PCR1 it is needed to store only the last sum of masses, besides the previous conjunctive rules result, while in WAO it is in addition needed to store the number of the steps. PCR1 and WAO rules become quasi-associative. In this work, we extend WAO (which herein gives no solution) for the degenerate case when all column sums of all non-empty sets are zero, and then the conflicting mass is transferred to the non-empty disjunctive form of all non-empty sets together; but if this disjunctive form happens to be empty, then one considers an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set. In addition to WAO, we propose a general formula for PCR1 (WAO for non-degenerate cases). Several numerical examples and comparisons with other rules for combination of evidence published in literature have been presented too. Another distinction between these alternative rules is that WAO is defined on the power set 2^Θ, while PCR1 is on the hyper-power set D^Θ. PCR1 and WAO are particular cases of the WO. In PCR1, the proportionalization is done for each non-empty set with respect to the non-zero sum of its corresponding mass matrix - instead of its mass column average as in WAO, but the results are the same as Ph. Smets has pointed out in non degenerate cases. In this paper, one has also proved that a nice feature of PCR1, is that it works in all cases; i.e. not only on non-degenerate cases but also on degenerate cases as well (degenerate cases might appear in dynamic fusion problems), while the WAO does not work in these cases since it gives the sum of masses less than 1. WAO and PCR1 provide both however a counter-intuitive result when one or several sources become totally ignorant that why improved versions of PCR1 have been developed in a companion paper.

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Uniform and Partially Uniform Redistribution Rules

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Abstract - ***This short paper introduces two new fusion rules for combining quantitative basic belief assignments. These rules although very simple have not been proposed in literature so far and could serve as useful alternatives because of their low computation cost with respect to the recent advanced Proportional Conflict Redistribution rules developed in the DSmT framework.***

Keywords: Uniform Redistribution Rule, Partially Uniform Redistribution Rule, information fusion, belief functions, Dezert-Smarandache Theory (DSmT).

1 Introduction

Since the development of DSmT (Dezert-Smarandache Theory) in 2002 [4, 5], a new look for information fusion in the framework of belief has been proposed which covers many aspects related to the fusion of uncertain and conflicting beliefs. Mainly, the fusion of quantitative or qualitative belief functions of highly uncertain and confliction sources of evidence with theoretical advances in belief conditioning rules. The Shafer's milestone book [3] introducing the concept of belief functions and Demspter's rule of combination of beliefs has been the important step towards non probabilistic reasoning approach, aside Zadeh's fuzzy logic [6, 8]. Since Shafer's seminal work, many alternatives have been proposed to circumvent limitations of Dempster's rule pointed out first by Zadeh in [7] (see [2] and [5] for a review). The Proportional Conflict Redistribution rule number 5 (PCR5) [5] is one of the most efficient alternative to Dempster's rule which can be used both in Dempster-Shafer Theory (DST) as well as in DSmT. The simple idea behind PCR5 is to redistribute every partial conflict only onto propositions which are truly involved in the partial conflict and proportionally to the corresponding belief mass assignment of each source generating this conflict. Although very efficient and appealing, the PCR5 rule suffers of its relative complexity in implementation and in some cases, it is required to use simpler (but less precise) rule of combination which requires only a low complexity. For this purpose, we herein present two new cheap alternatives for combination of basic belief assignments (bba's): the Uniform Redistribution Rule (URR) and the Partially Uniform Redistribution Rule (PURR). In the sequel, we assume the reader familiar with the basics of DSmT, mainly with the definition and notation of hyper-power set 2^G and also bba's defined over hyper-power set. Basics of DSmT can be found in chapter 1 of [4] which is freely downloadable on internet.

2 Uniform Redistribution Rule

Let's consider a finite and discrete frame of discernment Θ, its hyper-power set 2^G (i.e. Dedekind's lattice) and two quantitative basic belief assignments $m_1(.)$ and $m_2(.)$ defined on 2^G expressed by two independent sources of evidence.

The Uniform Redistribution Rule (URR) consists in redistributing the total conflicting mass k_{12} to all focal elements of G^Θ generated by the consensus operator. This way of redistributing mass is very simple and URR is different from Dempster's rule of combination [3], because Dempster's rule redistributes the total conflict proportionally with respect to the masses resulted from the conjunctive rule of non-empty sets. PCR5 and PCR4 [5] do proportional redistributions of partial conflicting masses to the sets involved in the conflict. Here it is the URR formula for two sources: $\forall A \neq \emptyset$, one has

$$m_{12URR}(A) = m_{12}(A) + \frac{1}{n_{12}} \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2) \tag{1}$$

where $m_{12}(A)$ is the result of the conjunctive rule applied to belief assignments $m_1(.)$ and $m_2(.)$, and $n_{12} = Card\{Z \in G^\Theta, m_1(Z) \neq 0 \text{ or } m_2(Z) \neq 0\}$.

For $s \geq 2$ sources to combine: $\forall A \neq \emptyset$, one has

$$m_{12\ldots sURR}(A) = m_{12\ldots s}(A) + \frac{1}{n_{12\ldots s}} \sum_{\substack{X_1,X_2,\ldots,X_s \in G^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = \emptyset}} \prod_{i=1}^{s} m_1(X_i) \tag{2}$$

where $m_{12\ldots s}(A)$ is the result of the conjunctive rule applied to $m_i(.)$, for all $i \in \{1, 2, \ldots, s\}$ and

$$n_{12\ldots s} = Card\{Z \in G^\Theta, m_1(Z) \neq 0 \text{ or } m_2(Z) \neq 0 \text{ or } \ldots \text{ or } m_s(Z) \neq 0\}$$

As alternative, we can also consider the cardinal of the ensemble of sets whose masses resulted from the conjunctive rule are non-null, i.e. the cardinality of the core of conjunctive consensus:

$$n^c_{12\ldots s} = \text{Card}\{Z \in G^\Theta, m_{12\ldots s}(Z) \neq 0\}$$

We denote this modified version of URR as MURR in the sequel.

3 Example for URR and MURR

Example for URR: Let's consider $\Theta = \{A, B, C\}$ with DSm hybrid model $A \cap C = C \cap (A \cup B) = \emptyset$ and the following two belief assignments

$$m_1(A) = 0.4 \qquad m_1(B) = 0.2 \qquad m_1(A \cup B) = 0.4$$

$$m_2(A) = 0.2 \qquad m_2(C) = 0.3 \qquad m_2(A \cup B) = 0.5$$

then the conjunctive operator provides for this DSm hybrid model a consensus on A, B, C, $A \cup B$, $A \cap B$ and $B \cap C$ with supporting masses

$$m_{12}(A) = 0.36 \quad m_{12}(B) = 0.10 \quad m_{12}(A \cup B) = 0.20$$

$$m_{12}(A \cap B) = 0.04 \quad m_{12}(B \cap C) = 0.06$$

and partial conflicts between two sources on $A \cap C$ and $C \cap (A \cup B)$ with

$$m_{12}(A \cap C) = 0.12 \quad m_{12}(C \cap (A \cup B)) = 0.12$$

Then with URR, the total conflicting mass $m_{12}(A \cap C) + m_{12}(C \cap (A \cup B)) = 0.12 + 0.12 = 0.24$ is uniformly (i.e. equally) redistributed to A, B, C and $A \cup B$ because the sources support only these propositions. That is $n_{12} = 4$ and thus $0.24/n_{12} = 0.06$ is added to $m_{12}(A)$, $m_{12}(B)$, $m_{12}(C)$ and $m_{12}(A \cup B)$ with URR. One finally gets:

$$m_{12URR}(A) = m_{12}(A) + \frac{0.24}{n_{12}} = 0.36 + 0.06 = 0.42$$

$$m_{12URR}(B) = m_{12}(B) + \frac{0.24}{n_{12}} = 0.10 + 0.06 = 0.16$$

$$m_{12URR}(C) = m_{12}(C) + \frac{0.24}{n_{12}} = 0.00 + 0.06 = 0.06$$

$$m_{12URR}(A \cup B) = m_{12}(A \cup B) + \frac{0.24}{n_{12}} = 0.20 + 0.06 = 0.26$$

while the others remain the same:

$$m_{12URR}(A \cap B) = 0.04$$

$$m_{12URR}(B \cap C) = 0.06$$

and of course

$$m_{12URR}(A \cap C) = m_{12URR}(C \cap (A \cup B)) = 0$$

Example for MURR: Let's consider the same frame, same model and same bba as in previous example. In this case the total conflicting mass 0.24 is uniformly redistributed to the sets A, B, $A \cup B$, $A \cap B$, and $B \cap C$, i.e.

to the sets whose masses, after applying the conjunctive rule to the given sources, are non-zero. Thus $n_{12} = 5$, and $0.24/5 = 0.048$. Hence:

$$m_{12MURR}(A) = 0.36 + 0.048 = 0.408$$
$$m_{12MURR}(B) = 0.10 + 0.048 = 0.148$$
$$m_{12MURR}(A \cup B) = 0.20 + 0.048 = 0.248$$
$$m_{12MURR}(A \cap B) = 0.04 + 0.048 = 0.088$$
$$m_{12MURR}(B \cap C) = 0.06 + 0.048 = 0.108$$

4 Partially Uniform Redistribution Rule

It is also possible to do a uniformly partial redistribution, i.e. to uniformly redistribute the conflicting mass only to the sets involved in the conflict. For example, if $m_{12}(A \cap B) = 0.08$ and $A \cap B = \emptyset$, then 0.08 is equally redistributed to A and B only, supposing A and B are both non-empty, so 0.04 assigned to A and 0.04 to B.

$\forall A \neq \emptyset$, one has the Partially Uniform Redistribution Rule (PURR) for two sources

$$m_{12PURR}(A) = m_{12}(A) + \frac{1}{2} \sum_{\substack{X_1, X_2 \in G^\Theta \\ X_1 \cap X_2 = \emptyset \\ X_1 = A \text{ or } X_2 = A}} m_1(X_1) m_2(X_2) \tag{3}$$

where $m_{12}(A)$ is the result of the conjunctive rule applied to belief assignments $m_1(.)$ and $m_2(.)$.

For $s \geq 2$ sources to combine: $\forall A \neq \emptyset$, one has

$$m_{12\ldots sPURR}(A) = m_{12\ldots s}(A) + \frac{1}{s} \sum_{\substack{X_1, X_2, \ldots, X_s \in G^\Theta \\ X_1 \cap X_2 \cap \ldots \cap X_s = \emptyset \\ \text{at leat one } X_j = A, j \in \{1, \ldots, s\}}} \text{Card}_A(\{X_1, \ldots, X_s\}) \prod_{i=1}^{s} m_1(X_i) \tag{4}$$

where $\text{Card}_A(\{X_1, \ldots, X_s\})$ is the number of A's occurring in $\{X_1, X_2, \ldots, X_s\}$.

If $A = \emptyset$, $m_{12PURR}(A) = 0$ and $m_{12\ldots sPURR}(A) = 0$.

5 Example for PURR

Let's take back the example of section 3. Based on PURR, $m_{12}(A \cap C) = 0.12$ is redistributed as follows: 0.06 to A and 0.06 to C. And $m_{12}(C \cap (A \cup B)) = 0.12$ is redistributed in this way: 0.06 to C and 0.06 to $A \cup B$. Therefore we finally get

$$m_{12PURR}(A) = m_{12}(A) + \frac{0.12}{2} = 0.36 + 0.06 = 0.42$$
$$m_{12PURR}(B) = m_{12}(B) = 0.10$$
$$m_{12PURR}(C) = m_{12}(C) + \frac{0.12}{2} + \frac{0.12}{2} = 0.12$$

$$m_{12PURR}(A \cup B) = m_{12}(A \cup B) + \frac{0.12}{2} = 0.20 + 0.06 = 0.26$$

while the others remain the same:

$$m_{12PURR}(A \cap B) = 0.04$$
$$m_{12PURR}(B \cap C) = 0.06$$

and of course

$$m_{12PURR}(A \cap C) = m_{12PURR}(C \cap (A \cup B)) = 0$$

6 Neutrality of vacuous belief assignment

Both URR (with MURR included) and PURR verify the neutrality of Vacuous Belief Assignment (VBA): since any bba $m_1(.)$ combined with the VBA defined on any frame $\Theta = \{\theta_1, \ldots, \theta_n\}$ by $m_{VBA}(\theta_1 \cup \ldots \cup \theta_n) = 1$, using the conjunctive rule, gives $m_1(.)$, so no conflicting mass is needed to transfer.

7 Conclusion

Two new simple rules of combination have been presented in the framework of DSmT which have a lower complexity than PCR5. These rules are very easy to implement but from a theoretical point of view remain less precise in their transfer of conflicting beliefs since they do not take into account the proportional redistribution with respect to the mass of each set involved in the conflict. So we cannot reasonably expect that URR or PURR outperforms PCR5 but they may hopefully appear as good enough in some specific fusion problems when the level of total conflict is not important. PURR does a more refined redistribution that URR and MURR but it requires a little more calculation.

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The Combination of Paradoxical, Uncertain and Imprecise Sources of Information based on DSmT and Neutro-Fuzzy Inference

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Abstract – *The management and combination of uncertain, imprecise, fuzzy and even paradoxical or high conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. In this chapter, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT) in the literature, developed for dealing with imprecise, uncertain and paradoxical sources of information. We focus our presentation here rather on the foundations of DSmT, and on the two important new rules of combination, than on browsing specific applications of DSmT available in literature. Several simple examples are given throughout the presentation to show the efficiency and the generality of this new approach. The last part of this chapter concerns the presentation of the neutrosophic logic, the neutro-fuzzy inference and its connection with DSmT. Fuzzy logic and neutrosophic logic are useful tools in decision making after fusioning the information using the DSm hybrid rule of combination of masses.*

Keywords: Dezert-Smarandache Theory, DSmT, Data Fusion, Fuzzy Logic, Neutrosophic Logic, Neutro-fuzzy inference, Plausible and Paradoxical Reasoning

1 Introduction

The management and combination of uncertain, imprecise, fuzzy and even paradoxical or high conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. The combination (fusion) of information arises in many fields of applications nowadays (especially in defense, medicine, finance, geo-science, economy, etc). When several sensors, observers or experts have to be combined together to solve a problem, or if one wants to update our current estimation of solutions for a given problem with some new information available, we need powerful and solid mathematical tools for the fusion, specially when the information one has to deal with is imprecise and uncertain. In this chapter, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT) in the literature, developed for dealing with imprecise, uncertain and paradoxical sources of information. Recent publications have shown the interest and the ability of DSmT to solve problems where other approaches fail, especially when conflict between sources becomes high. We focus our presentation here rather on the foundations of DSmT, and on the two important new rules of combination, than on browsing specific applications of DSmT available in literature. A particular attention is given to general (hybrid) rule of combination which deals with any model for fusion problems, depending on the nature of elements or hypotheses involved into them. The Shafer's model on which is based the Dempster-Shafer Theory (DST) appears only as a specific DSm hybrid model and can be easily handled by our approach as well. Several simple examples are given throughout the presentation to show the efficiency and the generality of this new approach. The last part of this work concerns the presentation of the neutrosophic logic, the neutro-fuzzy inference and its connection with DSmT. Fuzzy logic and neutrosophic logic are useful tools in decision making after fusioning the information using the DSm hybrid rule of combination of masses.

2 Foundations of the DSmT

The development of the DSmT (Dezert-Smarandache Theory of plausible and paradoxical reasoning [37]) arises from the necessity to overcome the inherent limitations of the DST (Dempster-Shafer Theory [31]) which are closely related with

the acceptance of Shafer's model for the fusion problem under consideration (i.e. the frame of *discernment* Θ defined as a finite set of *exhaustive* and *exclusive* hypotheses θ_i, $i = 1, \ldots, n$), the third middle excluded principle (i.e. the existence of the complement for any elements/propositions belonging to the power set of Θ), and the acceptance of Dempter's rule of combination (involving normalization) as the framework for the combination of independent sources of evidence. Discussions on limitations of DST and presentation of some alternative rules to the Dempster's rule of combination can be found in [50, 51, 52, 46, 53, 17, 47, 28, 39, 43, 20, 27, 22, 30, 23, 37] and therefore they will be not reported in details in this chapter due to space limitation. We argue that these three fundamental conditions of the DST can be removed and another new mathematical approach for combination of evidence is possible.

The basis of the DSmT is the refutation of the principle of the third excluded middle and Shafer's model, since for a wide class of fusion problems the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated. Many problems involving fuzzy continuous and relative concepts described in natural language and having no absolute interpretation like tallness/smallness, pleasure/pain, cold/hot, Sorites paradoxes, etc, enter in this category. DSmT starts with the notion of *free DSm model*, denoted $\mathcal{M}^f(\Theta)$, and considers Θ only as a frame of exhaustive elements θ_i, $i = 1, \ldots, n$ which can potentially overlap. This model is *free* because no other assumption is done on the hypotheses, but the weak exhaustivity constraint which can always been satisfied according the closure principle explained in [37]. No other constraint is involved in the free DSm model. When the free DSm model holds, the classic commutative and associative DSm rule of combination (corresponding to the conjunctive consensus defined on the free Dedekind's lattice) is performed.

Depending on the intrinsic nature of the elements of the fusion problem under consideration, it can however happen that the free model does not fit the reality because some subsets of Θ can contain elements known to be truly exclusive but also truly non existing at all at a given time (specially when working on dynamic fusion problem where the frame Θ varies with time with the revision of the knowledge available). These integrity constraints are then explicitly and formally introduced into the free DSm model $\mathcal{M}^f(\Theta)$ in order to adapt it properly to fit as close as possible with the reality and permit to construct a *hybrid DSm model* $\mathcal{M}(\Theta)$ on which the combination will be efficiently performed. Shafer's model, denoted $\mathcal{M}^0(\Theta)$, corresponds to a very specific hybrid DSm model including all possible exclusivity constraints. The DST has been developed for working only with $\mathcal{M}^0(\Theta)$ while the DSmT has been developed for working with any kind of hybrid model (including Shafer's model and the free DSm model), to manage as efficiently and precisely as possible imprecise, uncertain and potentially high conflicting sources of evidence while keeping in mind the possible dynamicity of the information fusion problematic. The foundations of the DSmT are therefore totally different from those of all existing approaches managing uncertainties, imprecisions and conflicts. DSmT provides a new interesting way to attack the information fusion problematic with a general framework in order to cover a wide variety of problems.

DSmT refutes also the idea that sources of evidence provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliabilty of sources, but also because of possible different and relative interpretation of Θ, e.g. what is considered as good for somebody can be considered as bad for somebody else. There is some unavoidable subjectivity in the belief assignments provided by the sources of evidence, otherwise it would mean that all bodies of evidence have a same objective and universal interpretation (or measure) of the phenomena under consideration, which unfortunately rarely occurs in reality, but when bba are based on some *objective probabilities* transformations. But in this last case, probability theory can handle properly and efficiently the information, and the DST, as well as the DSmT, becomes useless. If we now get out of the probabilistic background argumentation for the construction of bba, we claim that in most of cases, the sources of evidence provide their beliefs about elements of the frame of the fusion problem only based on their own limited knowledge and experience without reference to the (inaccessible) absolute truth of the space of possibilities. First applications of DSmT for target tracking, satellite surveillance, situation analysis and sensor allocation optimization can be found in [37].

2.1 Notion of hyper-power set D^Θ

One of the cornerstones of the DSmT is the free Dedekind lattice [14] denoted *hyper-power set* in the DSmT framework. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements[1]. The hyper-power set D^Θ is defined as the set of all composite propositions built from elements of Θ with $\cup$ and $\cap$ operators[2] such that:

1. $\emptyset, \theta_1, \ldots, \theta_n \in D^\Theta$.
2. If $A, B \in D^\Theta$, then $A \cap B \in D^\Theta$ and $A \cup B \in D^\Theta$.

[1] We do not assume here that elements θ_i are necessary exclusive. There is no restriction on θ_i but the exhaustivity.

[2] Θ generates D^Θ under operators $\cup$ and $\cap$

3. No other elements belong to D^Θ, except those obtained by using rules 1 or 2.

The dual (obtained by switching $\cup$ and $\cap$ in expressions) of D^Θ is itself. There are elements in D^Θ which are self-dual (dual to themselves), for example α_8 for the case when $n = 3$ in the following example. The cardinality of D^Θ is majored by 2^{2^n} when the cardinality of Θ equals n, i.e. $|\Theta| = n$. The generation of hyper-power set D^Θ is closely related with the famous Dedekind problem [14, 13] on enumerating the set of isotone Boolean functions. The generation of the hyper-power set is presented in [37]. Since for any given finite set Θ, $|D^\Theta| \geq |2^\Theta|$ we call D^Θ the *hyper-power set* of Θ.

Example of the first hyper-power sets D^Θ

- For the degenerate case ($n = 0$) where $\Theta = \{\}$, one has $D^\Theta = \{\alpha_0 \triangleq \emptyset\}$ and $|D^\Theta| = 1$.
- When $\Theta = \{\theta_1\}$, one has $D^\Theta = \{\alpha_0 \triangleq \emptyset, \alpha_1 \triangleq \theta_1\}$ and $|D^\Theta| = 2$.
- When $\Theta = \{\theta_1, \theta_2\}$, one has $D^\Theta = \{\alpha_0, \alpha_1, \ldots, \alpha_4\}$ and $|D^\Theta| = 5$ with $\alpha_0 \triangleq \emptyset$, $\alpha_1 \triangleq \theta_1 \cap \theta_2$, $\alpha_2 \triangleq \theta_1$, $\alpha_3 \triangleq \theta_2$ and $\alpha_4 \triangleq \theta_1 \cup \theta_2$.
- When $\Theta = \{\theta_1, \theta_2, \theta_3\}$, one has $D^\Theta = \{\alpha_0, \alpha_1, \ldots, \alpha_{18}\}$ and $|D^\Theta| = 19$ with

$$
\begin{array}{ll}
\alpha_0 \triangleq \emptyset & \\
\alpha_1 \triangleq \theta_1 \cap \theta_2 \cap \theta_3 & \alpha_{10} \triangleq \theta_2 \\
\alpha_2 \triangleq \theta_1 \cap \theta_2 & \alpha_{11} \triangleq \theta_3 \\
\alpha_3 \triangleq \theta_1 \cap \theta_3 & \alpha_{12} \triangleq (\theta_1 \cap \theta_2) \cup \theta_3 \\
\alpha_4 \triangleq \theta_2 \cap \theta_3 & \alpha_{13} \triangleq (\theta_1 \cap \theta_3) \cup \theta_2 \\
\alpha_5 \triangleq (\theta_1 \cup \theta_2) \cap \theta_3 & \alpha_{14} \triangleq (\theta_2 \cap \theta_3) \cup \theta_1 \\
\alpha_6 \triangleq (\theta_1 \cup \theta_3) \cap \theta_2 & \alpha_{15} \triangleq \theta_1 \cup \theta_2 \\
\alpha_7 \triangleq (\theta_2 \cup \theta_3) \cap \theta_1 & \alpha_{16} \triangleq \theta_1 \cup \theta_3 \\
\alpha_8 \triangleq (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3) & \alpha_{17} \triangleq \theta_2 \cup \theta_3 \\
\alpha_9 \triangleq \theta_1 & \alpha_{18} \triangleq \theta_1 \cup \theta_2 \cup \theta_3
\end{array}
$$

The cardinality of hyper-power set D^Θ for $n \geq 1$ follows the sequence of Dedekind's numbers [32], i.e. 1,2,5,19,167, 7580,7828353,... and analytical expression of Dedekind's numbers has been obtained recently by Tombak in [42] (see [37] for details on generation and ordering of D^Θ).

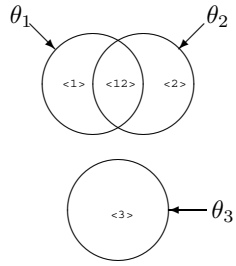
2.2 Notion of free and hybrid DSm models

Elements θ_i, $i = 1, \ldots, n$ of Θ constitute the finite set of hypotheses/concepts characterizing the fusion problem under consideration. D^Θ constitutes what we call the *free DSm model* $\mathcal{M}^f(\Theta)$ and allows to work with fuzzy concepts which depict a continuous and relative intrinsic nature. Such kinds of concepts cannot be precisely refined in an absolute interpretation because of the unapproachable universal truth.

However for some particular fusion problems involving discrete concepts, elements θ_i are truly exclusive. In such case, all the exclusivity constraints on θ_i, $i = 1, \ldots, n$ have to be included in the previous model to characterize properly the true nature of the fusion problem and to fit it with the reality. By doing this, the hyper-power set D^Θ reduces naturally to the classical power set 2^Θ and this constitutes the most restricted hybrid DSm model, denoted $\mathcal{M}^0(\Theta)$, coinciding with Shafer's model. As an exemple, let's consider the 2D problem where $\Theta = \{\theta_1, \theta_2\}$ with $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$ and assume now that θ_1 and θ_2 are truly exclusive (i.e. Shafer's model $\mathcal{M}^0$ holds), then because $\theta_1 \cap \theta_2 \overset{\mathcal{M}^0}{=} \emptyset$, one gets $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2 \overset{\mathcal{M}^0}{=} \emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} \equiv 2^\Theta$.

Between the class of fusion problems corresponding to the free DSm model $\mathcal{M}^f(\Theta)$ and the class of fusion problems corresponding to Shafer's model $\mathcal{M}^0(\Theta)$, there exists another wide class of hybrid fusion problems involving in Θ both fuzzy continuous concepts and discrete hypotheses. In such (hybrid) class, some exclusivity constraints and possibly some non-existential constraints (especially when working on dynamic[3] fusion) have to be taken into account. Each hybrid fusion problem of this class will then be characterized by a proper hybrid DSm model $\mathcal{M}(\Theta)$ with $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$ and $\mathcal{M}(\Theta) \neq \mathcal{M}^0(\Theta)$. As simple example of DSm hybrid model, let's consider the 3D case with the frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with the model $\mathcal{M} \neq \mathcal{M}^f$ in which we force all possible conjunctions to be empty, but $\theta_1 \cap \theta_2$. This hybrid DSm model is then represented with the following Venn diagram (where boundaries of intersection of θ_1 and θ_2 are not precisely defined if θ_1 and θ_2 represent only fuzzy concepts like *smallness* and *tallness* by example).

[3] i.e. when the frame Θ and/or the model $\mathcal{M}$ is changing with time.



2.3 Generalized belief functions

From a general frame Θ, we define a map $m(.) : D^\Theta \to [0,1]$ associated to a given body of evidence $\mathcal{B}$ as

$$m(\emptyset) = 0 \qquad \text{and} \qquad \sum_{A \in D^\Theta} m(A) = 1 \tag{1}$$

The quantity $m(A)$ is called the *generalized basic belief assignment/mass* (gbba) of A.

The *generalized belief and plausibility functions* are defined in almost the same manner as within the DST, i.e.

$$\text{Bel}(A) = \sum_{\substack{B \subseteq A \\ B \in D^\Theta}} m(B) \qquad\qquad \text{Pl}(A) = \sum_{\substack{B \cap A \neq \emptyset \\ B \in D^\Theta}} m(B) \tag{2}$$

These definitions are compatible with the definitions of classical belief functions in the DST framework when D^Θ reduces to 2^Θ for fusion problems where Shafer's model $\mathcal{M}^0(\Theta)$ holds. We still have $\forall A \in D^\Theta$, $\text{Bel}(A) \leq \text{Pl}(A)$. Note that when working with the free DSm model $\mathcal{M}^f(\Theta)$, one has always $\text{Pl}(A) = 1\ \forall A \neq \emptyset \in D^\Theta$ which is normal.

2.4 The classic DSm rule of combination

When the free DSm model $\mathcal{M}^f(\Theta)$ holds for the fusion problem under consideration, the classic DSm rule of combination $m_{\mathcal{M}^f(\Theta)} \equiv m(.) \triangleq [m_1 \oplus m_2](.)$ of two independent[4] sources of evidences $\mathcal{B}_1$ and $\mathcal{B}_2$ over the same frame Θ with belief functions $\text{Bel}_1(.)$ and $\text{Bel}_2(.)$ associated with gbba $m_1(.)$ and $m_2(.)$ corresponds to the conjunctive consensus of the sources. It is given by [37]:

$$\forall C \in D^\Theta, \qquad m_{\mathcal{M}^f(\Theta)}(C) \equiv m(C) = \sum_{\substack{A,B \in D^\Theta \\ A \cap B = C}} m_1(A) m_2(B) \tag{3}$$

Since D^Θ is closed under $\cup$ and $\cap$ set operators, this new rule of combination guarantees that $m(.)$ is a proper generalized belief assignment, i.e. $m(.) : D^\Theta \to [0,1]$. This rule of combination is commutative and associative and can always be used for the fusion of sources involving fuzzy concepts when free DSm model holds for the problem under consideration. This rule can be directly and easily extended for the combination of $k > 2$ independent sources of evidence [37].

This classic DSm rule of combination looks very expensive in terms of computations and memory size due to the huge number of elements in D^Θ when the cardinality of Θ increases. This remark is however valid only if the cores (the set of focal elements of gbba) $\mathcal{K}_1(m_1)$ and $\mathcal{K}_2(m_2)$ coincide with D^Θ, i.e. when $m_1(A) > 0$ and $m_2(A) > 0$ for all $A \neq \emptyset \in D^\Theta$. Fortunately, it is important to note here that in most of the practical applications the sizes of $\mathcal{K}_1(m_1)$ and $\mathcal{K}_2(m_2)$ are much smaller than $|D^\Theta|$ because bodies of evidence generally allocate their basic belief assignments only over a subset of the hyper-power set. This makes things easier for the implementation of the classic DSm rule (3). The DSm rule is actually very easy to implement. It suffices for each focal element of $\mathcal{K}_1(m_1)$ to multiply it with the focal elements of $\mathcal{K}_2(m_2)$ and then to pool all combinations which are equivalent under the algebra of sets.

While very costly in term on merory storage in the worst case (i.e. when all $m(A) > 0$, $A \in D^\Theta$ or $A \in 2^{\Theta_{ref}}$), the DSm rule however requires much smaller memory storage than for the DST working on the ultimate refinement $2^{\Theta_{ref}}$ of same initial frame Θ as shown in following table

[4] While independence is a difficult concept to define in all theories managing epistemic uncertainty, we follow here the interpretation of Smets in [38] and [39], p. 285 and consider that two sources of evidence are independent (i.e distinct and noninteracting) if each leaves one totally ignorant about the particular value the other will take.

$\lvert\Theta\rvert = n$	$\lvert D^\Theta\rvert$	$\lvert 2^{\Theta_{ref}}\rvert = 2^{2^n-1}$
2	5	$2^3 = 8$
3	19	$2^7 = 128$
4	167	$2^{15} = 32768$
5	7580	$2^{31} = 2147483648$

However in most fusion applications only a small subset of elements of D^Θ have a non null basic belief mass because all the commitments are just usually impossible to assess precisely when the dimension of the problem increases. Thus, it is not necessary to generate and keep in memory all elements of D^Θ or $2^{\Theta_{ref}}$ but only those which have a positive belief mass. However there is a real technical challenge on how to manage efficiently all elements of the hyper-power set. This problem is obviously much more difficult when trying to work on the refined frame of discernment $2^{\Theta_{ref}}$ if one prefer to apply Dempster-Shafer theory and use the Dempster's rule of combination. It is important to keep in mind that the ultimate refined frame consisting in exhaustive and exclusive finite set of refined hypotheses is just impossible to justify and to define precisely for all problems dealing with fuzzy and ill-defined continuous concepts. A full discussion and example on refinement can be found in [37].

2.5 The hybrid DSm rule of combination

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the true nature of the fusion problem under consideration which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such case, the hybrid DSm rule of combination based on the chosen hybrid DSm model $\mathcal{M}(\Theta)$ for $k \geq 2$ independent sources of information is defined for all $A \in D^\Theta$ as [37]:

$$m_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A)\Big[S_1(A) + S_2(A) + S_3(A)\Big] \tag{4}$$

where $\phi(A)$ is the *characteristic non-emptiness function* of a set A, i.e. $\phi(A) = 1$ if $A \notin \emptyset$ and $\phi(A) = 0$ otherwise, where $\emptyset \triangleq \{\emptyset_\mathcal{M}, \emptyset\}$. $\emptyset_\mathcal{M}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set. $S_1(A) \equiv m_{\mathcal{M}^f(\theta)}(A)$, $S_2(A)$, $S_3(A)$ are defined by

$$S_1(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1 \cap X_2 \cap \ldots \cap X_k) = A}} \prod_{i=1}^{k} m_i(X_i) \tag{5}$$

$$S_2(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in \emptyset \\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \emptyset) \wedge (A = I_t)]}} \prod_{i=1}^{k} m_i(X_i) \tag{6}$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ u(c(X_1 \cap X_2 \cap \ldots \cap X_k)) = A \\ (X_1 \cap X_2 \cap \ldots \cap X_k) \in \emptyset}} \prod_{i=1}^{k} m_i(X_i) \tag{7}$$

with $\mathcal{U} \triangleq u(X_1) \cup u(X_2) \cup \ldots \cup u(X_k)$ where $u(X)$ is the union of all θ_i that compose X, $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ is the total ignorance, and $c(X)$ is the canonical form[5] of X, i.e. its simplest form (for example if $X = (A \cap B) \cap (A \cup B \cup C)$, $c(X) = A \cap B$). $S_1(A)$ corresponds to the classic DSm rule for k independent sources based on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $S_3(A)$ transfers the sum of relatively empty sets directly onto the canonical disjunctive form of non-empty sets.

The hybrid DSm rule of combination generalizes the classic DSm rule of combination and is not equivalent to Dempter's rule. It works for any models (the free DSm model, Shafer's model or any other hybrid models) when manipulating *precise* generalized (or eventually classical) basic belief functions. An extension of this rule for the combination of *imprecise* generalized (or eventually classical) basic belief functions is presented in next section.

Note that in DSmT framework it is also possible to deal directly with complements if necessary depending on the problem under consideration and the information provided by the sources of evidence themselves. The first and simplest way is to work on Shafer's model when utimate refinement is possible. The second way is to deal with partially known

[5]The canonical form is introduced here in order to improve the original formula given in [37] for preserving the neutral impact of the vacuous belief mass $m(\Theta) = 1$ within complex hybrid models.

frame and introduce directly the complementary hypotheses into the frame itself. By example, if one knows only two hypotheses θ_1, θ_2 and their complements $\bar{\theta}_1$, $\bar{\theta}_2$, then can choose $\Theta = \{\theta_1, \theta_2, \bar{\theta}_1, \bar{\theta}_2\}$. In such case, we don't necessarily assume that $\bar{\theta}_1 = \theta_2$ and $\bar{\theta}_2 = \theta_1$ because $\bar{\theta}_1$ and $\bar{\theta}_2$ may include other unknown hypotheses we have no information about (case of partial known frame). More generally, in DSmT framework, it is not necessary that the frame is built on pure/simple (possibly vague) hypotheses θ_i as usually done in all theories managing uncertainty. The frame Θ can also contain directly as elements conjunctions and/or disjunctions (or mixed propositions) and negations/complements of pure hypotheses as well. The DSm rules also work in such non-classic frames because DSmT works on any distributive lattice built from Θ anywhere Θ is defined.

2.6 Examples of combination rules

Here are some numerical examples on results obtained by DSm rules of combination. More examples can be found in [37].

2.6.1 Example with $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$, two independent experts, and the two following bbas

$$m_1(\theta_1) = 0.6 \quad m_1(\theta_3) = 0.6 \quad m_2(\theta_2) = 0.6 \quad m_2(\theta_4) = 0.6$$

represented in terms of mass matrix

$$\mathbf{M} = \begin{bmatrix} 0.6 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

- The Dempster's rule can not be applied because: $\forall 1 \leq j \leq 4$, one gets $m(\theta_j) = 0/0$ (undefined!).
- But the classic DSm rule works because one obtains: $m(\theta_1) = m(\theta_2) = m(\theta_3) = m(\theta_4) = 0$, and $m(\theta_1 \cap \theta_2) = 0.12$, $m(\theta_1 \cap \theta_4) = 0.48$, $m(\theta_2 \cap \theta_3) = 0.08$, $m(\theta_3 \cap \theta_4) = 0.32$ (partial paradoxes/conflicts).
- Suppose now one finds out that all intersections are empty (Shafer's model), then one applies the hybrid DSm rule and one gets (index h stands here for *hybrid* rule): $m_h(\theta_1 \cup \theta_2) = 0.12$, $m_h(\theta_1 \cup \theta_4) = 0.48$, $m_h(\theta_2 \cup \theta_3) = 0.08$ and $m_h(\theta_3 \cup \theta_4) = 0.32$.

2.6.2 Generalization of Zadeh's example with $\Theta = \{\theta_1, \theta_2, \theta_3\}$

Let's consider $0 < \epsilon_1, \epsilon_2 < 1$ be two very tiny positive numbers (close to zero), the frame of discernment be $\Theta = \{\theta_1, \theta_2, \theta_3\}$, have two experts (independent sources of evidence s_1 and s_2) giving the belief masses

$$m_1(\theta_1) = 1 - \epsilon_1 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = \epsilon_1$$

$$m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1 - \epsilon_2 \quad m_2(\theta_3) = \epsilon_2$$

From now on, we prefer to use matrices to describe the masses, i.e.

$$\begin{bmatrix} 1-\epsilon_1 & 0 & \epsilon_1 \\ 0 & 1-\epsilon_2 & \epsilon_2 \end{bmatrix}$$

- Using Dempster's rule of combination, one gets

$$m(\theta_3) = \frac{(\epsilon_1 \epsilon_2)}{(1-\epsilon_1)\cdot 0 + 0 \cdot (1-\epsilon_2) + \epsilon_1\epsilon_2} = 1$$

  which is absurd (or at least counter-intuitive). Note that whatever positive values for ϵ_1, ϵ_2 are, Dempster's rule of combination provides always the same result (one) which is abnormal. The only acceptable and correct result obtained by Dempster's rule is really obtained only in the trivial case when $\epsilon_1 = \epsilon_2 = 1$, i.e. when both sources agree in θ_3 with certainty which is obvious.

- Using the DSm rule of combination based on free-DSm model, one gets $m(\theta_3) = \epsilon_1\epsilon_2$, $m(\theta_1 \cap \theta_2) = (1-\epsilon_1)(1-\epsilon_2)$, $m(\theta_1 \cap \theta_3) = (1-\epsilon_1)\epsilon_2$, $m(\theta_2 \cap \theta_3) = (1-\epsilon_2)\epsilon_1$ and the others are zero which appears more reliable/trustable.
- Going back to Shafer's model and using the hybrid DSm rule of combination, one gets $m(\theta_3) = \epsilon_1\epsilon_2$, $m(\theta_1 \cup \theta_2) = (1-\epsilon_1)(1-\epsilon_2)$, $m(\theta_1 \cup \theta_3) = (1-\epsilon_1)\epsilon_2$, $m(\theta_2 \cup \theta_3) = (1-\epsilon_2)\epsilon_1$ and the others are zero.

Note that in the special case when $\epsilon_1 = \epsilon_2 = 1/2$, one has

$$m_1(\theta_1) = 1/2 \quad m_1(\theta_2) = 0 \quad m_1(\theta_3) = 1/2 \qquad \text{and} \qquad m_2(\theta_1) = 0 \quad m_2(\theta_2) = 1/2 \quad m_2(\theta_3) = 1/2$$

Dempster's rule of combinations still yields $m(\theta_3) = 1$ while the hybrid DSm rule based on the same Shafer's model yields now $m(\theta_3) = 1/4$, $m(\theta_1 \cup \theta_2) = 1/4$, $m(\theta_1 \cup \theta_3) = 1/4$, $m(\theta_2 \cup \theta_3) = 1/4$ which is normal.

2.6.3 Comparison with Smets, Yager and Dubois & Prade rules

We compare the results provided by DSmT rules and the main common rules of combination on the following very simple numerical example where only 2 independent sources (a priori assumed equally reliable) are involved and providing their belief initially on the 3D frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$. It is assumed in this example that Shafer's model holds and thus the belief assignments $m_1(.)$ and $m_2(.)$ do not commit belief to internal conflicting information. $m_1(.)$ and $m_2(.)$ are chosen as follows:

$$\begin{array}{llll} m_1(\theta_1) = 0.1 & m_1(\theta_2) = 0.4 & m_1(\theta_3) = 0.2 & m_1(\theta_1 \cup \theta_2) = 0.1 \\ m_2(\theta_1) = 0.5 & m_2(\theta_2) = 0.1 & m_2(\theta_3) = 0.3 & m_2(\theta_1 \cup \theta_2) = 0.1 \end{array}$$

These belief masses are usually represented in the form of a belief mass matrix $\mathbf{M}$ given by

$$\mathbf{M} = \begin{bmatrix} 0.1 & 0.4 & 0.2 & 0.3 \\ 0.5 & 0.1 & 0.3 & 0.1 \end{bmatrix} \tag{8}$$

where index i for the rows corresponds to the index of the source no. i and the indexes j for columns of $\mathbf{M}$ correspond to a given choice for enumerating the focal elements of all sources. In this particular example, index $j = 1$ corresponds to θ_1, $j = 2$ corresponds to θ_2, $j = 3$ corresponds to θ_3 and $j = 4$ corresponds to $\theta_1 \cup \theta_2$.

Now let's imagine that one finds out that θ_3 is actually truly empty because some extra and certain knowledge on θ_3 is received by the fusion center. As example, θ_1, θ_2 and θ_3 may correspond to three suspects (potential murders) in a police investigation, $m_1(.)$ and $m_2(.)$ corresponds to two reports of independent witnesses, but it turns out that finally θ_3 has provided a strong alibi to the criminal police investigator once arrested by the policemen. This situation corresponds to set up a hybrid model $\mathcal{M}$ with the constraint $\theta_3 \overset{\mathcal{M}}{=} \emptyset$.

Let's examine the result of the fusion in such situation obtained by the Smets', Yager's, Dubois & Prade's and hybrid DSm rules of combinations. First note that, based on the free DSm model, one would get by applying the classic DSm rule (denoted here by index $DSmc$) the following fusion result

$$\begin{array}{rrrr} m_{DSmc}(\theta_1) = 0.21 & m_{DSmc}(\theta_2) = 0.11 & m_{DSmc}(\theta_3) = 0.06 & m_{DSmc}(\theta_1 \cup \theta_2) = 0.03 \\ m_{DSmc}(\theta_1 \cap \theta_2) = 0.21 & m_{DSmc}(\theta_1 \cap \theta_3) = 0.13 & m_{DSmc}(\theta_2 \cap \theta_3) = 0.14 & \\ m_{DSmc}(\theta_3 \cap (\theta_1 \cup \theta_2)) = 0.11 & & & \end{array}$$

But because of the exclusivity constraints (imposed here by the use of Shafer's model and by the non-existential constraint $\theta_3 \overset{\mathcal{M}}{=} \emptyset$), the total conflicting mass is actually given by

$$k_{12} = 0.06 + 0.21 + 0.13 + 0.14 + 0.11 = 0.65 \qquad \text{(conflicting mass)}$$

- If one applies **Dempster's rule** [31] (denoted here by index DS), one gets:

$$\begin{aligned} m_{DS}(\emptyset) &= 0 \\ m_{DS}(\theta_1) &= 0.21/[1 - k_{12}] = 0.21/[1 - 0.65] = 0.21/0.35 = 0.600000 \\ m_{DS}(\theta_2) &= 0.11/[1 - k_{12}] = 0.11/[1 - 0.65] = 0.11/0.35 = 0.314286 \\ m_{DS}(\theta_1 \cup \theta_2) &= 0.03/[1 - k_{12}] = 0.03/[1 - 0.65] = 0.03/0.35 = 0.085714 \end{aligned}$$

- If one applies **Smets' rule** [40, 41] (i.e. the non normalized version of Dempster's rule with the conflicting mass transferred onto the empty set), one gets:

$$\begin{aligned} m_S(\emptyset) = m(\emptyset) &= 0.65 \qquad \text{(conflicting mass)} \\ m_S(\theta_1) &= 0.21 \\ m_S(\theta_2) &= 0.11 \\ m_S(\theta_1 \cup \theta_2) &= 0.03 \end{aligned}$$

- If one applies **Yager's rule** [45, 46, 47], one gets:

$$\begin{aligned} m_Y(\emptyset) &= 0 \\ m_Y(\theta_1) &= 0.21 \\ m_Y(\theta_2) &= 0.11 \\ m_Y(\theta_1 \cup \theta_2) &= 0.03 + k_{12} = 0.03 + 0.65 = 0.68 \end{aligned}$$

- If one applies **Dubois & Prade's rule** [18], one gets because $\theta_3 \overset{\mathcal{M}}{=} \emptyset$:

$$\begin{aligned}
m_{DP}(\emptyset) &= 0 \qquad \text{(by definition of Dubois \& Prade's rule)} \\
m_{DP}(\theta_1) &= [m_1(\theta_1)m_2(\theta_1) + m_1(\theta_1)m_2(\theta_1 \cup \theta_2) + m_2(\theta_1)m_1(\theta_1 \cup \theta_2)] \\
&\quad + [m_1(\theta_1)m_2(\theta_3) + m_2(\theta_1)m_1(\theta_3)] \\
&= [0.1 \cdot 0.5 + 0.1 \cdot 0.1 + 0.5 \cdot 0.3] + [0.1 \cdot 0.3 + 0.5 \cdot 0.2] = 0.21 + 0.13 = 0.34 \\
m_{DP}(\theta_2) &= [0.4 \cdot 0.1 + 0.4 \cdot 0.1 + 0.1 \cdot 0.3] + [0.4 \cdot 0.3 + 0.1 \cdot 0.2] = 0.11 + 0.14 = 0.25 \\
m_{DP}(\theta_1 \cup \theta_2) &= [m_1(\theta_1 \cup \theta_2)m_2(\theta_1 \cup \theta_2)] + [m_1(\theta_1 \cup \theta_2)m_2(\theta_3) + m_2(\theta_1 \cup \theta_2)m_1(\theta_3)] \\
&\quad + [m_1(\theta_1)m_2(\theta_2) + m_2(\theta_1)m_1(\theta_2)] \\
&= [0.30.1] + [0.3 \cdot 0.3 + 0.1 \cdot 0.2] + [0.1 \cdot 0.1 + 0.5 \cdot 0.4] = [0.03] + [0.09 + 0.02] + [0.01 + 0.20] \\
&= 0.03 + 0.11 + 0.21 = 0.35
\end{aligned}$$

  Now if one adds up the masses, one gets $0 + 0.34 + 0.25 + 0.35 = 0.94$ which is less than 1. Therefore Dubois & Prade's rule of combination does not work when a singleton, or an union of singletons, becomes empty (in a dynamic fusion problem). The products of such empty-element columns of the mass matrix $\mathbf{M}$ are lost; this problem is fixed in DSmT by the sum $S_2(.)$ in (4) which transfers these products to the total or partial ignorances.

In this particular example, using the hybrid DSm rule, one transfers the product of the empty-element θ_3 column, $m_1(\theta_3)m_2(\theta_3) = 0.2 \cdot 0.3 = 0.06$, to $m_{DSmh}(\theta_1 \cup \theta_2)$, which becomes equal to $0.35 + 0.06 = 0.41$.

2.7 Fusion of imprecise beliefs

In many fusion problems, it seems very difficult (if not impossible) to have precise sources of evidence generating precise basic belief assignments (especially when belief functions are provided by human experts), and a more flexible plausible and paradoxical theory supporting imprecise information becomes necessary. In the previous sections, we presented the fusion of *precise* uncertain and conflicting/paradoxical generalized basic belief assignments (gbba) in the DSmT framework. We mean here by precise gbba, basic belief functions/masses $m(.)$ defined precisely on the hyper-power set D^Θ where each mass $m(X)$, where X belongs to D^Θ, is represented by only one real number belonging to $[0,1]$ such that $\sum_{X \in D^\Theta} m(X) = 1$. In this section, we present the DSm fusion rule for dealing with *admissible imprecise generalized basic belief assignments* $m^I(.)$ defined as real subunitary intervals of $[0,1]$, or even more general as real subunitary sets [i.e. sets, not necessarily intervals]. An imprecise belief assignment $m^I(.)$ over D^Θ is said admissible if and only if there exists for every $X \in D^\Theta$ at least one real number $m(X) \in m^I(X)$ such that $\sum_{X \in D^\Theta} m(X) = 1$. The idea to work with imprecise belief structures represented by real subset intervals of $[0,1]$ is not new and has been investigated in [21, 15, 16] and references therein. The proposed works available in the literature, upon our knowledge were limited only to sub-unitary interval combination in the framework of Transferable Belief Model (TBM) developed by Smets [40, 41]. We extend the approach of Lamata & Moral and Denœux based on subunitary interval-valued masses to subunitary set-valued masses; therefore the closed intervals used by Denœux to denote imprecise masses are generalized to any sets included in [0,1], i.e. in our case these sets can be unions of (closed, open, or half-open/half-closed) intervals and/or scalars all in $[0,1]$. Here, the proposed extension is done in the context of the DSmT framework, although it can also apply directly to fusion of imprecise belief structures within TBM as well if the user prefers to adopt TBM rather than DSmT.

Before presenting the general formula for the combination of generalized imprecise belief structures, we remind the following set operators involved in the formula. Several numerical examples are given in [37].

- **Addition of sets**

$$S_1 \boxplus S_2 = S_2 \boxplus S_1 \triangleq \{x \mid x = s_1 + s_2, s_1 \in S_1, s_2 \in S_2\} \quad \text{with} \quad \begin{cases} \inf(S_1 \boxplus S_2) = \inf(S_1) + \inf(S_2) \\ \sup(S_1 \boxplus S_2) = \sup(S_1) + \sup(S_2) \end{cases}$$

- **Subtraction of sets**

$$S_1 \boxminus S_2 \triangleq \{x \mid x = s_1 - s_2, s_1 \in S_1, s_2 \in S_2\} \quad \text{with} \quad \begin{cases} \inf(S_1 \boxminus S_2) = \inf(S_1) - \sup(S_2) \\ \sup(S_1 \boxminus S_2) = \sup(S_1) - \inf(S_2) \end{cases}$$

- **Multiplication of sets**

$$S_1 \boxdot S_2 \triangleq \{x \mid x = s_1 \cdot s_2, s_1 \in S_1, s_2 \in S_2\} \quad \text{with} \quad \begin{cases} \inf(S_1 \boxdot S_2) = \inf(S_1) \cdot \inf(S_2) \\ \sup(S_1 \boxdot S_2) = \sup(S_1) \cdot \sup(S_2) \end{cases}$$

2.7.1 DSm rule of combination for imprecise beliefs

We present the generalization of the DSm rules to combine any type of imprecise belief assignment which may be represented by the union of several sub-unitary (half-) open intervals, (half-)closed intervals and/or sets of points belonging to [0,1]. Several numerical examples are also given. In the sequel, one uses the notation (a, b) for an open interval, $[a, b]$ for a closed interval, and $(a, b]$ or $[a, b)$ for a half open and half closed interval. From the previous operators on sets, one can generalize the DSm rules (classic and hybrid) from scalars to sets in the following way [37] (chap. 6): $\forall A \neq \emptyset \in D^\Theta$,

$$m^I(A) = \boxed{\sum}_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1 \cap X_2 \cap \ldots \cap X_k) = A}} \boxed{\prod}_{i=1,\ldots,k} m_i^I(X_i) \tag{9}$$

where $\boxed{\sum}$ and $\boxed{\prod}$ represent the summation, and respectively product, of sets.

Similarly, one can generalize the hybrid DSm rule from scalars to sets in the following way:

$$m^I_{\mathcal{M}(\Theta)}(A) \triangleq \phi(A) \boxdot \left[S_1^I(A) \boxplus S_2^I(A) \boxplus S_3^I(A) \right] \tag{10}$$

$\phi(A)$ is the *characteristic non emptiness function* of the set A and $S_1^I(A)$, $S_2^I(A)$ and $S_3^I(A)$ are defined by

$$S_1^I(A) \triangleq \boxed{\sum}_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1 \cap X_2 \cap \ldots \cap X_k) = A}} \boxed{\prod}_{i=1,\ldots,k} m_i^I(X_i) \tag{11}$$

$$S_2^I(A) \triangleq \boxed{\sum}_{\substack{X_1,X_2,\ldots,X_k \in \emptyset \\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \emptyset) \wedge (A = I_t)]}} \boxed{\prod}_{i=1,\ldots,k} m_i^I(X_i) \tag{12}$$

$$S_3^I(A) \triangleq \boxed{\sum}_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta \\ (X_1 \cup X_2 \cup \ldots \cup X_k) = A \\ (X_1 \cap X_2 \cap \ldots \cap X_k) \in \emptyset}} \boxed{\prod}_{i=1,\ldots,k} m_i^I(X_i) \tag{13}$$

In the case when all sets are reduced to points (numbers), the set operations become normal operations with numbers; the sets operations are generalizations of numerical operations. When imprecise belief structures reduce to precise belief structure, DSm rules (9) and (10) reduce to their precise version (3) and (4) respectively.

2.7.2 Example

Here is a simple example of fusion with with multiple-interval masses. For simplicity, this example is a particular case when the theorem of admissibility (see [37] p. 138 for details) is verified by a few points, which happen to be just on the bounders. It is an extreme example, because we tried to comprise all kinds of possibilities which may occur in the imprecise or very imprecise fusion. So, let's consider a fusion problem over $\Theta = \{\theta_1, \theta_2\}$, two independent sources of information with the following imprecise admissible belief assignments

$A \in D^\Theta$	$m_1^I(A)$	$m_2^I(A)$
θ_1	$[0.1, 0.2] \cup \{0.3\}$	$[0.4, 0.5]$
θ_2	$(0.4, 0.6) \cup [0.7, 0.8]$	$[0, 0.4] \cup \{0.5, 0.6\}$

Table 1: Inputs of the fusion with imprecise bba

Using the DSm classic rule for sets, one gets

$$m^I(\theta_1) = ([0.1, 0.2] \cup \{0.3\}) \boxdot [0.4, 0.5] = ([0.1, 0.2] \boxdot [0.4, 0.5]) \cup (\{0.3\} \boxdot [0.4, 0.5]) = [0.04, 0.10] \cup [0.12, 0.15]$$

$$\begin{aligned} m^I(\theta_2) &= ((0.4, 0.6) \cup [0.7, 0.8]) \boxdot ([0, 0.4] \cup \{0.5, 0.6\}) \\ &= ((0.4, 0.6) \boxdot [0, 0.4]) \cup ((0.4, 0.6) \boxdot \{0.5, 0.6\}) \cup ([0.7, 0.8] \boxdot [0, 0.4]) \cup ([0.7, 0.8] \boxdot \{0.5, 0.6\}) \\ &= (0, 0.24) \cup (0.20, 0.30) \cup (0.24, 0.36) \cup [0, 0.32] \cup [0.35, 0.40] \cup [0.42, 0.48] = [0, 0.40] \cup [0.42, 0.48] \end{aligned}$$

$$
\begin{aligned}
m^I(\theta_1 \cap \theta_2) &= [([0.1,0.2] \cup \{0.3\}) \boxdot ([0,0.4] \cup \{0.5,0.6\})] \boxplus [[0.4,0.5] \boxdot ((0.4,0.6) \cup [0.7,0.8])] \\
&= [([0.1,0.2] \boxdot [0,0.4]) \cup ([0.1,0.2] \boxdot \{0.5,0.6\}) \cup (\{0.3\} \boxdot [0,0.4]) \cup (\{0.3\} \boxdot \{0.5,0.6\})] \\
&\quad \boxplus [([0.4,0.5] \boxdot (0.4,0.6)) \cup ([0.4,0.5] \boxdot [0.7,0.8])] \\
&= [[0,0.08] \cup [0.05,0.10] \cup [0.06,0.12] \cup [0,0.12] \cup \{0.15,0.18\}] \boxplus [(0.16,0.30) \cup [0.28,0.40]] \\
&= [[0,0.12] \cup \{0.15,0.18\}] \boxplus (0.16,0.40] = (0.16,0.52] \cup (0.31,0.55] \cup (0.34,0.58] = (0.16,0.58]
\end{aligned}
$$

Hence finally the fusion admissible result is given by:

$A \in D^\Theta$	$m^I(A) = [m_1^I \oplus m_2^I](A)$
θ_1	$[0.04,0.10] \cup [0.12,0.15]$
θ_2	$[0,0.40] \cup [0.42,0.48]$
$\theta_1 \cap \theta_2$	$(0.16,0.58]$
$\theta_1 \cup \theta_2$	0

Table 2: Fusion result with the DSm classic rule

If one finds out[6] that $\theta_1 \cap \theta_2 \overset{\mathcal{M}}{\equiv} \emptyset$ (this is our hybrid model $\mathcal{M}$ one wants to deal with), then one uses the hybrid DSm rule for sets (10): $m^I_{\mathcal{M}}(\theta_1 \cap \theta_2) = 0$ and $m^I_{\mathcal{M}}(\theta_1 \cup \theta_2) = (0.16,0.58]$, the others imprecise masses are not changed. In other words, one gets now with hybrid DSm rule applied to imprecise beliefs:

$A \in D^\Theta$	$m^I_{\mathcal{M}}(A) = [m_1^I \oplus m_2^I](A)$
θ_1	$[0.04,0.10] \cup [0.12,0.15]$
θ_2	$[0,0.40] \cup [0.42,0.48]$
$\theta_1 \cap \theta_2 \overset{\mathcal{M}}{\equiv} \emptyset$	0
$\theta_1 \cup \theta_2$	$(0.16,0.58]$

Table 3: Fusion result with the hybrid DSm rule for $\mathcal{M}$

Let's check now the admissibility conditions and theorem. For the source 1, there exist the precise masses $(m_1(\theta_1) = 0.3) \in ([0.1,0.2] \cup \{0.3\})$ and $(m_1(\theta_2) = 0.7) \in ((0.4,0.6) \cup [0.7,0.8])$ such that $0.3 + 0.7 = 1$. For the source 2, there exist the precise masses $(m_1(\theta_1) = 0.4) \in ([0.4,0.5])$ and $(m_2(\theta_2) = 0.6) \in ([0,0.4] \cup \{0.5,0.6\})$ such that $0.4 + 0.6 = 1$. Therefore both sources associated with $m_1^I(.)$ and $m_2^I(.)$ are admissible imprecise sources of information.

It can be easily checked that the DSm classic fusion of $m_1(.)$ and $m_2(.)$ yields the paradoxical basic belief assignment $m(\theta_1) = [m_1 \oplus m_2](\theta_1) = 0.12$, $m(\theta_2) = [m_1 \oplus m_2](\theta_2) = 0.42$ and $m(\theta_1 \cap \theta_2) = [m_1 \oplus m_2](\theta_1 \cap \theta_2) = 0.46$. One sees that the admissibility theorem is satisfied since $(m(\theta_1) = 0.12) \in (m^I(\theta_1) = [0.04,0.10] \cup [0.12,0.15])$, $(m(\theta_2) = 0.42) \in (m^I(\theta_2) = [0,0.40] \cup [0.42,0.48])$ and $(m(\theta_1 \cap \theta_2) = 0.46) \in (m^I(\theta_1 \cap \theta_2) = (0.16,0.58])$ such that $0.12 + 0.42 + 0.46 = 1$. Similarly if one finds out that $\theta_1 \cap \theta_2 = \emptyset$, then one uses the hybrid DSm rule and one gets: $m(\theta_1 \cap \theta_2) = 0$ and $m(\theta_1 \cup \theta_2) = 0.46$; the others remain unchanged. The admissibility theorem still holds, because one can pick at least one number in each subset $m^I(.)$ such that the sum of these numbers is 1. This approach can be also used in the similar manner to obtain imprecise pignistic probabilities from $m^I(.)$ for decision-making under uncertain, paradoxical and imprecise sources of information as well. The generalized pignistic transformation (GPT) is presented in next section.

2.8 The generalized pignistic transformation (GPT)

2.8.1 The classical pignistic transformation

We follow here the Smets' vision which considers the management of information as a two 2-levels process: credal (for combination of evidences) and pignistic[7] (for decision-making) , i.e "*when someone must take a decision, he must then construct a probability function derived from the belief function that describes his credal state. This probability function is then used to make decisions*" [39] (p. 284). One obvious way to build this probability function corresponds to the so-called Classical Pignistic Transformation (CPT) defined in the DST framework (i.e. based on the Shafer's model assumption) as [41]:

$$
P\{A\} = \sum_{X \in 2^\Theta} \frac{|X \cap A|}{|X|} m(X) \tag{14}
$$

[6] We consider now a dynamic fusion problem.

[7] Pignistic terminology has been coined by Philippe Smets and comes from *pignus*, a bet in Latin.

where $|A|$ denotes the number of worlds in the set A (with convention $|\emptyset|/|\emptyset| = 1$, to define $P\{\emptyset\}$). $P\{A\}$ corresponds to $BetP(A)$ in Smets' notation [41]. Decisions are achieved by computing the expected utilities of the acts using the subjective/pignistic $P\{.\}$ as the probability function needed to compute expectations. Usually, one uses the maximum of the pignistic probability as decision criterion. The max. of $P\{.\}$ is often considered as a prudent betting decision criterion between the two other alternatives (max of plausibility or max. of credibility which appears to be respectively too optimistic or too pessimistic). It is easy to show that $P\{.\}$ is indeed a probability function (see [40]).

2.8.2 Notion of DSm cardinality

One important notion involved in the definition of the Generalized Pignistic Transformation (GPT) is the *DSm cardinality*. The *DSm cardinality* of any element A of hyper-power set D^Θ, denoted $\mathcal{C}_\mathcal{M}(A)$, corresponds to the number of parts of A in the corresponding fuzzy/vague Venn diagram of the problem (model $\mathcal{M}$) taking into account the set of integrity constraints (if any), i.e. all the possible intersections due to the nature of the elements θ_i. This *intrinsic cardinality* depends on the model $\mathcal{M}$ (free, hybrid or Shafer's model). $\mathcal{M}$ is the model that contains A, which depends both on the dimension $n = |\Theta|$ and on the number of non-empty intersections present in its associated Venn diagram (see [37] for details). The DSm cardinality depends on the cardinal of $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ and on the model of D^Θ (i.e., the number of intersections and between what elements of Θ - in a word the structure) at the same time; it is not necessarily that every singleton, say θ_i, has the same DSm cardinal, because each singleton has a different structure; if its structure is the simplest (no intersection of this elements with other elements) then $\mathcal{C}_\mathcal{M}(\theta_i) = 1$, if the structure is more complicated (many intersections) then $\mathcal{C}_\mathcal{M}(\theta_i) > 1$; let's consider a singleton θ_i: if it has 1 intersection only then $\mathcal{C}_\mathcal{M}(\theta_i) = 2$, for 2 intersections only $\mathcal{C}_\mathcal{M}(\theta_i)$ is 3 or 4 depending on the model $\mathcal{M}$, for m intersections it is between $m+1$ and 2^m depending on the model; the maximum DSm cardinality is 2^{n-1} and occurs for $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ in the free model $\mathcal{M}^f$; similarly for any set from D^Θ: the more complicated structure it has, the bigger is the DSm cardinal; thus the DSm cardinality measures the complexity of en element from D^Θ, which is a nice characterization in our opinion; we may say that for the singleton θ_i not even $|\Theta|$ counts, but only its structure (= how many other singletons intersect θ_i). Simple illustrative examples are given in Chapter 3 and 7 of [37]. One has $1 \leq \mathcal{C}_\mathcal{M}(A) \leq 2^n - 1$. $\mathcal{C}_\mathcal{M}(A)$ must not be confused with the classical cardinality $|A|$ of a given set A (i.e. the number of its distinct elements) - that's why a new notation is necessary here. $\mathcal{C}_\mathcal{M}(A)$ is very easy to compute by programming from the algorithm of generation of D^Θ given explicated in [37].

As example, let's take back the example of the simple hybrid DSm model described in section 2.2, then one gets the following list of elements (with their DSm cardinal) for the restricted D^Θ taking into account the integrity constraints of this hybrid model:

$A \in D^\Theta$	$\mathcal{C}_\mathcal{M}(A)$
$\alpha_0 \triangleq \emptyset$	0
$\alpha_1 \triangleq \theta_1 \cap \theta_2$	1
$\alpha_2 \triangleq \theta_3$	1
$\alpha_3 \triangleq \theta_1$	2
$\alpha_4 \triangleq \theta_2$	2
$\alpha_5 \triangleq \theta_1 \cup \theta_2$	3
$\alpha_6 \triangleq \theta_1 \cup \theta_3$	3
$\alpha_7 \triangleq \theta_2 \cup \theta_3$	3
$\alpha_8 \triangleq \theta_1 \cup \theta_2 \cup \theta_3$	4

Eaxmple of DSm cardinals: $\mathcal{C}_\mathcal{M}(A)$ for hybrid model $\mathcal{M}$

2.8.3 The Generalized Pignistic Transformation

To take a rational decision within the DSmT framework, it is necessary to generalize the Classical Pignistic Transformation in order to construct a pignistic probability function from any generalized basic belief assignment $m(.)$ drawn from the DSm rules of combination. Here is the simplest and direct extension of the CPT to define the Generalized Pignistic Transformation:

$$\forall A \in D^\Theta, \qquad P\{A\} = \sum_{X \in D^\Theta} \frac{\mathcal{C}_\mathcal{M}(X \cap A)}{\mathcal{C}_\mathcal{M}(X)} m(X) \tag{15}$$

where $\mathcal{C}_\mathcal{M}(X)$ denotes the DSm cardinal of proposition X for the DSm model $\mathcal{M}$ of the problem under consideration.

The decision about the solution of the problem is usually taken by the maximum of pignistic probability function $P\{.\}$. Let's remark the close ressemblance of the two pignistic transformations (14) and (15). It can be shown that (15) reduces to (14) when the hyper-power set D^Θ reduces to classical power set 2^Θ if we adopt Shafer's model. But (15) is a generalization of (14) since it can be used for computing pignistic probabilities for any models (including Shafer's model). It has been proved in [37] (Chap. 7) that $P\{.\}$ is indeed a probability function.

3 Fuzzy Inference for Information Fusion

We further connect the fusion rules of combination with fuzzy and neutrosophic operators. Lets first replace the Conjunctive Rule and Disjunctive Rule with the fuzzy T-norm and T-conorm versions respectively. These rules started from the T-norm and T-conorm respectively in fuzzy and neutrosophic logics, where the *and* logic operator $\wedge$ corresponds in fusion to the conjunctive rule, while the *or* logic operator $\vee$ corresponds to the disjunctive rule. While the logic operators deal with degrees of truth and degrees of falsehood, the fusion rules deal with degrees of belief and degrees of disbelief of hypotheses.

3.1 T-Norm

A T-norm is a function $T_n : [0,1]^2 \mapsto [0,1]$, defined in fuzzy set theory and fuzzy logic to represent the *intersection* of two fuzzy sets and the fuzzy logical operator *and* respectively. Extended to the fusion theory the T-norm will be a substitute for the conjunctive rule. The T-norm satisfies the conditions:

a) Boundary Conditions: $T_n(0,0) = 0, T_n(x,1) = x$

b) Commutativity: $T_n(x,y) = T_n(y,x)$

c) Monotonicity: If $x \leq u$ and $y \leq v$, then $T_n(x,y) \leq T_n(u,v)$

d) Associativity: $T_n(T_n(x,y),z) = T_n(x,T_n(y,z))$

There are many functions which satisfy the T-norm conditions. We present below the most known ones:

- The Algebraic Product T-norm: $T_{\text{n-algebraic}}(x,y) = x \cdot y$
- The Bounded T-norm: $T_{\text{n-bounded}}(x,y) = \max\{0, x+y-1\}$
- The Default (min) T-norm (introduced by Zadeh): $T_{\text{n-min}}(x,y) = \min\{x,y\}$

3.2 T-conorm

A T-conorm is a function $T_c : [0,1]^2 \mapsto [0,1]$, defined in fuzzy set theory and fuzzy logic to represent the *union* of two fuzzy sets and the fuzzy logical operator *or* respectively. Extended to the fusion theory the T-conorm will be a substitute for the disjunctive rule. The T-conorm satisfies the conditions:

a) Boundary Conditions: $T_c(1,1) = 1, T_c(x,0) = x$

b) Commutativity: $T_c(x,y) = T_c(y,x)$

c) Monotonicity: if $x \leq u$ and $y \leq v$, then $T_c(x,y) \leq T_c(u,v)$

d) Associativity: $T_c(T_c(x,y),z) = T_c(x,T_c(y,z))$

There are many functions which satisfy the T-conorm conditions. We present below the most known ones:

- The Algebraic Product T-conorm: $T_{\text{c-algebraic}}(x,y) = x + y - x \cdot y$
- The Bounded T-conorm: $T_{\text{c-bounded}}(x,y) = \min\{1, x+y\}$
- The Default (max) T-conorm (introduced by Zadeh): $T_{\text{c-max}}(x,y) = \max\{x,y\}$

Then, the *T-norm Fusion rule* is defined as follows: $m_{\cap 12}(A) = \sum_{\substack{Y,Y\in\Theta \\ X\cap Y=A}} T_n(m_1(X), m_2(Y))$ and the *T-conorm Fusion rule* is defined as follows: $m_{\cup 12}(A) = \sum_{\substack{Y,Y\in\Theta \\ X\cup Y=A}} T_c(m_1(X), m_2(Y))$.

The min T-norm rule yields results, very closed to Conjunctive Rule. It satisfies the principle of neutrality of the vacuous bba, reflects the majority opinion, converges towards idempotence. It is simpler to apply, but needs normalization. What is missed it is a strong justification of the way of presenting the fusion process. But we think, the consideration between two sources of information as a vague relation, characterized with the particular way of association between focal elements, and corresponding degree of association (interaction) between them is reasonable. Min rule can be interpreted as an optimistic lower bound for combination of bba and the below Max rule as a prudent/pessimistic upper bound. The T-norm and T-conorm are commutative, associative, isotone, and have a neutral element.

4 Degrees of intersection, union, inclusion

In order to improve many fusion rules we can insert a degree of intersection, a degree of union, or a degree of inclusion. These are defined as follows:

4.1 Degree of Intersection

The degree of intersection measures the percentage of overlapping region of two sets X_1, X_2 with respect to the whole reunited regions of the sets using the cardinal of sets not the fuzzy set point of view:

$$d(X_1 \cap X_2) = \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|}$$

where $|X|$ means cardinal of the set X.

For the minimum intersection/overlapping, i.e. when $X_1 \cap X_2 = \emptyset$, the degree of intersection is 0, while for the maximum intersection/overlapping, i.e. when $X_1 = X_2$, the degree of intersection is 1.

4.2 Degree of Union

The degree of intersection measures the percentage of non-overlapping region of two sets X_1, X_2 with respect to the whole reunited regions of the sets using the cardinal of sets not the fuzzy set point of view:

$$d(X_1 \cup X_2) = \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|}$$

For the maximum non-overlapping, i.e. when $X_1 \cap X_2 = \emptyset$, the degree of union is 1, while for the minimum non-overlapping, i.e. when $X_1 = X_2$, the degree of union is 0. The sum of degrees of intersection and union is 1 since they complement each other.

4.3 Degree of inclusion

The degree of inclusion measures the percentage of the included region X_1 with respect to the includant region X_2: Let $X_1 \subseteq X_2$, then

$$d(X_1 \subseteq X_2) = \frac{|X_1|}{|X_2|}$$

$d(\emptyset \subseteq X_2) = 0$ because nothing (i.e. empty set) is included in X_2, while $d(X_2 \subseteq X_2) = 1$ because X_2 is fulfilled by inclusion. By definition $d(\emptyset \subseteq \emptyset) = 1$. We can generalize the above degree for $n \geq 2$ sets.

4.4 Improvements of belief and plausibility functions

Thus the Bel(.) and Pl(.) functions can incorporate in their formulas the above degrees of inclusion and intersection respectively:

- Belief function improved: $\forall A \in D^\Theta \setminus \emptyset, \text{Bel}_d(A) = \sum_{\substack{X \in D^\Theta \\ X \subseteq A}} \frac{|X|}{|A|} m(X)$
- Plausibility function improved: $\forall A \in D^\Theta \setminus \emptyset, \text{Pl}_d(A) = \sum_{\substack{X \in D^\Theta \\ X \cap A \neq \emptyset}} \frac{|X \cap A|}{|X \cup A|} m(X)$

4.5 Improvements of fusion rules

- Disjunctive rule improved:

$$\forall A \in D^\Theta \setminus \emptyset, \qquad m_{\cup d}(A) = k_{\cup d} \cdot \sum_{\substack{X_1, X_2 \in D^\Theta \\ X_1 \cup X_2 = A}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2)$$

  where $k_{\cup d}$ is a constant of normalization.

- Dezert-Smarandache classical rule improved:

$$\forall A \in D^\Theta \setminus \emptyset, \qquad m_{DSmCd}(A) = k_{DSmCd} \cdot \sum_{\substack{X_1, X_2 \in D^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2)$$

  where k_{DSmCd} is a constant of normalization. This rule is similar with the Zhangs Center Combination rule [54] extended on the Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$ and using another definition for the degree of intersection (here $\mathcal{C}$ denotes the complement).

- Dezert-Smarandache hybrid rule improved:

$$\forall A \in D^\Theta \setminus \emptyset, \qquad m_{DSmHd}(A) = k_{DSmCd} \cdot \{ \sum_{\substack{X_1, X_2 \in D^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2)$$

$$+ \sum_{\substack{X_1, X_2 \in \emptyset \\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \emptyset) \wedge (A = I_t)]}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_1, X_2 \in D^\Theta \\ u(c(X_1 \cap X_2)) = A \\ (X_1 \cap X_2) \in \emptyset}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2) \}$$

where k_{DSmHd} is a constant of normalization.

5 Neutrosophic Inference for Information Fusion

Similarly to the fuzzy improvement of the fusion rules we can now consider the neutrosophic improvement of the fusion rules of combination. Lets now replace the Conjunctive Rule and Disjunctive Rule with the neutrosophic N-norm and N-conorm versions respectively [44].

5.1 Neutrosophy

Neutrosophic Logic, Neutrosophic Set, and Neutrosophic Probability started from Neutrosophy [33, 36, 34, 35]. Neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. It is an extension of dialectics. Its fundamental theory is that every idea $< A >$ tends to be neutralized, diminished, balanced by $< NonA >$ ideas (not only $< AntiA >$ as Hegel asserted) - as a state of equilibrium, where $< NonA >=$ what is not $< A >$, $< AntiA >=$ the opposite of $< A >$, and $< NeutA >=$ what is neither $< A >$ nor $< AntiA >$.

5.2 Nonstandard analysis

5.2.1 Short introduction

Abraham Robinson developed the nonstandard analysis in sixties [29]. x is called *infinitesimal* if $|x| < 1/n$ for any positive n. A *left monad* is defined by $({}^-a) = \{a - x | x \in \mathbb{R}^\star, x > 0 \text{ infinitesimal}\} = a - \epsilon$ and a right monad by $(b^+) = \{b + x | x \in \mathbb{R}^\star, x > 0 \text{ infinitesimal}\} = b + \epsilon$ where $\epsilon > 0$ is infinitesimal; a, b are called *standard parts*, ϵ is called *nonstandard* part. A bimonad is defined as $({}^-a^+) = ({}^-a) \cup (a^+)$.

5.2.2 Operations with nonstandard finite real numbers

$$^-a \star b =^- (a \star b) \qquad a \star b^+ = (a \star b)^+ \qquad ^-a \star b^+ =^- (a \star b)^+$$

- the left monads absorb themselves: $^-a \star^- b =^- (a \star b)$
- the right monads absorb themselves: $a^+ \star b^+ = (a \star b)^+$

where $\star$ operation can be addition, subtraction, multiplication, division and power. The operations with real standard or non-standard subsets are defined according definitions given in section 2.7.

5.3 Neutrosophic logic

Lets consider the nonstandard unit interval $]^-0, 1^+[$, with left and right borders vague, imprecise. Let T, I, F be standard or nonstandard subsets of $]^-0, 1^+[$. Then: Neutrosophic Logic (NL) is a logic in which each proposition is $T\%$ true, $I\%$ indeterminate, and $F\%$ false, where:

$$-0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^+$$

T, I, F are not necessary intervals, but any sets (discrete, continuous, open or closed or half-open/half-closed interval, intersections or unions of the previous sets, etc.).

For example: proposition P is between 30-40% or 45-50% true, 20% indeterminate, and 60% or between 66-70% false (according to various analyzers or parameters). NL is a generalization of Zadehs fuzzy logic (FL), especially of Atanassovs intuitionistic fuzzy logic (IFL) [1, 2, 7], and other logics.

5.4 Differences between Neutrosophic Logic and Intuitionistic Fuzzy Logic

a) In NL there is no restriction on T, I, F, while in IFL the sum of components (or their superior limits) $= 1$; thus NL can characterize the incomplete information (sum < 1), paraconsistent information (sum > 1).

b) NL can distinguish, in philosophy, between absolute truth [NL(absolute truth)$= 1^+$] and relative truth [NL(relative truth)$= 1$], while IFL cannot; *absolute truth* is truth in all possible worlds (Leibniz), *relative truth* is truth in at least one world.

c) In NL the components can be nonstandard, in IFL they dont.

d) NL, like *dialetheism* [some contradictions are true], can deal with paradoxes, NL(paradox) $= (1, I, 1)$, while IFL cannot.

5.5 Neutrosophic Logic generalizes many logics

Let the components reduced to scalar numbers, t, i, f, with $t + i + f = n$; NL generalizes:

- the *Boolean logic* (for $n = 1$ and $i = 0$, with f, f either 0 or 1);
- the *multi-valued logic*, which supports the existence of many values between true and false - Lukasiewicz, 3 values [24, 25]; Post, m values - (for $n = 1$, $Ii = 0$, $0 \leq t, f \leq 1$);
- the *intuitionistic logic*, which supports incomplete theories, where $A \wedge \neg A$ not always true, and $\exists x P(x)$ needs an algorithm constructing x [9, 10, 11, 12, 19] (for $0 < n < 1$ and $i = 0$, $0 \leq t, f \leq 1$);
- the *fuzzy logic*, which supports degrees of truth [48] (for $n = 1$ and $i = 0$, $0 \leq t, f \leq 1$);
- the *intuitionistic fuzzy logic*, which supports degrees of truth and degrees of falsity while whats left is considered indeterminacy [2] (for $n = 1$);
- the *paraconsistent logic*, which supports conflicting information, and anything follows from contradictions fails, i.e. $\neg A \wedge A \supset B$ fails; $\neg A \wedge A$ is not always false (for $n > 1$ and $i = 0$, with both $0 < t, f < 1$);
- the *dialetheism*, which says that some contradictions are true, $\neg A \wedge A =$ true (for $t = f = 1$ and $i = 0$; some paradoxes can be denoted this way too);
- the *faillibilism*, which says that uncertainty belongs to every proposition (for $i > 0$).

5.6 Neutrosophic Logic connectors

One notes the neutrosophic logical values of the propositions A_1 and A_2 by $NL(A_1) = (T_1, I_1, F_1)$ and $NL(A_2) = (T_2, I_2, F_2)$. If, after calculations, in the below operations one obtains values < 0 or > 1, then one replaces them with $^-0$ or 1^+ respectively.

5.6.1 Negation

$$NL(\neg A_1) = (\{1^+\} \boxminus T_1, \{1^+\} \boxminus I_1, \{1^+\} \boxminus F_1)$$

5.6.2 Conjunction

$$NL(A_1 \wedge A_2) = (T_1 \boxdot T_2, I_1 \boxdot I_2, F_1 \boxdot F_2)$$

5.6.3 Weak or inclusive disjunction

$$NL(A_1 \vee A_2) = (T_1 \boxplus T_2 \boxminus (T_1 \boxdot T_2), I_1 \boxplus I_2 \boxminus (I_1 \boxdot I_2), F_1 \boxplus F_2 \boxminus (F_1 \boxdot F_2))$$

Many properties of the classical logic operators do not apply in neutrosophic logic. Neutrosophic logic operators (connectors) can be defined in many ways according to the needs of applications or of the problem solving.

5.7 Neutrosophic Set

Let U be a universe of discourse, M a set included in U. An element x from U is noted with respect to the neutrosophic set M as $x(T, I, F)$ and belongs to M in the following way: it is $t\%$ true in the set (degree of membership), $i\%$ indeterminate (unknown if it is in the set) (degree of indeterminacy), and $f\%$ false (degree of non-membership), where t varies in T, i varies in I, f varies in F. This definition is analogue to NL, and similarly NS generalizes the fuzzy set (FS), especially the intuitionistic fuzzy set (IFS), intuitionistic set (IS), paraconsistent set (PS) For example: $x(50, 20, 40) \in A$ means: with a belief of 50% x is in A, with a belief of 40% x is not in A, and the 20% is undecidable

5.7.1 Neutrosophic Set Operators

Let A_1 and A_2 be two sets over the universe U. An element $x(T_1, I_1, F_1) \in A_1$ and $x(T_2, I_2, F_2) \in A_2$ [neutrosophic membership appurtenance to A_1 and respectively to A_2]. NS operators (similar to NL connectors) can also be defined in many ways.

5.7.2 Complement

If $x(T1, I1, F1) \in A_1$ then $x(\{1^+\} \boxminus T_1, \{1^+\} \boxminus I_1, \{1^+\} \boxminus F_1)) \in \mathcal{C}(A_1)$.

5.7.3 Intersection

If $x(T_1, I_1, F_1) \in A_1$ and $x(T_2, I_2, F_2) \in A_2$ then $x(T_1 \boxdot T_2, I_1 \boxdot I_2, F_1 \boxdot F_2) \in A_1 \cap A_2$.

5.7.4 Union

If $x(T_1, I_1, F_1) \in A_1$ and $x(T_2, I_2, F_2) \in A_2$ then $x(T_1 \boxplus T_2 \boxminus (T_1 \boxdot T_2), I_1 \boxplus I_2 \boxminus (I_1 \boxdot I_2), F_1 \boxplus F_2 \boxminus (F_1 \boxdot F_2)) \in A_1 \cup A_2$.

5.7.5 Difference

If $x(T_1, I_1, F_1) \in A_1$ and $x(T_2, I_2, F_2) \in A_2$ then $x(T_1 \boxminus (T_1 \boxdot T_2), I_1 \boxminus (I_1 \boxdot I_2), F_1 \boxminus (F_1 \boxdot F_2)) \in A_1 \setminus A_2$.

5.8 Differences between Neutrosophic Set and Intuitionistic Fuzzy Set

a) In NS there is no restriction on T, I, F, while in IFS the sum of components (or their superior limits) $= 1$; thus NL can characterize the incomplete information (sum < 1), paraconsistent information (sum > 1).

b) NS can distinguish, in philosophy, between absolute membership [NS(absolute membership)$= 1^+$] and relative membership [NS(relativemembership)$= 1$], while IFS cannot; absolute membership is membership in all possible worlds, relative membership is membership in at least one world.

c) In NS the components can be nonstandard, in IFS they dont.

d) NS, like dialetheism [some contradictions are true], can deal with paradoxes, NS(paradox element) $= (1, I, 1)$, while IFS cannot.

e) NS operators can be defined with respect to T, I, F while IFS operators are defined with respect to T and F only

f) I can be split in NS in more subcomponents (for example in Belnaps four-valued logic [8] indeterminacy is split into uncertainty and contradiction), but in IFS it cannot.

5.9 N-norm

Here each element x and y has three components: $x(t_1, i_1, f_1)$, $y(t_2, i_2, f_2)$. We define :

$$\begin{cases} \max\{x, y\} = (\max\{t_1, t_2\}, \max\{i_1, i_2\}, \max\{f_1, f_2\}) \\ \min\{x, y\} = (\min\{t_1, t_2\}, \min\{i_1, i_2\}, \min\{f_1, f_2\}) \end{cases}$$

An *N-norm* is a function $N_n : ([^-0, 1^+] \boxdot [^-0, 1^+] \boxdot [^-0, 1^+])^2 \mapsto [^-0, 1^+]$, defined in neutrosophic set theory and neutrosophic logic to represent the *intersection* of two neutrosophic sets and the neutrosophic logical operator *and* respectively. Extended to the fusion theory the N-norm will be a substitute for the conjunctive rule. The N-norm satisfies the conditions:

a) Boundary Conditions: $N_n(0, 0) = 0, N_n(x, 1) = x$.

b) Commutativity: $N_n(x, y) = N_n(y, x)$.

c) Monotonicity: If $x \leq u$ and $y \leq v$, then $N_n(x, y) \leq N_n(u, v)$.

d) Associativity: $N_n(N_n(x, y), z) = N_n(x, N_n(y, z))$.

There are many functions which satisfy the N-norm conditions. We present below the most known ones:

- The Algebraic Product N-norm: $N_{\text{n-algebraic}}(x, y) = x \boxdot y$
- The Bounded N-norm: $N_{\text{n-bounded}}(x, y) = \max\{0, x \boxplus y \boxminus 1\}$
- The Default (min) N-norm: $N_{\text{n-min}}(x, y) = \min\{x, y\}$.

5.10 N-conorm

An N-conorm is a function, $N_c : ([^-0,1^+] \boxdot [^-0,1^+] \boxdot [^-0,1^+])^2 \mapsto [^-0,1^+]$, defined in neutrosophic set theory and neutrosophic logic to represent the *union* of two neutrosophic sets and the neutrosophic logical operator *or* respectively. Extended to the fusion theory the N-conorm will be a substitute for the disjunctive rule. The N-conorm satisfies the conditions:

a) Boundary Conditions: $N_c(1,1) = 1$, $N_c(x,0) = x$.

b) Commutativity: $N_c(x,y) = N_c(y,x)$.

c) Monotonicity: if $x \leq u$ and $y \leq v$, then $N_c(x,y) \leq N_c(u,v)$.

d) Associativity: $N_c(N_c(x,y),z) = N_c(x,N_c(y,z))$.

There are many functions which satisfy the N-conorm conditions. We present below the most known ones:

- The Algebraic Product N-conorm: $N_{\text{c-algebraic}}(x,y) = x \boxplus y \boxminus (x \boxdot y)$
- The Bounded N-conorm: $N_{\text{c-bounded}}(x,y) = \min\{1, x \boxplus y\}$
- The Default (max) N-conorm: $N_{\text{c-max}}(x,y) = \max\{x,y\}$.

Then, the *N-norm Fusion rule* and the *N-conorm Fusion rule* are defined as follows:

$$m_{Nn12}(A) = \sum_{\substack{X,Y\in\Theta \\ Y\cap Y=A}} N_n(m_1(X), m_2(Y)) \qquad m_{Nc12}(A) = \sum_{\substack{X,Y\in\Theta \\ Y\cup Y=A}} N_c(m_1(X), m_2(Y))$$

6 Examples of N-norm and N-conorm Fusion rules

Suppose one has the frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and two sources S_1 and S_2 that provide respectively the following information (triple masses): $m_1(\theta_1) = (0.6, 0.1, 0.3)$, i.e. S_1 believes in θ_1 with 60%, doesnt believe in θ_1 with 30%, and is undecided about θ_1 with 10%. Similarly, one considers also

$$m_1(\theta_2) = (0.8, 0, 0.2) \qquad m_2(\theta_1) = (0.5, 0.3, 0.2) \qquad m_2(\theta_2) = (0.7, 0.2, 0.1)$$

Since one can have all kind of information (i.e. incomplete, paraconsistent, complete) the sum of an hypothesis components may be < 1, > 1, or $= 1$. We can normalize the hypothesis components by dividing each component by the sum of the components.

6.1 Both Sources are right

If we consider that both sources are right, then one uses the N-norm (lets take, as an example, the Algebraic Product) and one gets[8]:

$$\begin{aligned} m_{\text{Nn12}}(\theta_1) &= m_1(\theta_1) \boxdot m_2(\theta_1) = (0.6, 0.1, 0.3) \boxdot (0.5, 0.3, 0.2) \\ &= (0.6 \cdot 0.5, 0.1 \cdot 0.3, 0.3 \cdot 0.2) = (0.30, 0.03, 0.06) \widetilde{\equiv} (0.769231, 0.076923, 0.153846) \end{aligned}$$

$$\begin{aligned} m_{\text{Nn12}}(\theta_2) &= m_1(\theta_2) \boxdot m_2(\theta_2) = (0.8, 0, 0.2) \boxdot (0.7, 0.2, 0.1) \\ &= (0.8 \cdot 0.7, 0 \cdot 0.2, 0.2 \cdot 0.1) = (0.56, 0, 0.02) \widetilde{\equiv} (0.965517, 0, 034483) \end{aligned}$$

$$\begin{aligned} m_{\text{Nn12}}(\theta_1 \cap \theta_2) &= [m_1(\theta_1) \boxdot m_2(\theta_2)] \boxplus [m_2(\theta_1) \boxdot m_1(\theta_2)] \\ &= [(0.6, 0.1, 0.3) \boxdot (0.7, 0.2, 0.1)] \boxplus [(0.8, 0, 0.2) \boxdot (0.5, 0.3, 0.2)] \\ &= (0.42, 0.02, 0.03) \boxplus (0.40, 0, 0.04) = (0.82, 0.02, 0.07) \widetilde{\equiv} (0.901099, 0.021978, 0.076923) \end{aligned}$$

If one finds out that $\theta_1 \cap \theta_2 = \emptyset$, then one uses the DSm hybrid rule adjusted with the N-norm in order to transfer the conflicting mass to $m_{\text{Nn12}}(\theta_1 \cup \theta_2) = (0.901099, 0.021978, 0.076923)$.

[8] where $\widetilde{\equiv}$ denotes *equality after normalization*

6.2 One Source is right and another one is not, but we dont know which one

We use the N-conorm (lets take, as an example, the Algebraic Product) and one gets:

$$\begin{aligned} m_{\text{Nc12}}(\theta_1) &= m_1(\theta_1) \boxplus m_2(\theta_1) \boxminus [m_1(\theta_1) \boxdot m_2(\theta_1)] \\ &= (0.6, 0.1, 0.3) \boxplus (0.5, 0.3, 0.2) \boxminus [(0.6, 0.1, 0.3) \boxdot (0.5, 0.3, 0.2)] \\ &= (0.6 + 0.5 - 0.6 \cdot 0.5, 0.1 + 0.3 - 0.1 \cdot 0.3, 0.3 + 0.2 - 0.3 \cdot 0.2) \\ &= (0.80, 0.37, 0.44) \widetilde{\equiv} (0.496894, 0.229814, 0.273292) \end{aligned}$$

$$\begin{aligned} m_{\text{Nc12}}(\theta_2) &= m_1(\theta_2) \boxplus m_2(\theta_2) \boxminus [m_1(\theta_2) \boxdot m_2(\theta_2)] \\ &= (0.8, 0, 0.2) \boxplus (0.7, 0.2, 0.1) \boxminus [(0.8, 0, 0.2) \boxdot (0.7, 0.2, 0.1)] \\ &= (0.8 + 0.7 - 0.8 \cdot 0.7, 0 + 0.2 - 0 \cdot 0.2, 0.2 + 0.1 - 0.2 \cdot 0.1) \\ &= (0.94, 0.20, 0.28) \widetilde{\equiv} (0.661972, 0.140845, 0.197183) \end{aligned}$$

$$\begin{aligned} m_{\text{Nc12}}(\theta_1 \cap \theta_2) &= [m_1(\theta_1) \boxplus m_2(\theta_2) \boxminus (m_1(\theta_1) \boxdot m_2(\theta_2))] \boxplus [m_1(\theta_2) \boxplus m_2(\theta_1) \boxminus (m_1(\theta_2) \boxdot m_2(\theta_1))] \\ &= [(0.6, 0.1, 0.3) \boxplus (0.7, 0.2, 0.1) \boxminus ((0.6, 0.1, 0.3) \boxdot (0.7, 0.2, 0.1))] \\ &\quad \boxplus [(0.8, 0, 0.2) \boxplus (0.5, 0.3, 0.2) \boxminus ((0.8, 0, 0.2) \boxdot (0.5, 0.3, 0.2))] \\ &= (0.88, 0.28, 0.37) \boxplus (0.90, 0.30, 0.36) \\ &= (1.78, 0.58, 0.73) \widetilde{\equiv} (0.576052, 0.187702, 0.236246). \end{aligned}$$

7 Conclusion

A general presentation of foundation of DSmT and its connection with neutrosophic logic has been proposed in this chapter. We proposed new rules of combination for uncertain, imprecise and highly conflicting sources of information. Several applications of DSmT have been proposed recently in the literature and show the efficiency of this new approach over classical rules based mainly on the Demspter's rule in the DST framework. In the last past of this chapter, we showed that the combination of paradoxical, uncertain and imprecise sources of information can also be done using fuzzy and neutrosophic logics or sets together with DSmT and other fusion rules or theories. The T-norms/conorm and N-norms/conorms help in redefining new fusion rules of combination or in improving the existing ones.

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Importance of Sources using the Repeated Fusion Method and the Proportional Conflict Redistribution Rules #5 and #6

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Abstract.
We present in this paper some examples of how to compute by hand the PCR5 fusion rule for three sources, so the reader will better understand its mechanism.
We also take into consideration the importance of sources, which is different from the classical discounting of sources.

1. Introduction.

Discounting of Sources.
Discounting a source $m_1(.)$ with the coefficient $0 \leq \alpha \leq 1$ and a source $m_2(.)$ with a coefficient $0 \leq \beta \leq 1$ (because we are not very confident in them), means to adjust them to $m_1'(.)$ and $m_2'(.)$ such that:
$m_1'(A) = \alpha \cdot m_1(A)$ for $A \neq \Theta$ (total ignorance), and $m_1'(\Theta) = \alpha \cdot m_1(\Theta) + 1 - \alpha$,
and $m_2'(A) = \beta \cdot m_2(A)$ for $A \neq \Theta$ (total ignorance), and $m_2'(\Theta) = \beta \cdot m_2(\Theta) + 1 - \beta$.

Importance of Sources using Repeated Fusion.
But if a source is more important than another one (since a such source comes from a more important person with a decision power, let's say an executive director), for example if source *m2(.)* is twice more important than source $m_1(.)$, then we can combine $m_1(.)$ with $m_2(.)$ and with $m_2(.)$, so we repeated $m_2(.)$ twice. Doing this procedure, the source which is repeated (combined) more times than another source attracts the result towards its masses – see an example below.
Jean Dezert has criticized this method since if a source is repeated say 4 times and other source is repeated *6* times, then combining *4* times $m_1(.)$ with 6 times $m_2(.)$ will give a result different from combining *2* times $m_1(.)$ with *3* times $m_2(.)$, although $4/6 = 2/3$. In order to avoid this, we take the simplified fraction n/p, where $gcd(n, p) = 1$, where *gcd* is the greatest common divisor of the natural numbers n and p.
This method is still controversial since after a large number of combining n times $m_1(.)$ with p times $m_2(.)$ for $n+p$ sufficiently large, the result is not much different from a previous one which combines n_1 times $m_1(.)$ with p_1 times $m_2(.)$ for n_1+p_1 sufficiently large but a little less than $n+p$, so the method is not well responding for large numbers.

A more efficacy method of importance of sources consists in taking into consideration the discounting on the empty set and then the normalization (see especially paper [4] and also [1]).

2. Using m_{PCR5} for 3 Sources.

Example calculated by hand for combining three sources using *PCR5* fusion rule.

Let's say that $m_2(.)$ is 2 times more important than $m_1(.)$; therefore we fusion $m_1(.)$, $m_2(.)$, $m_2(.)$.

	A	B	$A \cup B$	$A \cap B = \Phi$
m_1	0.1	0.7	0.2	
m_2	0.4	0.1	0.5	
m_2	0.4	0.1	0.5	
m_{122}	0.193	0.274	0.050	0.483

$$\left[\begin{array}{l} \dfrac{x_{1A}}{0.1} = \dfrac{y_{1B}}{0.7} = \dfrac{z_{2A\cup B}}{0.5} = \dfrac{0.005}{0.7} = \dfrac{0.05}{7} \\ x_{1A} = 0.000714 \\ y_{1B} = 0.000714 \\ z_{2A\cup B} = 0.003572 \end{array}\right.$$

$$\left[\begin{array}{l} \dfrac{x_{2A}}{0.4} = \dfrac{y_{2B}}{0.7} = \dfrac{z_{2A\cup B}}{0.5} = \dfrac{0.14}{1.6} = \dfrac{0.07}{0.8} = \dfrac{0.7}{8} \\ x_{2A} = 0.035000 \\ y_{2B} = 0.061250 \\ z_{2A\cup B} = 0.043750 \end{array}\right.$$

$$\left[\begin{array}{l} \dfrac{x_{3A}}{0.4} = \dfrac{y_{3B}}{0.1} = \dfrac{z_{3A\cup B}}{0.2} = \dfrac{0.008}{0.7} = \dfrac{0.08}{7} \\ x_{3A} \cong 0.004571 \\ y_{3B} \cong 0.001143 \\ z_{3A\cup B} \cong 0.002286 \end{array}\right.$$

$$\left[\begin{array}{l} \dfrac{x_{4A}}{0.4} = \dfrac{y_{4B}}{0.1} = \dfrac{z_{4A\cup B}}{0.2} = \dfrac{(0.4)(0.1)(0.2)}{0.7} = \dfrac{0.008}{0.7} = \dfrac{0.08}{7} \\ x_{4A} \cong 0.004571 \\ y_{4B} \cong 0.001143 \\ z_{4A\cup B} \cong 0.002286 \end{array}\right.$$

$$\begin{cases} \dfrac{x_{5A}}{0.4} = \dfrac{y_{5B}}{0.7} = \dfrac{z_{5A\cup B}}{0.5} = \dfrac{0.14}{1.6} = \dfrac{1.4}{16} \\ x_{5A} \cong 0.035000 \\ y_{5B} \cong 0.061250 \\ z_{5A\cup B} \cong 0.043750 \end{cases}$$

$$\begin{cases} \dfrac{x_{6A}}{0.1} = \dfrac{y_{6B}}{0.1} = \dfrac{z_{6A\cup B}}{0.5} = \dfrac{0.005}{0.7} = \dfrac{0.05}{7} \\ x_{6A} \cong 0.000714 \\ y_{6B} \cong 0.000714 \\ z_{6A\cup B} \cong 0.003572 \end{cases}$$

$$\begin{cases} \dfrac{x_{7A}}{0.1} = \dfrac{y_{7B}}{(0.1)(0.1)} = \dfrac{(0.1)(0.1)(0.1)}{0.1+0.01} = \dfrac{0.001}{0.11} \\ x_{6A} \cong 0.000909 \\ y_{6B} \cong 0.000091 \end{cases}$$

$$\begin{cases} \dfrac{x_{8A}}{0.4} = \dfrac{y_{8B}}{(0.7)(0.1)} = \dfrac{(0.4)(0.7)(0.1)}{0.1+0.01} = \dfrac{0.028}{0.47} = \dfrac{2.8}{47} \\ x_{8A} \cong 0.023830 \\ y_{8B} \cong 0.004170 \end{cases}$$

$$x_{9A} = x_{8A} \cong 0.023830$$
$$y_{9B} = y_{8B} \cong 0.004170$$

$$\begin{cases} \dfrac{x_{10A}}{(0.1)(0.4)} = \dfrac{y_{10B}}{0.1} = \dfrac{(0.1)(0.4)(0.1)}{0.04+0.1} = \dfrac{0.004}{0.14} = \dfrac{0.4}{14} = \dfrac{0.2}{7} \\ x_{10A} \cong 0.001143 \\ y_{8B} \cong 0.002857 \end{cases}$$

$$x_{11A} = x_{10A} \cong 0.001143$$
$$y_{11B} = y_{10B} \cong 0.002857$$

$$\begin{cases} \dfrac{x_{12A}}{(0.1)(0.4)} = \dfrac{y_{12B}}{0.1} = \dfrac{(0.4)(0.4)(0.7)}{0.16+0.7} = \dfrac{0.112}{0.86} = \dfrac{11.2}{86} \\ x_{12A} \cong 0.020837 \\ y_{12B} \cong 0.091163 \end{cases}$$

	A	B	A∪B
m_{122}^{PCR5}	0.345262	0.505522	0.149216

If we didn't double $m_2(.)$ in the fusion rule, we'd get a different result.
Let's suppose we only fusion $m_1(.)$ with $m_2(.)$:

	A	B	A∪B	A∩B=Φ
m_1	0.1	0.7	0.2	
m_2	0.4	0.1	0.5	
m_{12}	0.17	0.44	0.10	0.29
m_{12}^{PCR5}	0.322	0.668	0.100	0

And now we compare the fusion results:

	A	B	A∪B	
m_{122}^{PCR5}	0.345	0.506	0.149	- *three sources*(sec*ond – source – doubled*)*; importance of sources considered;*
m_{12}^{PCR5}	0.322	0.668	0.100	- *two sources; importance of sources not considered*.

The more times we repeat $m_2(.)$ the closer $m_{12...2}^{PCR5}(A) \rightarrow m_2(A)=0.4$, $m_{12...2}^{PCR5}(B) \rightarrow m_2(B)=0.1$, and $m_{12...2}^{PCR5}(A \cup B) \rightarrow m_2(A \cup B)=0.5$. Therefore, doubling, tripling, etc. a source, the mass of each element in the frame of discernment tends towards the mass value of that element in the repeated source (since that source is considered to have more importance than the others).

For the readers who want to do the previous calculation with a computer, here it is the m_{PCR5}
Formula for 3 Sources:

$$m_{PCR5}(A) = m_{123} + \sum_{\substack{X,Y \in G^{\Theta} \\ A \neq X \neq Y \neq A \\ A \cap X \cap Y = \Phi}} \left(\frac{m_1(A)^2 m_2(X) m_3(Y)}{m_1(A)+m_2(X)+m_3(Y)} + \right.$$

$$\left. + \frac{m_1(Y) m_2(A)^2 m_3(X)}{m_1(Y)+m_2(A)+m_3(X)} + \frac{m_1(X) m_2(Y) m_3(A)^2}{m_1(X)+m_2(Y)+m_3(A)} \right) +$$

$$+\sum_{\substack{X\in G^{\Theta}\\ A\cap X=\Phi}}\left(\frac{m_1(A)^2 m_2(X)m_3(X)}{m_1(A)+m_2(X)+m_3(X)}+\frac{m_1(X)m_2(A)^2 m_3(X)}{m_1(X)+m_2(A)+m_3(X)}+\frac{m_1(X)m_2(X)m_3(A)^2}{m_1(X)+m_2(X)+m_3(A)}\right)+$$

$$+\sum_{\substack{X\in G^{\Theta}\\ A\cap X=\Phi}}\left(\frac{m_1(A)^2 m_2(A)^2 m_3(X)}{m_1(A)+m_2(A)+m_3(X)}+\frac{m_1(X)m_2(A)^2 m_3(A)^2}{m_1(X)+m_2(A)+m_3(A)}+\frac{m_1(A)^2 m_2(X)m_3(A)^2}{m_1(A)+m_2(X)+m_3(A)}\right)$$

3. Similarly, let's see the m_{PCR6} Formula for 3 Sources:

$$m_{PCR6}(A)=m_{123}+\sum_{\substack{X,Y\in G^{\Theta}\\ A\neq X\neq Y\neq A\\ A\cap X\cap Y=\Phi}}\left(\frac{m_1(A)^2 m_2(X)m_3(Y)}{m_1(A)+m_2(X)+m_3(Y)}+\right.$$

$$\left.+\frac{m_1(Y)m_2(A)^2 m_3(X)}{m_1(Y)+m_2(A)+m_3(X)}+\frac{m_1(X)m_2(Y)m_3(A)^2}{m_1(X)+m_2(Y)+m_3(A)}\right)+$$

$$+\sum_{\substack{X\in G^{\Theta}\\ A\cap X=\Phi}}\left(\frac{m_1(A)^2 m_2(X)m_3(X)}{m_1(A)+m_2(X)+m_3(X)}+\frac{m_1(X)m_2(A)^2 m_3(X)}{m_1(X)+m_2(A)+m_3(X)}+\frac{m_1(X)m_2(X)m_3(A)^2}{m_1(X)+m_2(X)+m_3(A)}\right)+$$

$$+\sum_{\substack{X\in G^{\Theta}\\ A\cap X=\Phi}}\left(\frac{m_1(A)^2 m_2(A)m_3(X)+m_1(A)m_2(A)^2 m_3(X)}{m_1(A)+m_2(A)+m_3(X)}+\right.$$

$$+\frac{m_1(X)m_2(A)^2 m_3(A)+m_1(X)m_2(A)m_3(A)^2}{m_1(X)+m_2(A)+m_3(A)}+$$

$$\left.+\frac{m_1(A)^2 m_2(X)m_3(A)+m_1(A)m_2(X)m_3(A)^2}{m_1(A)+m_2(X)+m_3(A)}\right)$$

4. A General Formula for $PCR6$ for $s\geq 2$ Sources.

$$m_{PCR6}(A)=m_{12\ldots s}+\sum_{\substack{X_1,X_2,\ldots,X_{s-1}\in G^{\Theta}\\ X_i\neq A,i\in\{1,2,\ldots,s-1\}\\ \left(\bigcap_{i=1}^{s-1}X_i\right)\cap A=\Phi}}\sum_{k=1}^{s-1}\sum_{(i_1,i_2,\ldots,i_s)\in P(1,2,\ldots,s)}\left[m_{i_1}(A)+m_{i_2}(A)+\ldots+m_{i_k}(A)\right]\cdot$$

$$\cdot\frac{m_{i_1}(A)m_{i_2}(A)\ldots m_{i_k}(A)m_{i_{k+1}}(X_1)\ldots m_{i_s}(X_{s-k})}{m_{i_1}(A)+m_{i_2}(A)+\ldots+m_{i_k}(A)+m_{i_{k+1}}(X_1)+\ldots+m_{i_s}(X_{s-k})}$$

where *P(1, 2, ..., s)* is the set of all permutations of the elements *{1, 2, ..., s}*.

It should be observed that $X_1, X_2, \ldots, X_{s-1}$ may be different from each other, or some of them equal and others different, etc.

We wrote this *PCR6* general formula in the style of *PCR5*, different from Arnaud Martin & Christophe Oswald's notations, but actually doing the same thing. In order not to complicate the formula of *PCR6*, we did not use more summations or products after the third Sigma.

As a particular case:

$$m_{PCR6}(A) = m_{123} + \sum_{\substack{X_1,X_2 \in G^\Theta \\ X_1 \neq A, X_2 \neq A \\ X_1 \cap X_1 \cap A = \Phi}} \sum_{k=1}^{2} \sum_{(i_1,i_2,i_3) \in P(1,2,3)} \frac{\left[m_{i_1}(A) + \ldots + m_{i_k}(A) \right] m_{i_1}(A) \ldots m_{i_k}(A) m_{i_{k+1}}(X_1) \ldots m_{i_3}(X_2)}{m_{i_1}(A) + \ldots + m_{i_k}(A) + m_{i_{k+1}}(X_1) + \ldots + m_{i_3}(X_2)}$$

where $P(1,2,3)$ is the set of permutations of the elements $\{1,2,3\}$.

It should also be observed that X_1 may be different from or equal to X_2.

Conclusion.

The aim of this paper was to show how to manually compute *PCR5* for *3* sources on some examples, thus better understanding its essence. And also how to take into consideration the *importance of sources* doing the Repeated Fusion Method. We did not present the Method of Discounting to the Empty Set in order to emphasize the importance of sources, which is better than the first one, since the second method was the main topic of paper [4].

We also presented the *PCR5* formula for *3* sources (a particular case when *n=3*), and the general formula for *PCR6* in a different way but yet equivalent to Martin-Oswald's *PCR6* formula [2].

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A Class of DSm Conditioning Rules[1]

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Abstract:

In this paper we introduce two new DSm fusion conditioning rules with example, and as a generalization of them a class of DSm fusion conditioning rules, and then extend them to a class of DSm conditioning rules.

Keywords: conditional fusion rules, Dempster's conditioning rule, Dezert-Smarandache Theory, DSm conditioning rules

0. Introduction

In order to understand the material in this paper, it is first necessary to define the terms that we'll be using:

- Frame of discernment = the set of all hypotheses.
- Ignorance is the mass (belief) assigned to a union of hypotheses.
- Conflicting mass is the mass resulted from the combination of many sources of information of the hypotheses whose intersection is empty.
- Fusion space = is the space obtained by combining these hypotheses using union, intersection, or complement – depending on each fusion theory.
- Dempster-Shafer Theory is a fusion theory, i.e. method of examination of hypotheses based on measures and combinations of beliefs and plausibility in each hypothesis, beliefs provided by many sources of information such as sensors, humans, etc.
- Transferable Belief Model is also a fusion theory, an alternative of DST, whose method is of transferring the conflicting mass to the empty set.

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- Dezert-Smarandache Theory is a fusion theory, which is a natural extension of DST and works for high conflicting sources of information, and overcomes the cases where DST doesn't work.
- Power set = is the fusion space of Dempster-Shafer Theory (DST) and Transferable Belief Model (TBM) theory; the power set is the set of all subsets of the frame of discernment, i.e. all hypotheses and all possible unions of hypotheses. {In the fusion theory union of hypotheses means uncertainty about these hypotheses.}
- Hyper-power set = the fusion set of Dezert-Smarandache Theory (DSmT); the hyper-power set is the set formed by all unions and intersections of hypotheses. {By intersection of two or more hypotheses we understand the common part of these hypotheses – if any. In the case when their intersection is empty, we consider these hypotheses disjoint.}
- Super-power set = the fusion space for the Unification of Fusion Theories and rules; the super-power set is the set formed by all unions, intersection, and complements of the hypotheses. {By a complement of a hypothesis we understand the opposite of that hypothesis.}
- Basic belief assignment (bba), also called mass and noted by m(.), is a subjective probability (belief) function that a source assigns to some hypotheses or their combinations. This function is defined on the fusion space and whose values are in the interval [0, 1].

In the first section, we consider a frame of discernment and then we present the three known fusion spaces. The first fusion space, the power set, is used by Dempster-Shafer Theory (DST) and the Transferable Believe Model (TBM). The second fusion space, which is larger, the hyper-power set, is used by Dezert-Smarandache Theory (DSmT), while the third fusion space, the super-power set, is the most general one, and it is used in the Unification of Fusion Theories and Rules.

In the second section we present Dempster's conditioning rule and the Bel(.) and Pl(.) functions.

In order to overcome some difficult corner cases where Dempster's Conditioning Rule doesn't work, we design the first simple DSm conditioning rule and the second simple DSm conditioning rule in section 3. These rules are referring to the fact that: if a source provides us some evidence (i.e. a basic belief assignment), but later we find out that the true hypothesis is in a subset A of the fusion space, then we need to compute the conditional belief m(.|A).

In section 4 we give a Class of DSm Conditioning Rules that generalizes two simple DSm conditioning rules cover.

In section 5 we present two examples in military about target attribute identification.

1. Mathematical Preliminaries.

Let $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n \geq 2$, be a frame of discernment.

As fusion space, Shafer uses the power set 2^{Θ}, which means Θ closed under union of sets, $(\Theta, \cup)$, and it is a Boolean algebra. In Dempster-Shafer Theory (DST) all hypotheses θ_i are considered mutually exclusive, i.e. $\theta_i \cap \theta_j = \phi$ for any $i \neq j$, and exhaustive.

Dezert extended the power set to a hyper-power set D^{Θ} in Dezert-Smarandache Theory (DSmT), which means Θ closed under union and intersection of sets $(\Theta, \cup, \cap)$ and it is a distribute lattice; in this case the hypotheses are not necessarily exclusive, so there could be two or more hypotheses whose intersections are non-empty. Each model in DSmT is characterized by empty and non-empty intersections. If all intersections are empty, we get Shafer's model used in DST; if some intersection are empty and others are not, we have a hybrid model; and if all intersection are non-empty we have a free model.

Further on Smarandache [3] extended the hyper-power to a super-power set S^{Θ}, as in UFT (Unification of Fusion Theories), which means Θ closed under union, intersection, and complement of sets $(\Theta, \cup, \cap, \mathcal{C})$, that is a Boolean algebra.

We note by G any of these three fusion spaces, power set, hyper-power set, or super-power set.

2. Dempster's Conditioning Rule (DCR).

Let's have a bba (basic believe assignment, also called mass):

$m_1: G^{\Theta} \rightarrow [0, 1]$, where $\sum_{X \in G^{\Theta}} m_1(X) = 1$.

In the main time we find out that the truth is in $B \in G^{\Theta}$. We therefore need to adjust our bba according to the new evidence, so we need to compute the conditional bba $m_1(X|B)$ for all $X \in G^{\Theta}$.

Dempster's conditioning rule means to simply fuse the mass $m_1(.)$ with $m_2(B) = 1$ using Dempster's classical fusion rule.

A similar procedure can be done in DSmT, TBM, etc. by combining $m_1(.)$ with $m_2(B) = 1$ using other fusion rule.

In his book Shafer gave the conditional formulas for believe and plausible functions Bel(.) and respectively Pl(.) only, not for the mass m(.).

In general we know that:

$$\text{Bel(A)} = \sum_{X \subseteq A} m_1(X)$$

and

$$\text{Pl(A)} = \sum_{X \cap A \neq \phi} m_1(X) .$$

Let $m_1(.)$ and $m_2(.)$ be two bba's defined on G^Θ. The conjunctive rule for combining these bba's is the following:

$$(m_1 + m_2)(A) = \sum_{\substack{X,Y \in G^\Theta \\ X \cap Y = A}} m_1(X) m_2(Y)$$

In order to compute in DST the subjective conditional probability of B given A, i.e. m(A|B), Shafer combines the masses $m_1(.)$ and $m_2(B)=1$ using Dempster's rule (pp. 71-72 in [2]) and he gets:

$$m(A|B) = \frac{\sum_{X \cap Y = A} m_1(X) m_2(Y)}{1 - \sum_{X \cap Y = \phi} m_1(X) m_2(Y)} \text{ (which is Dempster's rule)}$$

$$= \frac{\sum_{X \cap B = A} m_1(X) m_2(B)}{1 - \sum_{X \cap B = \phi} m_1(X) m_2(B)} \text{ (since only } m_2(B) \neq 0, \text{ all other values of } m_2(Y) = 0 \text{ for } Y \neq B)$$

$$= \frac{\sum_{X \cap B = A} m_1(X)}{1 - \sum_{X \cap B = \phi} m_1(X)} = \frac{\sum_{X \cap B = A} m_1(X)}{\sum_{X \cap B \neq \phi} m_1(X)} \text{ which is exactly what Milan Daniel got in [1], but with}$$

different notations.

Therefore, **Dempster's Conditioning Rule** (DCR) referred to masses {not to Bel(.) or to Pl(.) functions as designed by Shafer} is the following:

$$\forall A \in 2^\Theta \setminus \phi \text{ we have } m_{DCR}(A|B) = \frac{\sum_{X \cap B = A} m_1(X)}{\sum_{X \cap B \neq \phi} m_1(X)} .$$

With M. Daniel's notations, Dempster's Conditioning Rule becomes:

$$\forall X \in D^\Theta \setminus \phi \text{ we have } m_{DCR}(X|A) = \frac{\sum_{Y \cap A = X} m_1(Y)}{\sum_{Y \cap A \neq \phi} m_1(Y)} .$$

DCR doesn't work when Pl(A) = 0 since its denominator becomes null.

3. Two DSm Conditioning Rules.

We can overcome this undefined division by constructing a **DSm first simple conditioning rule** in the super-power set:

$\forall X \in S^{\Theta} \setminus \phi$ we have $m_{DSmT1}(X|A) = \sum_{(Y \cap A = X) or (Y \cap A = \phi and X = A)} m(Y)$

which works in any case.

In the corner case when Pl(A) = 0, we get $m_{DSmT1}(A|A) = 1$ and all other $m_{DSmT1}(X|A) = 0$ for $X \neq A$.

The DSm first simple conditioning rule transfers the masses which are outside of A (i.e. the masses m(Y) with $Y \cap A = \phi$) to A in order to keep the normalization of m(.), in order to avoid doing normalization by division as DCR does.

Another way will be to uniformly split the total mass which is outside of A:

$$K_{cond} = \sum_{Y \cap A = \phi} m(Y)$$

to the non-empty sets of $\mathcal{P}(A)$, i.e. sets whose mass is non-zero, where $\mathcal{P}(A)$ is the set of all parts of A.

So, a **DSm second simple conditioning rule** is:

$$m_{DSmT2}(X \mid A) = \sum_{Y \cap A = X} m(Y) + \frac{1}{C_{P(A)}} \cdot \sum_{Y \cap A = \phi} m(Y)$$

where $C_{P(A)}$ is the cardinal of the set of elements from P(A) whose masses are not zero, i.e.

$$C_{P(A)} = \text{Card}\{Z \mid Z \in S^{\Theta}, Z \subseteq A, \sum_{Y \cap A = Z} m(Y) \neq 0\}.$$

In the corner edge when $C_{P(A)} = 0$, we replace it with the number of singletons included in A if any, the number of unions of singletons included in A if any, and A itself.

4. A Class of DSm Conditioning Rules.

In this way we can design a **class of DSm conditioning rules** taking into consideration not only masses, but also other parameters that might influence the decision-maker in calculating the subjective conditioning probability, and which is a generalization of Dempster's conditioning rule:

$$m_{DSmTclass}(X \mid A) = \frac{\sum_{Y \cap A = X} \frac{\alpha(Y)}{\beta(Y)}}{\sum_{Y \cap A = \phi} \frac{\alpha(Y)}{\beta(Y)}}$$

with $\alpha(Y) = \alpha_1(Y) \cdot \alpha_2(Y) \cdot \ldots \cdot \alpha_p(Y)$, where all $\alpha_i(Y)$, $1 \leq i \leq p$, are parameters that Y is directly proportional to;

and $\beta(Y) = \beta_1(Y) \cdot \beta_2(Y) \cdot \ldots \cdot \beta_r(Y)$, where all $\beta_j(Y)$, $1 \leq j \leq r$, are parameters that Y is inversely proportional to.

5. Examples of Conditioning Rules.

Example 5.1.

Let $m_1(.)$ be defined on the frame {F = friend, E = enemy, N = neutral}, where the hypotheses F, E, N are mutually exclusive, in the following way (see the second row):

	ϕ	F	E	N	$F \cup E$	$F \cup E \cup N$	$N \cap (F \cup E)$
m_1	0	0.2	0.1	0.3	0.1	0.3	0
m_2	0	0	0	0	1	0	0
$m_1 + m_2$	0	0.2	0.1	0	0.4	0	0.3
$m_{DCR}(X \mid F \cup E)$	0	2/7	1/7	0	4/7	0	0
$m_{TBM}(X \mid F \cup E)$	0.3	0.2	0.1	0	0.4	0	0
$m_{DSmT1}(X \mid F \cup E)$	0	0.2	0.1	0	0.7	0	0
$m_{DSmT2}(X \mid F \cup E)$	0	0.3	0.2	0	0.5	0	0

Table 1

Suppose the truth is in the set $F \cup E$. First we combine $m_1(.)$ with $m_2(E) = 1$ using the conjunctive rule, and its result $m_1 + m_2$ is in the fourth row in Table 1. All below conditioning rules are referred to the result of this conjunctive rule, and they differ through the way the conflicting mass, i.e. mass of empty intersections, is transferred to the other elements.

In DCR, since $N \cap (F \cup E) = \phi$ the conflicting mass $m_1(N) \cdot m_2(F \cup E) = 0.3 \cdot 1 = 0.3$, is transferred to the non-empty sets F, E, and $F \cup E$ proportionally with respect to their masses acquired after applying the conjunctive rule ($m_1 + m_2$), i.e. with respect to 0.2, 0.1, and

respectively 0.4. Thus, we get $m_{DST}(X|F \cup E)$ as in the fifth row of Table 1, where $X \in \{ \phi, F, E, N, F \cup E, F \cup E \cup N, N \cap (F \cup E)\}$.

In Smets' TBM (Transferable Believe Model), the conflicting mass, 0.3, is transferred to the empty set, since TBM considers an open world (non-exhaustive hypotheses). See row # 6.

With DSm first conditioning rule (row # 7) the conflicting mass 0.3 is transferred to the whole set that the truth belongs to, $F \cup E$. So, $m_{DSmT1}(F \cup E | F \cup E) = (m_1 + m_2)(F \cup E) + 0.3 = 0.4+0.3 = 0.7$.

In DSm second conditioning rule (row # 8) the conflicting mass 0.3 is uniformly transferred to the non-empty sets F, E, and $F \cup E$, therefore each such set receives $0.3/3 = 0.1$.

Example 5.2.

Let $m_1(.)$ be defined on the frame {A = Airplane, T = tank, S = ship, M = submarine}, where the hypotheses A, B, C, D are mutually exclusive, in the following way (see the second row):

	ϕ	A	T	S	M	$A \cup S$	$T \cup M$	$A \cap (T \cup M)$	$S \cap (T \cup M)$	$(A \cup S) \cap (T \cup M)$
m_1	0	0.4	0	0.5	0	0.1	0			
m_2	0	0	0	0	0	0	1			
$m_1 + m_2$	0	0	0	0	0	0	0	0.4	0.5	0.1
$m_{DCR}(X \mid T \cup M)$	0	N/A	N/A	N/A	N/A	N/A	N/A			
$m_{TBM}(X \mid T \cup M)$	1	0	0	0	0	0	0	0	0	0
$m_{DSmT1}(X \mid T \cup M)$	0	0	0	0	0	0	1	0	0	0
$m_{DSmT2}(X \mid T \cup M)$	0	0	1/3	0	1/3	0	1/3	0	0	0

Table 2

Suppose the truth is in $T \cup M$. Since the sets $A \cap (T \cup M)$, $S \cap (T \cup M)$, and $(A \cup S) \cap (T \cup M)$ are empty, their masses 0.4, 0.5, and respectively 0.1 have to be transferred to non-empty sets belonging to $P(T \cup M)$, where $P(T \cup M)$ means the set of all subsets of $T \cup M$.

In this case, DCR does not work since it gets an undefined division 0/0.

In Smets' TBM (Transferable Believe Model), the total conflicting mass, $0.4 + 0.5 + 0.1 = 1$, is transferred to the empty set, since TBM considers an open world (non-exhaustive hypotheses). See row # 6.

With DSm first conditioning rule (row # 7) the total conflicting mass, 1, is transferred to the whole set that the truth belongs to, $T \cup D$. So, $m_{DSmT1}(T \cup D | T \cup D) = (m_1 + m_2)(T \cup D) + 1 = 1$.

In DSm second conditioning rule (row # 8) the total conflicting mass is 1. Since $C_{(B \cup D)} = 0$, the total conflicting mass 1 is uniformly transferred to the sets T, D, and $T \cup D$ {i.e. the singletons and unions of singletons included in $T \cup D$}, therefore each such set receives 1/3.

Conclusion.

We have examined Dempster's Conditioning Rule in terms of bba. We saw that in the second military example, using DCR for target identification, the procedure failed mathematically. That's why we designed two DSm simple conditioning rules and could complete the procedure of target identification. We have compared these approaching of target identification using DCR, TBM conditioning, and the two DSm conditioning rules that got better results than DCR and TBM. We also observed from these examples that the two DSm simple conditioning rules give almost similar results.

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Extension of Inagaki General Weighted Operators

and

A New Fusion Rule Class of Proportional Redistribution of Intersection Masses

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Abstract.
In this paper we extend Inagaki Weighted Operators fusion rule (WO) [see 1, 2] in information fusion by doing redistribution of not only the conflicting mass, but also of masses of non-empty intersections, that we call Double Weighted Operators (DWO).
Then we propose a new fusion rule Class of Proportional Redistribution of Intersection Masses (CPRIM), which generates many interesting particular fusion rules in information fusion.
Both formulas are presented for 2 and for n ≥ 3 sources.
An application and comparison with other fusion rules are given in the last section.

Keywords: Inagaki Weighted Operator Rule, fusion rules, proportional redistribution rules, DSm classic rule, DSm cardinal, Smarandache codification, conflicting mass

ACM Classification: I.4.8.

1. Introduction.

Let $\theta = \{\theta_1, \theta_2, ..., \theta_n\}$, for $n \geq 2$, be the frame of discernment, and $S^\theta = (\theta, \cup, \cap, \tau)$ its super-power set, where τ(x) means complement of x with respect to the total ignorance.

Let I_t = total ignorance = $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$, and Φ be the empty set.

$S^\theta = 2^{\wedge}\theta_{refined} = 2^{\wedge}(2^{\wedge}\theta) = D^{\theta \cup \theta c}$, when refinement is possible, where $\theta_c = \{\tau(\theta_1), \tau(\theta_2), \ldots, \tau(\theta_n)\}$.

We consider the general case when the domain is S^θ, but S^θ can be replaced by $D^\theta = (\theta, \cup, \cap)$ or by $2^\theta = (\theta, \cup)$ in all formulas from below.

Let $m_1(\cdot)$ and $m_2(\cdot)$ be two normalized masses defined from S^θ to $[0,1]$.

We use the conjunction rule to first combine $m_1(\cdot)$ with $m_2(\cdot)$ and then we redistribute the mass of $m(X \cap Y) \neq 0$, when $X \cap Y = \Phi$.

Let's denote $m_{2\cap}(A) = \left(m_1 \oplus m_2\right)(A) = \sum_{\substack{X,Y \in S^\theta \\ (X \cap Y)=A}} m_1(X) m_2(Y)$ using the conjunction rule.

Let's note the set of intersections by:

$$S_\cap = \left\{ \begin{array}{l} X \in S^\theta \mid X = y \cap z, \text{ where } y, z \in S^\theta \setminus \{\Phi\}, \\ X \text{ is in a canonical form, and} \\ X \text{ contains at least an } \cap \text{ symbol in its formula} \end{array} \right\}. \quad (1)$$

In conclusion, $S_\cap$ is a set of formulas formed with singletons (elements from the frame of discernment), such that each formula contains at least an intersection symbol $\cap$, and each formula is in a canonical form (easiest form).

For example: $A \cap A \notin S_\cap$ since $A \cap A$ is not a canonical form, and $A \cap A = A$. Also, $(A \cap B) \cap B$ is not in a canonical form but $(A \cap B) \cap B = A \cap B \in S_\cap$.

Let

$S_\cap^\Phi =$ the set of all empty intersections from $S_\cap$,

and

$S_{\cap,r}^{non\Phi} =$ {the set of all non-empty intersections from $S_\cap^{non\Phi}$ whose masses are redistributed to other sets, which actually depends on the sub-model of each application}.

2. Extension of Inagaki General Weighted Operators (WO).

Inagaki general weighted operator (WO) is defined for two sources as:

$$\forall\, A \in 2^\theta \setminus \{\Phi\},\ m_{(WO)}(A) = \sum_{\substack{X,Y \in 2^\theta \\ (X \cap Y)=A}} m_1(X) m_2(Y) + W_m(A) \cdot m_{2\cap}(\Phi), \quad (2)$$

where

$$\sum_{X \in 2^\theta} W_m(X) = 1 \text{ and all } W_m(\cdot) \in [0,1]. \quad (3)$$

So, the conflicting mass is redistributed to non-empty sets according to these weights $W_m(\cdot)$.

In the extension of this WO, which we call the Double Weighted Operator (DWO), we redistribute not only the conflicting mass $m_{2\cap}(\Phi)$ but also the mass of some (or all) non-empty intersections, i.e. those from the set $S_{\cap,r}^{non\Phi}$, to non-empty sets from S^θ according to some weights $W_m(\cdot)$ for the conflicting mass (as in WO), and respectively according to the weights $V_m(.)$ for the non-conflicting mass of the elements from the set $S_{\cap,r}^{non\Phi}$:

$$\forall\, A \in \left(S^\theta \setminus S_{\cap,r}^{non\Phi}\right) \setminus \{\Phi\},\ m_{DWO}(A) = \sum_{\substack{X,Y \in S^\theta \\ (X \cap Y)=A}} m_1(X) m_2(Y) + W_m(A) \cdot m_{2\cap}(\Phi) + V_m(A) \cdot \sum_{z \in S_{\cap,r}^{non\Phi}} m_{2\cap}(z),$$

(4)

where

$$\sum_{X\in S^{\theta}} W_m(X)=1 \text{ and all } W_m(\cdot)\in[0,1], \text{ as in (3)}$$

and

$$\sum_{z\in S_{\cap,r}^{non\Phi}} V_m(z)=1 \text{ and all } V_m(\cdot)\in[0,1]. \tag{5}$$

In the free and hybrid modes, if no non-empty intersection is redistributed, i.e. $S_{\cap,r}^{non\Phi}$ contains no elements, DWO coincides with WO.

In the Shafer's model, always DWO coincides with WO.

For $s\geq 2$ sources, we have a similar formula:

$$\forall\, A\in \left(S^{\theta}\setminus S_{\cap,r}^{non\Phi}\right)\setminus\{\Phi\},\ m_{DWO}(A)=\sum_{\substack{X_1,X_2,\ldots,X_n\in S^{\theta}\\ \bigcap_{i=1}^{s} X_i=A}}\prod_{i=1}^{s} m_i(X_i)+W_m(A)\cdot m_{s\cap}(\Phi)+V_m(A)\cdot\sum_{z\in S_{\cap,r}^{non\Phi}} m_{s\cap}(z) \tag{6}$$

with the same restrictions on $W_m(\cdot)$ and $V_m(\cdot)$.

3. A Fusion Rule Class of Proportional Redistribution of Intersection Masses

For $A\in \left(S^{\theta}\setminus S_{\cap,r}^{non\Phi}\right)\setminus\{\Phi, I_t\}$ for two sources we have:

$$m_{CPRIM}(A)=m_{2\cap}(A)+f(A)\cdot\sum_{\substack{X,Y\in S^{\theta}\\ \{\Phi=X\cap Y \text{ and } A\subseteq M\}\\ \text{or } \{\Phi\neq X\cap Y\in S_{\cap,r}^{non\Phi} \text{ and } A\subseteq N\}}}\frac{m_1(X)m_2(Y)}{\sum_{z\subseteq M} f(z)}, \tag{7}$$

where $f(X)$ is a function directly proportional to X, $f: S^{\theta}\to[0,\infty]$. (8)

For example, $f(X)=m_{2\cap}(X)$, or (9)

$f(X)=card(X)$, or

$f(X)=\dfrac{card(X)}{card(M)}$ (ratio of cardinals), or

$f(X)=m_{2\cap}(X)+card(X)$, etc.;

and M is a subset of S^{θ}, for example: (10)

$M=\tau(X\cup Y)$, or

$M=(X\cup Y)$, or

M is a subset of $X\cup Y$, etc.,

where N is a subset of S^{θ}, for example: (11)

$N=X\cup Y$, or

N is a subset of $X\cup Y$, etc.

And

$$m_{CPRIM}(I_t) = m_{2\cap}(I_t) + \sum_{\substack{X,Y\in S^\theta \\ \left\{X\cap Y=\Phi \text{ and } (M=\Phi \text{ or } \sum_{z\subseteq M} f(z)=0)\right\}}} m_1(X)m_2(Y)\,. \qquad (12)$$

These formulas are easily extended for any $s \geq 2$ sources $m_1(\cdot), m_2(\cdot), ..., m_s(\cdot)$.

Let's denote, using the conjunctive rule:

$$m_{s\cap}(A) = \left(m_1 \oplus m_2 \oplus ... \oplus m_s\right)(A) \quad = \sum_{\substack{X_1,X_2,...,X_s\in S^\wedge\Theta \\ \bigcap_{i=1}^{s} X_i = A}} \prod_{i=1}^{s} m_i(x_i) \qquad (13)$$

$$m_{CPRIM}(A) = m_{s\cap}(A) + \mathrm{f(A)}\cdot \sum_{\substack{X_1,X_2,...,X_n\in S^\theta \\ \left\{\Phi=\bigcap_{i=1}^{s} X_i \text{ and } A\subseteq M\right\} \\ \text{or } \left\{\Phi\neq\bigcap_{i=1}^{s} X_i\in S_{\cap,r}^{non\Phi} \text{ and } A\subseteq N\right\}}} \frac{\prod_{i=1}^{s} m_i(X_i)}{\sum_{z\subseteq M} f(z) \neq 0} \qquad (14)$$

where $f(\cdot)$, M, and N are similar to the above where instead of $X \bigcup Y$ (for two sources) we take $X_1 \bigcup X_2 \bigcup ... \bigcup X_s$ (for s sources), and instead of $m_{2\cap}(X)$ for two sources we take $m_{s\cap}(X)$ for s sources.

4. Application and Comparison with other Fusion Rules.

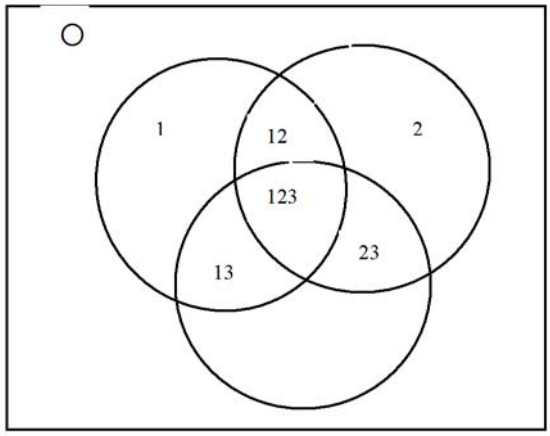
Let's consider the frame of discernment Θ = {A, B, C}, and two independent sources $m_1(.)$ and $m_2(.)$ that provide the following masses:

	A	B	C	A∪B∪C
$m_1(.)$	0.3	0.4	0.2	0.1
$m_2(.)$	0.5	0.2	0.1	0.2

Now, we apply the conjunctive rule and we get:

	A	B	C	A∪B∪C	A∩B	A∩C	B∩C
$m_{12\cap}(.)$	0.26	0.18	0.07	0.02	0.26	0.13	0.08

Suppose that all intersections are non-empty {this case is called: free DSm (Dezert-Smarandache) Model}. See below the Venn Diagram using the Smarandache codification [3]:



Applying DSm Classic rule, which is a generalization of classical conjunctive rule from the fusion space $(\Theta, \bigcup)$, called *power set*, when all hypotheses are supposed exclusive (i.e. all intersections are empty) to the fusion space $(\Theta, \bigcup, \bigcap)$, called *hyper-power set*, where hypotheses are not necessarily exclusive (i.e. there exist non-empty intersections), we just get:

	A	B	C	$A \bigcup B \bigcup C$	$A \cap B$	$A \cap C$	$B \cap C$
$m_{DSmC}(.)$	0.26	0.18	0.07	0.02	0.26	0.13	0.08

DSmC and the Conjunctive Rule have the same formula, but they work on different fusion spaces.

Inagaki rule was defined on the fusion space $(\Theta, \bigcup)$. In this case, since all intersections are empty, the total conflicting mass, which is $m_{12\cap}(A \cap B) + m_{12\cap}(A \cap C) + m_{12\cap}(B \cap C) = 0.26 + + 0.13 + 0.08 = 0.47$, and this is redistributed to the masses of A, B, C, and $A \bigcup B \bigcup C$ according to some weights w_1, w_2, w_3, and w_4 respectively, depending to each particular rule, where:
$0 \leq w_1, w_2, w_3, w_4 \leq 1$ and $w_1 + w_2 + w_3 + w_4 = 1$. Hence

	A	B	C	$A \bigcup B \bigcup C$
$m_{Inagaki}(.)$	$0.26+0.47w_1$	$0.18+0.47w_2$	$0.07+0.47w_3$	$0.02+0.47w_4$

Yet, Inagaki rule can also be straightly extended from the power set to the hyper-power set.

Suppose in DWO the user finds out that the hypothesis $B \cap C$ is not plausible, therefore $m_{12\cap}(B \cap C) = 0.08$ has to be transferred to the other non-empty elements: A, B, C, $A \bigcup B \bigcup C$, $A \cap B$, $A \cap C$, according to some weights v_1, v_2, v_3, v_4, v_5, and v_6 respectively, depending to the particular version of this rule is chosen, where:
$0 \leq v_1, v_2, v_3, v_4, v_5, v_6 \leq 1$ and $v_1 + v_2 + v_3 + v_4 + v_5 + v_6 = 1$. Hence

	A	B	C	$A \bigcup B \bigcup C$	$A \cap B$	$A \cap C$
$m_{DWO}(.)$	$0.26+0.08v_1$	$0.18+0.08v_2$	$0.07+0.08v_3$	$0.02+0.08v_4$	$0.26+0.08v_5$	$0.13+0.08v_6$

Now, since CPRIM is a particular case of DWO, but CPRIM is a class of fusion rules, let's consider a sub-particular case for example when the redistribution of $m_{12\cap}(B \cap C) = 0.08$ is done proportionally with respect to the DSm cardinals of B and C which are both equal to 4. DSm

cardinal of a set is equal to the number of disjoint parts included in that set upon the Venn Diagram (see it above).
Therefore 0.08 is split equally between B and C, and we get:

	A	B	C	$A \cup B \cup C$	$A \cap B$	$A \cap C$
$m_{CPRIMcard}(.)$	0.26	0.18+0.04=0.22	0.07+0.04=0.11	0.02	0.26	0.13

Applying one or another fusion rule is still debating today, and this depends on the hypotheses, on the sources, and on other information we receive.

5. Conclusion.
A generalization of Inagaki rule has been proposed in this paper, and also a new class of fusion rules, called **Class of Proportional Redistribution of Intersection Masses (CPRIM)**, which generates many interesting particular fusion rules in information fusion.

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α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM)

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Abstract.
In this paper we introduce a new procedure called α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM), which is as an alternative and extension of Saaty's Analytical Hierarchy Process (AHP). It works for any set of preferences that can be transformed into a system of homogeneous linear equations. A degree of consistency (and implicitly a degree of inconsistency) of a decision-making problem are defined. α-D MCDM is generalized to a set of preferences that can be transformed into a system of linear and/or non-linear homogeneous and/or non-homogeneous equations and/or inequalities.
Many consistent, weak inconsistent, and strong inconsistent examples are given.

Keywords:
Multi-Criteria Decision Making (MCDM), Analytical Hierarchy Process (AHP), α-Discounting Method, Fairness Principle, parameterize, paiwise comparison, n-wise comparison, consistent MCDM problem, weak or strong inconsistent MCDM problem

1. Introduction.

α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM) is an alternative and extension of Saaty's Analytical Hierarchy Process (AHP). It works not only for preferences that are pairwise comparisons of criteria as AHP does, but for preferences of any n-wise (with $n \geq 2$) comparisons of criteria that can be expressed as linear homogeneous equations.
The general idea of α-D MCDM is to assign null-null positive parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ to the coefficients in the right-hand side of each preference that diminish or increase them in order to transform the above linear homogeneous system of equations which has only the null-solution, into a system having
After finding the general solution of this system, the principles used to assign particular values to all parameters α's is the second important part of α-D, yet to be deeper investigated in the future.
In the current paper we herein propose the Fairness Principle, i.e. each coefficient should be discounted with the same percentage (we think this is fair: not making any favouritism or unfairness to any coefficient), but the reader can propose other principles.
For consistent decision-making problems with pairwise comparisons, α-Discounting Method together with the Fairness Principle give the same result as AHP.

But for weak inconsistent decision-making problem, α -Discounting together with the Fairness Principle give a different result from AHP.
α-Discounting/Fairness-Principle together give a justifiable result for strong inconsistent decision-making problems with two preferences and two criteria; but for more than two preferences with more than two criteria and the Fairness Principle has to be replaced by another principle of assigning numerical values to all parameters α's.

Since Saaty's AHP is not the topic of this paper, we only recall the main steps of applying this method, so the results of α-D MCDM and of AHP could be compared.
AHP works for only for pairwise comparisons of criteria, from which a square Preference Matrix, A (of size $n \times n$), is built. Then one computes the maximum eigenvalue λ_{max} of A and its corresponding eigenvector.
If λ_{max} is equal to the size of the square matrix, then the decision-making problem is consistent, and its corresponding normalized eigenvector (Perron-Frobenius vector) is the priority vector.
If If λ_{max} is strictly greater than the size of the square matrix, then the decision-making problem is inconsistent. One raise to the second power matrix A, and again the resulted matrix is raised to the second power, etc. obtaining the sequence of matrices A^2, A^4, A^8, …, etc. In each case, one computes the maximum eigenvalue and its associated normalized eigenvector, until the difference between two successive normalized eigenvectors is smaller than a given threshold. The last such normalized eigenvector will be the priority vector.
Saaty defined the Consistency Index as:

$$\mathrm{CI(A)} = \frac{\lambda_{max}(A) - n}{n-1},$$

where n is the size of the square matrix A.

2. α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM).

2.1. Description of α-D MCDM.

The general idea of this paper is to discount the coefficients of an inconsistent problem to some percentages in order to transform it into a consistent problem.

Let the Set of Criteria be $C = \{C_1, C_2, \ldots, C_n\}$, with $n \geq 2$,
and the Set of Preferences be $P = \{P_1, P_2, \ldots, P_m\}$, with $m \geq 1$.
Each preference P_i is a linear homogeneous equation of the above criteria $C_1, C_2, \ldots, C_n$:

$$P_i = f(C_1, C_2, \ldots, C_n).$$

We need to construct a basic belief assignment (bba):

$$m: C \rightarrow [0, 1]$$

such that $m(C_i) = x_i$, with $0 \leq x_i \leq 1$, and

$$\sum_{i=1}^{n} m(x_i) = 1.$$

We need to find all variables x_i in accordance with the set of preferences P.
Thus, we get an m×n linear homogeneous system of equations whose associated matrix is

$$A = (a_{ij}),\ 1 \le i \le m \text{ and } 1 \le j \le n.$$

In order for this system to have non-null solutions, the rank of the matrix A should be strictly less than n.

2.2. Classification of Linear Decision-Making Problems.

a) We say that a **linear decision-making problem is consistent** if, by any substitution of a variable x_i from an equation into another equation, we get a result in agreement with all equations.

b) We say that a **linear decision-making problem is weakly inconsistent** if, by at least one substitution of a variable x_i from an equation into another equation, we get a result in disagreement with at least another equation in the following ways:

(WD1) $\left\{\begin{array}{l} x_i = k_1 \cdot x_j, k > 1; \\ x_i = k_2 \cdot x_j, k_2 > 1, k_2 \neq k_1 \end{array}\right\}$

or

(WD2) $\left\{\begin{array}{l} x_i = k_1 \cdot x_j, 0 < k < 1; \\ x_i = k_2 \cdot x_j, 0 < k_2 < 1, k_2 \neq k_1 \end{array}\right\}$

or

(WD3) $\{x_i = k \cdot x_i, k \neq 1\}$

(WD1)-(WD3) are weak disagreements, in the sense that for example a variable x > y always, but with different ratios (for example: x=3y and x=5y).
All disagreements in this case should be like (WD1)-(WD3).

c) We say that a **linear decision-making problem is strongly inconsistent** if, by at least one substitution of a variable x_i from an equation into another equation, we get a result in disagreement with at least another equation in the following way:

(SD4) $\left\{\begin{array}{l} x_i = k_1 \cdot x_j; \\ x_i = k_2 \cdot x_j, \end{array}\right\}$ with $0 < k_1 < 1 < k_2$ or $0 < k_2 < 1 < k_1$ (i.e. from one equation one gets $x_i < x_j$ while from the other equation one gets the opposite inequality: $x_j < x_i$).

At least one inconsistency like (SD4) should exist, no matter if other types of inconsistencies like (WD1)-(WD3) may occur or not.

Compute the determinant of A.

a) If det(A)=0, the decision problem is consistent, since the system of equations is dependent.

It is not necessarily to parameterize the system. {In the case we have parameterized, we can use the Fairness Principle – i.e. setting all parameters equal $\alpha_1 = \alpha_2 = \ldots = \alpha_p = \alpha > 0$}.
Solve this system; find its general solution.
Replace the parameters and secondary variables, getting a particular solution.
Normalize this particular solution (dividing each component by the sum of all components).
Wet get the priority vector (whose sum of its components should be 1).

b) If $\det(A) \neq 0$, the decision problem is inconsistent, since the homogeneous linear system has only the null-solution.

b1) If the inconsistency is weak, then parameterize the right-hand side coefficients, and denote the system matrix $A(\alpha)$.
Compute $\det(A(\alpha)) = 0$ in order to get the parametric equation.
If the Fairness Principle is used, set all parameters equal, and solve for $\alpha > 0$.
Replace α in $A(\alpha)$ and solve the resulting dependent homogeneous linear system.
Similarly as in a), replace each secondary variable by 1, and normalize the particular solution in order to get the priority vector.

b2) If the inconsistency is strong, the Fairness Principle may not work properly. Another approachable principle might by designed.
Or, get more information and revise the strong inconsistencies of the decision-making problem.

2.3. Comparison between AHP and α-D MCDM:

a) α-D MCDM's general solution includes all particular solutions, that of AHP as well;
b) α-D MCDM uses all kind of comparisons between criteria, not only paiwise comparisons;
c) for consistent problems, AHP and α-D MCDM/Fairness-Principle give the same result;
d) for large inputs, in α-D MCDM we can put the equations under the form of a matrix (depending on some parameters alphas), and then compute the determinant of the matrix which should be zero; after that, solve the system (all can be done on computer using math software); the software such as MATHEMATICA and APPLE for example can do these determinants and calculate the solutions of this linear system;
e) α-D MCDM can work for larger classes of preferences, i.e. preferences that can be transformed in homogeneous linear equations, or in non-linear equations and/or inequalities – see more below.

2.4. Generalization of α-D MCDM.

Let each preference be expressed as a linear or non-linear equation or inequality. All preferences together will form a system of linear/non-linear equations/inequalities, or a mixed system of equations and inequalities.
Solve this system, looking for a strictly positive solution (i.e. all unknowns $x_i > 0$). Then normalize the solution vector.
If there are more such numerical solutions, do a discussion: analyze the normalized solution vector in each case.
If there is a general solution, extract the best particular solution.

If there is no strictly positive solution, parameterize the coefficients of the system, find the parametric equation, and look for some principle o apply in order to find the numerical values of the parameters α 's. A discussion might also be involved. We may get undetermined solutions.

3. Degrees of Consistency and Inconsistency in α -D MCDM/Fairness-Principle.

For α -D MCDM/Fairness-Principle in consistent and weak consistent decision-making problems, we have the followings:

a) If $0 < \alpha < 1$, then α is the **degree of consistency** of the decision-making problem, and $\beta = 1-\alpha$ is the **degree of inconsistency** of the decision-making problem.
b) If $\alpha > 1$, then $1/\alpha$ is the **degree of consistency** of the decision-making problem, and $\beta = 1-1/\alpha$ is the **degree of inconsistency** of the decision-making problem.

4. Principles of α-D MCDM (Second Part).

a) In applications, for the second part of α -D Method, the Fairness Principle can be replaced by other principles.
Expert's Opinion. For example, if we have information that a preference's coefficient should be discounted twice more than another coefficient (due to an expert's opinion), and another preference's coefficient should be discounted a third of another one, then appropriately we set for example: $\alpha_1 = 2\alpha_2$ and respectively $\alpha_3 = (1/3)\alpha_4$, etc. in the parametric equation.
b) For α -D/Fairness-Principle or Expert's Opinion.
Another idea herein is to set a **threshold of consistency** t_c (or implicitly **a threshold of inconsistency t_i**). Then, if the degree of consistency is smaller than a required t_c, the Fairness Principle or Expert's Opinion (whichever was used) should be discharged, and another principle of finding all parameters α 's should be designed; and similarly if the degree of inconsistency is bigger than t_i.
c) One may measure the system's accuracy (or error) for the case when all m preferences can be transformed into equations; for example, preference P_i is transformed into an equation $f_i(x_1, x_2, \ldots, x_n)=0$; then we need to find the unknowns $x_1, x_2, \ldots, x_n$ such that:

$$e(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{m} |f_i(x_1, x_2, ..., x_n)| \text{ is minimum,}$$

where "e" means error.
Calculus theory (partial derivatives) can be used to find the minimum (if this does exist) of a function of n variables, $e(x_1, x_2, \ldots, x_n)$, with e: $R_+^n \rightarrow R_+$.
For consistent decision-making problems the system's accuracy/error is zero, so we get the exact result.
We prove this through the fact that the normalized priority vector $[a_1\ a_2\ \ldots\ a_n]$, where $x_i=a_i > 0$ for all i, is a particular solution of the system $f_i(x_1, x_2, \ldots, x_n)=0$ for i=1, 2, …, m; therefore:

$$\sum_{i=1}^{m} |f_i(a_1, a_2, ..., a_n)| = \sum_{i=1}^{m} |0| = 0.$$

But, for inconsistent decision-making problems we find approximations for the variables.

5. **Extension of α-D MCDM (Non-Linear α-D MCDM).**

It is not difficult to generalize the α-D MCDM for the case when the preferences are non-linear homogeneous (or even non-homogeneous) equations.
This non-linear system of preferences has to be dependent (meaning that its general solution – its main variables - should depend upon at least one secondary variable).
If the system is not dependent, we can parameterize it in the same way. Then, again, in the second part of this Non-Linear α-D MCDM we assign some values to each of the secondary variables (depending on extra-information we might receive), and we also need to design a principle which will help us to find the numerical values for all parameters. We get a particular solution (such extracted from the general solution), which normalized will produce our priority vector.
Yet, the Non-Linear α-D MCDM is more complicated, and depends on each non-linear decision-making problem.

Let's see some examples.

6. Consistent Example 1.

6.4. Let the Set of Preferences be: $\{C1, C2, C3\}$,
and The Set of Criteria be:

1. $C1$ is 4 times as important as $C2$.
2. $C2$ is 3 times as important as $C3$.
3. $C3$ is one twelfth as important as $C1$.

Let $m(C1) = x$, $m(C2) = y$, $m(C3) = z$.

The linear homogeneous system associated to this decision-making problem is:

$$\begin{cases} x = 4y \\ y = 3z \\ z = \dfrac{x}{12} \end{cases}$$

whose associated matrix A_1 is:

$$\begin{pmatrix} 1 & -4 & 0 \\ 0 & 1 & -3 \\ -1/12 & 0 & 1 \end{pmatrix}$$, whence $\det(A_1) = 0$, so the DM problem is consistent.

Solving this homogeneous linear system we get its general solution that we set as a vector [12z 3z z], where z can be any real number (z is considered a secondary variable, while x=12z and y=3z are main variables).

Replacing z=1, the vector becomes [12 3 1], and then normalizing (dividing by 12+3+1=16 each vector component) we get the priority vector: [12/16 3/16 1/16], so the preference will be on C1.

6.5. Using AHP, we get the same result.

The preference matrix is:

$$\begin{pmatrix} 1 & 4 & 12 \\ 1/4 & 1 & 3 \\ 1/12 & 1/3 & 1 \end{pmatrix}$$

whose maximum eigenvalue is $\lambda_{max} = 3$ and its corresponding normalized eigenvector (Perron-Frobenius vector) is [12/16 3/16 1/16].

6.6. Using Mathematica 7.0 Software:

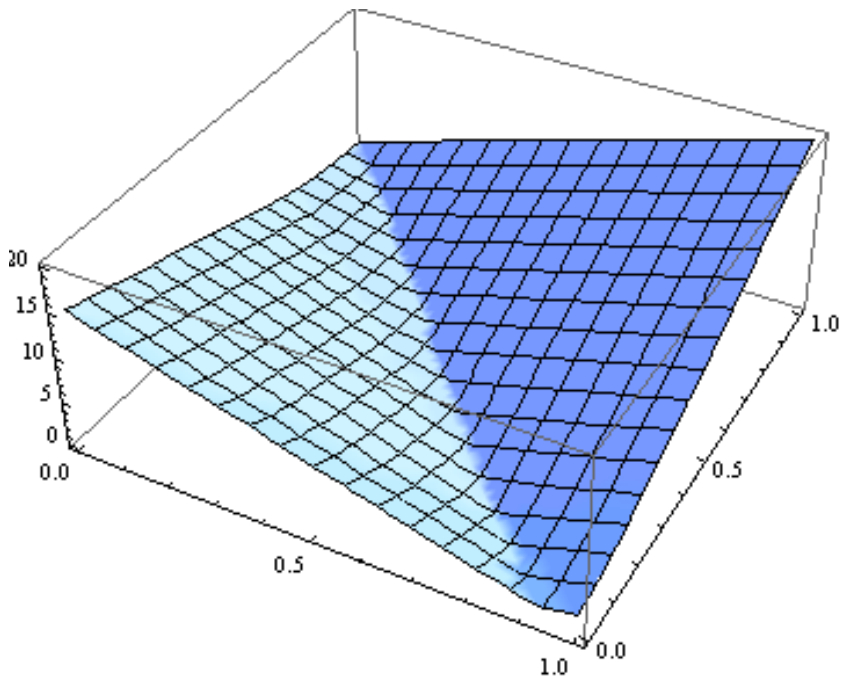
Using MATHEMTICA 7.0 software, we graph the function:

$$h(x,y) = |x-4y|+|3x+4y-3|+|13x+12y-12|, \text{ with } x,y \in [0,1],$$

which represents the consistent decision-making problem's associated system:

$$x/y=4,\ y/z=3,\ x/z=12, \text{ and } x+y+z=1,\ x>0,\ y>0,\ z>0.$$

In[1]:=
Plot3D[Abs[x-4y]+Abs[3x+4y-3]+Abs[13x+12y-12],{x,0,1},{y,0,1}]



The minimum of this function is zero, and occurs for x=12/16, y=3/16.

If we consider the original function of three variables associated with h(x,y) we have:

$$H(x,y,z) = |x-4y|+|y-3z|+|x-12z|,\ x+y+z=1,\ \text{with } x,y,z \in [0,1],$$

we also get the minimum of H(x,y,z) being zero, which occurs for x=12/16, y=3/16, z=1/16.

7. Weak Inconsistent Examples where AHP Doesn't Work.

The Set of Preferences is: $\{C1, C2, C3\}$.

7.4. Weak Inconsistent Example 2.

7.4.1. α-D MCDM method.

The Set of Criteria is:

1. $C1$ is 2 times as important as $C2$ and 3 times as important as $C3$ put together.
2. $C2$ is half as important as $C1$.
3. $C3$ is one third as important as $C1$.

Let $m(C1) = x$, $m(C2) = y$, $m(C3) = z$;

$$\begin{cases} x = 2y + 3z \\ y = \dfrac{x}{2} \\ z = \dfrac{x}{3} \end{cases}$$

AHP cannot be applied on this example because of the form of the first preference, which is not a pairwise comparison.

If we solve this homogeneous linear system of equations as it is we get x=y=z=0, since its associated matrix is:

$$\begin{pmatrix} 1 & -2 & -3 \\ -1/2 & 1 & 0 \\ -1/3 & 0 & 1 \end{pmatrix} = -1 \neq 0$$

but the null solution is not acceptable since the sum x+y+z has to be 1.
Let's parameterise each right-hand side coefficient and get the general solution of the above system:

$$\begin{cases} x = 2\alpha_1 y + 3\alpha_2 z & (1) \\ y = \dfrac{\alpha_3}{2} x & (2) \\ z = \dfrac{\alpha_4}{3} x & (3) \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$.

Replacing (2) and (3) in (1) we get

$$x = 2\alpha_1 \left(\frac{\alpha_3}{2} x \right) + 3\alpha_2 \left(\frac{\alpha_4}{3} x \right)$$

$$1 \cdot x = \left(\alpha_1 \alpha_3 + \alpha_2 \alpha_4 \right) \cdot x$$

whence

$$\alpha_1 \alpha_3 + \alpha_2 \alpha_4 = 1 \qquad \text{(parametric equation)} \qquad (4)$$

The general solution of the system is:

$$\begin{cases} y = \dfrac{\alpha_3}{2} x \\ z = \dfrac{\alpha_4}{3} x \end{cases}$$

whence the priority vector: $\left[x \quad \frac{\alpha_3}{2} x \quad \frac{\alpha_4}{3} x \right] \to \left[1 \quad \frac{\alpha_3}{2} \quad \frac{\alpha_4}{3} \right]$.

Fairness Principle: discount all coefficients with the same percentage: so, replace $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha > 0$ in (4) we get $\alpha^2 + \alpha^2 = 1$, whence $\alpha = \dfrac{\sqrt{2}}{2}$.

Priority vector becomes: $\left[1 \quad \frac{\sqrt{2}}{4} \quad \frac{\sqrt{2}}{6} \right]$

and normalizing it:

$$\begin{matrix} \left[0.62923 \right. & 0.22246 & \left. 0.14831 \right] \\ C1 & C2 & C3 \\ x & y & z \end{matrix}$$

Preference will be on C1, the largest vector component.

Let's verify it:

$\dfrac{x}{y} \cong 0.35354$ instead of 0.50, i.e. $\dfrac{\sqrt{2}}{2} = 70.71\%$ of the original.

$\dfrac{z}{x} \cong 0.23570$ instead of 0.33333, i.e. 70.71% of the original.

$x \cong 1.41421y + 2.12132z$ instead of $2y + 3z$, i.e. 70.71% of 2 respectively 70.71% of 3.

So, it was a fair discount for each coefficient.

7.4.2. Using Mathematica 7.0 Software:

Using MATHEMTICA 7.0 software, we graph the function:

$$g(x,y) = |4x-y-3|+|x-2y|+|4x+3y-3|, \text{ with } x,y \in [0,1],$$

which represents the weak inconsistent decision-making problem's associated system:

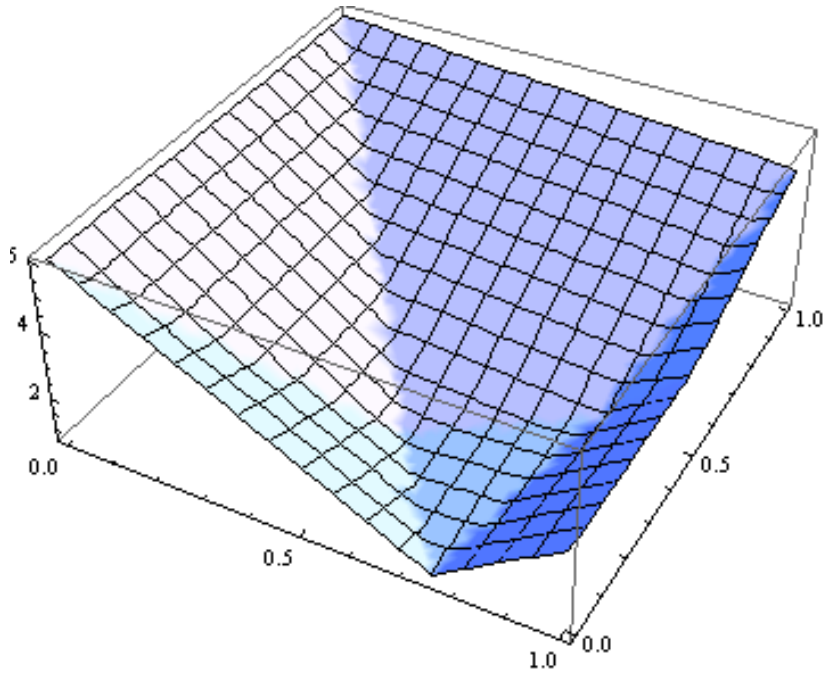
$$x-2y-3z=0,\ x-2y=0,\ x-3z=0, \text{ and } x+y+z=1,\ x>0,\ y>0,\ z>0.$$

by solving $z=1-x-y$ and replacing it in

$$G(x,y,z)= |x-2y-3z|+|x-2y|+|x-3z| \text{ with } x>0,\ y>0,\ z>0,$$

In[2]:=

```
Plot3D[Abs[4x-y-3]+Abs[x-2y]+Abs[4x+3y-3],{x,0,1},{y,0,1}]
```



Then find the minimum of g(x,y) if any:

In[3]:=

```
FindMinValue[{Abs[4x-y-3]+Abs[x-2y]+Abs[4x+3y-3],x+y≤1,x>0,y>0},{x,y}]
```

The following result is returned:

Out[3]:= 0.841235.

FindMinValue::eit: The algorithm does not converge to the tolerance of 4.806217383937354`*^-6 in 500 iterations. The best estimated solution, with feasibility residual, KKT residual, or complementary residual of {0.0799888,0.137702,0.0270028}, is returned.

7.1.2. Matrix Method of using α -Discounting.

The determinant of the homogeneous linear system (1), (2), (3) is:

$$\begin{vmatrix} 1 & -2\alpha_1 & -3\alpha_2 \\ -\frac{1}{2}\alpha_3 & 1 & 0 \\ -\frac{1}{3}\alpha_4 & 0 & 1 \end{vmatrix} = (1+0+0)-(\alpha_2\alpha_4+\alpha_1\alpha_3)=0$$

or

$\alpha_1\alpha_3+\alpha_2\alpha_4=1$ (parametric equation).

The determinant has to be zero in order for the system to have non-null solutions.

The rank of the matrix is 2.

So, we find two variables, for example it is easier to solve for y and z from the last two equations, in terms of x:

$$\begin{cases} y=\frac{1}{2}\alpha_3 x \\ z=\frac{1}{3}\alpha_4 x \end{cases}$$

and the procedure follows the same steps as in the previous one.

Let's change Example 1 in order to study various situations.

7.2. **Weak Inconsistent Example 3**, which is more weakly inconsistent than Example 2.

1. Same as in Example 1.
2. $C2$ is 4 times as important as $C1$
3. Same as in Example 1.

$$\begin{cases} x=2\alpha_1 y+3\alpha_2 z \\ y=4\alpha_3 x \\ z=\frac{\alpha_4}{3}x \end{cases}$$

$$x=2\alpha_1(4\alpha_3 x)+3\alpha_2\left(\frac{\alpha_4}{3}\right)x$$

$$1\cdot x=(8\alpha_1\alpha_3+\alpha_2\alpha_4)$$

$8\alpha_1\alpha_3+\alpha_2\alpha_4=1$ (parametric equation)

$\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha>0.$

$$9\alpha^2 = 1 \Rightarrow \alpha = \frac{1}{3}$$

$$\left[\begin{array}{ccc} x & 4\alpha_3 x & \frac{\alpha_4}{3}x \end{array}\right] \to \left[\begin{array}{ccc} 1 & 4\alpha_3 & \frac{\alpha_4}{3} \end{array}\right]$$

$$\left[\begin{array}{ccc} 1 & \frac{4}{3} & \frac{1}{9} \end{array}\right] = \left[\begin{array}{ccc} \frac{9}{9} & \frac{12}{9} & \frac{1}{9} \end{array}\right];$$

normalized: $\left[\begin{array}{ccc} \frac{9}{22} & \frac{12}{22} & \frac{1}{22} \end{array}\right]$.

$\frac{y}{x} = 1.333$ instead of 4;

$\frac{z}{x} = 0.111$ instead of 0.3333;

$x = 0.667y + 1 \cdot z$ instead of $2y + 3z$.

Each coefficient was reduced at $\frac{1}{3}(= 33.33\%)$.

The bigger is the inconsistency $(\beta \to 1)$, the bigger is the discounting $(\alpha \to 0)$.

7.3. **Weak Inconsistent Example 4**, which is even more inconsistent than Example 3.

1. Same as in Example 1.
2. Same as in Example 2.
3. $C3$ is 5 times as important as $C1$.

$$\begin{cases} x = 2\alpha_1 y + 3\alpha_2 z \\ y = 4\alpha_3 x \\ z = 5\alpha_4 x \end{cases}$$

$$x = 2\alpha_1 (4\alpha_3 x) + 3\alpha_2 (5\alpha_4 x)$$

$$1 \cdot x = (8\alpha_1\alpha_3 + 15\alpha_2\alpha_4)x$$

whence $8\alpha_1\alpha_3 + 15\alpha_2\alpha_4 = 1$ (parametric equation).

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha > 0,\ 23\alpha^2 = 1,\ \alpha = \frac{\sqrt{23}}{23}$$

$$\left[\begin{array}{ccc} 1 & 4\alpha_3 & 5\alpha_4 \end{array}\right] \to \left[\begin{array}{ccc} 1 & \frac{4\sqrt{23}}{23} & \frac{5\sqrt{23}}{23} \end{array}\right]$$

Normalized: $\left[\begin{array}{ccc} 0.34763 & 0.28994 & 0.36243 \end{array}\right]$

$\frac{y}{x} \cong 0.83405$ instead of 4, i.e. reduced at $\frac{\sqrt{23}}{23} = 20.85\%$

$\frac{z}{x} \cong 1.04257$ instead of 5

$x \cong 0.41703y + 0.62554 \cdot z$ instead of $2x + 3y$.

Each coefficient was reduced at $\alpha = \frac{\sqrt{23}}{23} \cong 20.85\%$.

7.4. **Consistent Example 5.**

When we get $\alpha = 1$, we have a consistent problem.

Suppose the preferences:

1. Same as in Example 1
2. $C2$ is one fourth as important as $C1$
3. $C2$ is one sixth as important as $C3$.

The system is:

$$\begin{cases} x = 2y + 3z \\ y = \frac{x}{4} \\ z = \frac{x}{6} \end{cases}$$

7.4.1. First Method of Solving this System.

Replacing the second and third equations of this system into the first, we get:

$$x = 2\left(\frac{x}{4}\right) + 3\left(\frac{x}{6}\right) = \frac{x}{2} + \frac{x}{2} = x,$$

which is an identity (so, no contradiction).

General solution:

$$\left[x \quad \frac{x}{4} \quad \frac{x}{6} \right]$$

Priority vector:

$$\left[1 \quad \frac{1}{4} \quad \frac{1}{6} \right]$$

Normalized is:

$$\left[\frac{12}{17} \quad \frac{3}{17} \quad \frac{2}{17} \right]$$

7.4.2. Second Method of Solving this System.

Let's parameterize:

$$\begin{cases} x = 2\alpha_1 y + 3\alpha_2 z \\ y = \frac{\alpha_3}{4} x \\ z = \frac{\alpha_4}{6} x \end{cases}$$

Replacing the last two equations into the first we get:

$$x = 2\alpha_1\left(\frac{\alpha_3}{4}x\right) + 3\alpha_2\left(\frac{\alpha_4}{6}x\right) = \frac{\alpha_1\alpha_3}{2}x + \frac{\alpha_2\alpha_4}{2}x$$

$$1 \cdot x = \frac{\alpha_1\alpha_3 + \alpha_2\alpha_4}{2} \cdot x,$$

whence $1 = \frac{\alpha_1\alpha_3 + \alpha_2\alpha_4}{2}$ or $\alpha_1\alpha_3 + \alpha_2\alpha_4 = 2$.

Consider the fairness principle: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha > 0$, then $2\alpha^2 = 2$, $\alpha = \pm 1$, but we take only the positive value $\alpha = 1$ (as expected for a consistent problem).

Let's check:

$$\frac{y}{x} = \frac{\frac{3}{17}}{\frac{12}{17}} = \frac{1}{4}, \text{ exactly as in the original system;}$$

$$\frac{z}{x} = \frac{\frac{2}{17}}{\frac{12}{17}} = \frac{1}{6}, \text{ exactly as in the original system;}$$

$$x = 2y + 3z \text{ since } x = 2\left(\frac{x}{4}\right) + 3\left(\frac{x}{6}\right);$$

hence all coefficients were left at $\alpha = 1$ (=100%) of the original ones.

No discount was needed.

7.5. General Example 6.

Let's consider the general case:

$$\begin{cases} x = a_1 y + a_2 z \\ y = a_3 x \\ z = a_4 x \end{cases}$$

where $a_1, a_2, a_3, a_4 > 0$

Let's parameterize:

$$\begin{cases} x = a_1\alpha_1 y + a_2\alpha_2 z \\ y = a_3\alpha_3 x \\ z = a_4\alpha_4 x \end{cases}$$

with $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$.

Replacing the second and third equations into the first, we get:

$$x = a_1\alpha_1\left(a_3\alpha_3 x\right) + a_2\alpha_2\left(a_4\alpha_4 x\right)$$

$$x = a_1 a_3\alpha_1\alpha_3 x + a_2 a_4\alpha_2\alpha_4 x$$

whence

$$a_1 a_3\alpha_1\alpha_3 + a_2 a_4\alpha_2\alpha_4 = 1 \text{ (parametric equation)}$$

The general solution of the system is: $\left(x,\ a_3\alpha_3 x,\ a_4\alpha_4 x\right)$

The priority vector is $\left[1 \;\; a_3\alpha_3 \;\; a_4\alpha_4\right]$.

Consider the fairness principle: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha > 0$

we get:

$$\alpha^2 = \frac{1}{a_1a_3 + a_2a_4},$$

so,

$$\alpha = \frac{1}{\sqrt{a_1a_3 + a_2a_4}}$$

i) If $\alpha \in [0,1]$, then α is the degree of consistency of the problem, while $\beta = 1-\alpha$ is the degree of the inconsistency of the problem.

ii) If $\alpha > 1$, then $\frac{1}{\alpha}$ is the degree of consistency, while $\beta = 1 - \frac{1}{\alpha}$ is the degree of inconsistency.

When the degree of consistency $\to 0$, the degree of inconsistency $\to 1$, and reciprocally.

Discussion of the General Example 6.

Suppose the coefficients a_1, a_2, a_3, a_4 become big such that $a_1a_3 + a_2a_4 \to \infty$, then $\alpha \to 0$, and $\beta \to 1$.

Particular Example 7.

Let's see a particular case when a_1, a_2, a_3, a_4 make $a_1a_3 + a_2a_4$ big:

$$a_1 = 50, \;\; a_2 = 20, \;\; a_3 = 100, \;\; a_4 = 250,$$

then $\alpha = \frac{1}{\sqrt{50 \cdot 100 + 20 \cdot 250}} = \frac{1}{\sqrt{10000}} = \frac{1}{100} = 0.01$ = degree of consistency,

whence $\beta = 0.99$ degree of inconsistency.

The priority vector for Particular Example 7 is $\left[1 \;\; 100(0.01) \;\; 250(0.01)\right] = \left[1 \;\; 1 \;\; 2.5\right]$ which normalized is:

$\left[\frac{2}{9} \;\; \frac{2}{9} \;\; \frac{5}{9}\right]$.

Particular Example 8.

Another case when a_1, a_2, a_3, a_4 make the expression $a_1a_3 + a_2a_4$ a tiny positive number:

$a_1 = 0.02, \;\; a_2 = 0.05, \;\; a_3 = 0.03, \;\; a_4 = 0.02$, then

$$\alpha = \frac{1}{\sqrt{0.02 \cdot (0.03) + 0.05 \cdot (0.02)}} = \frac{1}{0.04} = 25 > 1.$$

Then $\frac{1}{\alpha} = \frac{1}{25} = 0.04$ is the degree of consistency of the problem, and 0.96 the degree of inconsistency.

The priority vector for example 5.2 is $\left[1 \;\; a_3\alpha \;\; a_4\alpha\right] = \left[1 \;\; 0.03(25) \;\; 0.02(25)\right] = \left[1 \;\; 0.75 \;\; 0.50\right]$ which normalized is $\left[\frac{4}{9} \;\; \frac{3}{9} \;\; \frac{3}{9}\right]$.

Let's verify:

$\frac{y}{x}=\frac{3}{9}\div\frac{4}{9}=0.75$ instead of 0.03, i.e. $\alpha=25$ times larger (or 2500%);

$\frac{z}{x}=\frac{2}{9}\div\frac{4}{9}=0.50$ instead of 0.02, i.e. 25 larger;

$x=0.50y+1.25z$ instead of $x=0.02y+0.05z$ (0.50 is 25 times larger than 0.02, and 1.25 is 25 times larger than 0.05) because $\frac{4}{9}=0.50\left(\frac{3}{9}\right)+1.25\left(\frac{2}{9}\right)$.

8.1. Jean Dezert's Weak Inconsistent Example 9.

Let $\alpha_1,\alpha_2,\alpha_3>0$ be the parameters. Then:

$$\left.\begin{cases}(5)\quad \frac{y}{x}=3\alpha_1\\ (6)\quad \frac{x}{z}=4\alpha_2\\ (7)\quad \frac{y}{z}=5\alpha_3\end{cases}\right\}\Rightarrow \frac{y}{x}\cdot\frac{x}{z}=(3\alpha_1)\cdot(4\alpha_2)\Rightarrow\frac{y}{z}=12\alpha_1\alpha_2$$

In order for $\frac{y}{z}=12\alpha_1\alpha_2$ to be consistent with $\frac{y}{z}=5\alpha_3$ we need to have $12\alpha_1\alpha_2=5\alpha_3$ or

$$2.4\alpha_1\alpha_2=\alpha_3 \quad \text{(Parametric Equation)} \qquad (8)$$

Solving this system:

$$\begin{cases}\frac{y}{x}=3\alpha_1\Rightarrow y=3\alpha_1\cdot x\\ \frac{x}{z}=4\alpha_2\Rightarrow x=4\alpha_2\cdot z\\ \frac{y}{z}=5\alpha_3\Rightarrow y=12\alpha_1\alpha_2 z\end{cases}$$

we get the general solution:

$$\left[4\alpha_2 z \quad 5(2.4\alpha_1\alpha_2)z \quad z\right]$$

$$\left[4\alpha_2 z \quad 12\alpha_1\alpha_2 z \quad z\right]$$

General normalized priority vector is:

$$\left[\frac{4\alpha_2}{4\alpha_2+12\alpha_1\alpha_2+1} \quad \frac{12\alpha_1\alpha_2}{4\alpha_2+12\alpha_1\alpha_2+1} \quad \frac{1}{4\alpha_2+12\alpha_1\alpha_2+1}\right]$$

where $\alpha_1,\alpha_2>0$; ($\alpha_3=2.4\alpha_1\alpha_2$).

Which α_1 and α_2 give the best result? How to measure it? This is the greatest challenge!

α-Discounting Method includes all solutions (all possible priority vectors which make the matrix consistent).

Because we have to be consistent with all proportions (i.e. using the Fairness Principle of finding the parameters' numerical values), there should be the same discounting of all three proportions (5), (6), and (7), whence

$$\alpha_1 = \alpha_2 = \alpha_3 > 0 \qquad (9)$$

The parametric equation (8) becomes $2.4\alpha_1^2 = \alpha_1$ or $2.4\alpha_1^2 - \alpha_1 = 0$, $\alpha_1\left(2.4\alpha_1 - 1\right) = 0$, whence $\alpha_1 = 0$ or $\alpha_1 = \frac{1}{2.4} = \frac{5}{12}$.

$\alpha_1 = 0$ is not good, contradicting (9).

Our system becomes now:

$$\begin{cases} \frac{y}{x} = 3 \cdot \frac{5}{12} = \frac{15}{12} & (10) \\ \frac{x}{z} = 4 \cdot \frac{5}{12} = \frac{20}{12} & (11) \\ \frac{y}{z} = 5 \cdot \frac{5}{12} = \frac{25}{12} & (12) \end{cases}$$

We see that (10) and (11) together give

$$\frac{y}{x} \cdot \frac{x}{z} = \frac{15}{12} \cdot \frac{20}{12} \text{ or } \frac{y}{z} = \frac{25}{12},$$

so, they are now consistent with (12).

From (11) we get $x = \frac{20}{12} z$ and from (12) we get $y = \frac{25}{12} z$.

The priority vector is:

$$\left[\frac{20}{12} z \quad \frac{25}{12} z \quad 1z\right]$$

which is normalized to:

$$\frac{\frac{20}{12}}{\frac{20}{12} + \frac{25}{12} + 1} = \frac{\frac{20}{12}}{\frac{20}{12} + \frac{25}{12} + \frac{12}{12}} = \frac{20}{57}, \quad \frac{\frac{25}{12}}{\frac{57}{12}} = \frac{25}{57}, \quad \frac{1}{\frac{57}{12}} = \frac{12}{57}, \text{ i.e.}$$

$$\begin{matrix} C_1 & C_2 & C_3 \end{matrix}$$

$$\left[\frac{20}{57} \quad \frac{25}{57} \quad \frac{12}{57}\right]^T \qquad (13)$$

$$\begin{matrix} C_1 & C_2 & C_3 \\ \cong [0.3509 & 0.4386 & 0.2105]^T \end{matrix}$$

the highest priority

Let's study the result:

$$\begin{matrix} C_1 & C_2 & C_3 \\ \left[\frac{20}{57} & \frac{25}{57} & \frac{12}{57}\right]^T \\ x & y & z \end{matrix}$$

Ratios:	Percentage of Discounting:
$\frac{y}{x}=\frac{\frac{25}{57}}{\frac{20}{57}}=\frac{25}{20}=1.25$ instead of 3;	$\frac{\frac{25}{20}}{3}=\frac{5}{12}=\alpha_1=41.\bar{6}\%$
$\frac{x}{z}=\frac{\frac{20}{57}}{\frac{12}{57}}=\frac{20}{12}=\frac{5}{3}=1.\bar{6}$ instead of 4;	$\frac{\frac{20}{12}}{4}=\frac{5}{12}=\alpha_1=41.\bar{6}\%$
$\frac{y}{z}=\frac{\frac{25}{57}}{\frac{12}{57}}=\frac{25}{12}=2.08\bar{3}$ instead of 5;	$\frac{\frac{25}{12}}{5}=\frac{5}{12}=\alpha_1=41.\bar{6}\%$

Hence all original proportions, which were respectively equal to 3, 4, and 5 in the problem, were reduced by multiplication with the same factor $\alpha_1=\frac{5}{12}$, i.e. by getting $41.\bar{6}\%$ of each of them.

So, it was fair to reduce each factor to the same percentage $41.\bar{6}\%$ of itself. But this is not the case in Saaty's method: its normalized priority vector is

$$\begin{matrix} C_1 & C_2 & C_3 \\ [0.2797 & 0.6267 & 0.0936]^T, \\ x & y & z \end{matrix}$$

where:

Ratios:	Percentage of Discounting:
$\frac{y}{x}=\frac{0.6267}{02797}\cong 2.2406$ instead of 3;	$\frac{2.2406}{3}\cong 74.6867\%$

$\frac{x}{z}=\frac{0.2797}{0.0936}\cong 2.9882$ instead of 4; $\frac{2.9882}{4}\cong 74.7050\%$

$\frac{y}{z}=\frac{06267}{0.0936}\cong 6.6955$ instead of 5; $\frac{6.6955}{5}\cong 133.9100\%$

Why, for example, the first proportion, which was equal to 3, was discounted to 74.6867% of it, while the second proportion, which was equal to 4, was discounted to another percentage (although close) 74.7050% of it?

Even more dough we have for the third proportion's coefficient, which was equal to 5, but was increased to 133.9100% of it, while the previous two proportions were decreased; what is the justification for these?

That's why we think our α-D/Fairness-Principle is better justified.

We can solve this same problem using matrices. (5), (6), (7) can be written in another way to form a linear parameterized homogeneous linear system:

$$\begin{cases} 3\alpha_1 x - y = 0 \\ x - 4\alpha_2 z = 0 \\ y - 5\alpha_3 z = 0 \end{cases} \quad (14)$$

Whose associated matrix is:

$$P_1=\begin{bmatrix} 3\alpha_1 & -1 & 0 \\ 1 & 0 & -4\alpha_2 \\ 0 & 1 & -5\alpha_3 \end{bmatrix} \quad (15)$$

a) If $\det(P_1)\neq 0$ then the system (10) has only the null solution $x=y=z=0$.

b) Therefore, we need to have $\det(P_1)=0$, or $(3\alpha_1)(4\alpha_2)-5\alpha_3=0$, or $2.4\alpha_1\alpha_2-\alpha_3=0$, so we get the same parametric equation as (8).

In this case the homogeneous parameterized linear system (14) has a triple infinity of solutions.

This method is an extension of Saaty's method, since we have the possibility to manipulate the parameters α_1,α_2, and α_3. For example, if a second source tells us that $\frac{x}{z}$ has to be discounted 2 times as much as $\frac{y}{x}$, and $\frac{y}{z}$ should be discounted 3 times less than $\frac{y}{x}$, then we set $\alpha_2=2\alpha_1$, and respectively $\alpha_3=\frac{\alpha_1}{3}$, and the original (5), (6), (7) system becomes:

$$\begin{cases} \frac{y}{x}=3\alpha_1 \\ \frac{x}{z}=4\alpha_2=4(2\alpha_1)=8\alpha_1 \\ \frac{y}{z}=5\alpha_3=5\left(\frac{\alpha_1}{3}\right)=\frac{5}{3}\alpha_1 \end{cases} \quad (16)$$

and we solve it in the same way.

8.2. **Weak Inconsistent Example 10.**

Let's complicate Jean Dezert's Weak Inconsistent Example 6.1. with one more preference: C_2 is 1.5 times as much as C_1 and C_3 together. The new system is:

$$\begin{cases} \frac{y}{x} = 3 \\ \frac{x}{z} = 4 \\ \frac{y}{z} = 5 \\ y = 1.5(x+z) \\ x, y, z \in [0,1] \\ x + y + x = 1 \end{cases} \tag{17}$$

We parameterized it:

$$\begin{cases} \frac{y}{x} = 3\alpha_1 \\ \frac{x}{z} = 4\alpha_2 \\ \frac{y}{z} = 5\alpha_3 \\ y = 1.5\alpha_4(x+z) \\ x, y, z \in [0,1] \\ x + y + x = 1 \end{cases} \tag{18}$$

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$$

Its associated matrix is:

$$P_2 = \begin{bmatrix} 3\alpha_1 & -1 & 0 \\ 1 & 0 & -4\alpha_2 \\ 0 & 1 & -5\alpha_3 \\ 1.5\alpha_4 & -1 & 1.5\alpha_4 \end{bmatrix} \tag{19}$$

The rank of matrix P_2 should be strictly less than 3 in order for the system (18) to have non-null solution.

If we take the first three rows in (19) we get the matrix P_1, whose determinant should be zero, therefore one also gets the previous parametric equation $2.4\alpha_1\alpha_2 = \alpha_3$.

If we take rows 1, 3, and 4, since they all involve the relations between C_2 and the other criteria C_1 and C_3 we get

$$P_3 = \begin{bmatrix} 3\alpha_1 & -1 & 0 \\ 0 & 1 & -5\alpha_3 \\ 1.5\alpha_4 & -1 & 1.5\alpha_4 \end{bmatrix} \tag{20}$$

whose determinant should also be zero:

$$\det(P_3)=\left[3\alpha_1\ \ (1.5\alpha_4)+5\alpha_3(1.5\alpha_4)+0\right]-\left[0+3\alpha_1(5\alpha_3)+0\right]=$$
$$=4.5\alpha_1\alpha_4+7.5\alpha_3\alpha_4-15\alpha_1\alpha_3=0 \quad (21)$$

If we take

$$P_4=\begin{bmatrix}1 & 0 & -4\alpha_2\\ 0 & 1 & -5\alpha_3\\ 1.5\alpha_4 & -1 & 1.5\alpha_4\end{bmatrix} \quad (22)$$

then

$$\det(P_4)=\left[1.5\alpha_4\ +0+0\right]-\left[-6\alpha_2\alpha_4+5\alpha_3+0\right]=1.5\alpha_4+6\alpha_2\alpha_4-5\alpha_3=0 \quad (23)$$

If we take

$$P_5=\begin{bmatrix}3\alpha_1 & -1 & 0\\ 1 & 0 & -4\alpha_2\\ 1.5\alpha_4 & -1 & 1.5\alpha_4\end{bmatrix} \quad (24)$$

then

$$\det(P_5)=\left[0\ +0+6\alpha_2\alpha_4\right]-\left[0+12\alpha_1\alpha_2-1.5\alpha_4\right]=6\alpha_2\alpha_4-12\alpha_1\alpha_2+1.5\alpha_4=0 \quad (25)$$

So, these four parametric equations form a parametric system:

$$\begin{cases}2.4\alpha_1\alpha_2-\alpha_3=0\\ 4.5\alpha_1\alpha_4+7.5\alpha_3\alpha_4-15\alpha_1\alpha_3=0\\ 1.5\alpha_4+6\alpha_2\alpha_4-5\alpha_3=0\\ 6\alpha_2\alpha_4-12\alpha_1\alpha_2+1.5\alpha_4=0\end{cases} \quad (26)$$

which should have a non-null solution.

If we consider $\alpha_1=\alpha_2=\alpha_3=\frac{5}{12}>0$ as we got at the beginning, then substituting all α's into the last three equations of the system (26) we get:

$$4.5\left(\frac{5}{12}\right)\alpha_4+7.5\left(\frac{5}{12}\right)\alpha_4-15\left(\frac{5}{12}\right)\left(\frac{5}{12}\right)=0\Rightarrow\alpha_4=0.5208\bar{3}=\frac{25}{48}$$

$$1.5\alpha_4+6\left(\frac{5}{12}\right)\alpha_4-5\left(\frac{5}{12}\right)=0\Rightarrow\alpha_4=0.5208\bar{3}$$

$$6\left(\frac{5}{12}\right)\alpha_4-12\left(\frac{5}{12}\right)\left(\frac{5}{12}\right)+1.5\alpha_4=0\Rightarrow\alpha_4=0.5208\bar{3}$$

α_4 could not be equal to $\alpha_1=\alpha_2=\alpha_3$ since it is an extra preference, because the number of rows was bigger than the number of columns.

So the system is consistent, having the same solution as previously, without having added the fourth preference $y=1.5(x+z)$.

9.1. Jean Dezert's Strong Inconsistent Example 11.

The preference matrix is:

$$M_1 = \begin{pmatrix} 1 & 9 & \frac{1}{9} \\ \frac{1}{9} & 1 & 9 \\ 9 & \frac{1}{9} & 1 \end{pmatrix}$$

so,

$$\begin{cases} x = 9y, x > y \\ x = \frac{1}{9}z, x < z \\ y = 9z, y > z \end{cases}$$

The other three equations: $y = \frac{1}{9}x,\ \ z = 9x,\ \ z = \frac{1}{9}y$ result directly from the previous three ones, so we can eliminate them.

From x>y and y>z (first and third above inequalities) we get x>z, but the second inequality tells us the opposite: x<z; that's why we have a strong contradiction/inconsistency. Or, if we combine all three we have x>y>z>x… strong contradiction again.

Parameterize:

$$\begin{cases} x = 9\alpha_1 y & (27) \\ x = \frac{1}{9}\alpha_2 z & (28) \\ y = 9\alpha_3 z & (29) \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3 > 0$.

From (27) we get: $y = \frac{1}{9\alpha_1}x$, from (28) we get $z = \frac{1}{9\alpha_2}x$, which is replaced in (29) and we get:

$y = 9\alpha_3\left(\frac{9}{\alpha_2}x\right) = \frac{81\alpha_3}{\alpha_2}x$.

So $\frac{1}{9\alpha_1}x = \frac{81\alpha_3}{\alpha_2}x$ or $\alpha_2 = 729\alpha_1\alpha_3$ (parametric equation).

The general solution of the system is:

$$\left(x,\ \frac{1}{9\alpha_1}x,\ \frac{9}{\alpha_2}x\right)$$

The general priority vector is:

$$\left[1\ \ \frac{1}{9\alpha_1}\ \ \frac{9}{\alpha_2}\right].$$

Consider the fairness principle, then $\alpha_1 = \alpha_2 = \alpha_3 = \alpha > 1$ are replaced into the parametric equation: $\alpha = 729\alpha^2$, whence $\alpha = 0$ (not good) and $\alpha = \frac{1}{729} = \frac{1}{9^3}$.

The particular priority vector becomes $\left[1 \quad 9^2 \quad 9^4\right] = \left[1 \quad 81 \quad 6561\right]$ and normalized

$$\left[\frac{1}{6643} \quad \frac{81}{6643} \quad \frac{6561}{6643}\right]$$

Because the consistency is $\alpha = \frac{1}{729} = 0.00137$ is extremely low, we can disregard this solution (and the inconsistency is very big $\beta = 1 - \alpha = 0.99863$).

9.1.2. **Remarks:**

a) If in M_1 we replace all six 9's by a bigger number, the inconsistency of the system will increase. Let's use 11.
   Then $\alpha = \frac{1}{11^3} = 0.00075$ (consistency), while inconsistency $\beta = 0.99925$.

b) But if in M_1 we replace all 9's by the smaller positive number greater than 1, the consistency decreases. Let's use 2. Then $\alpha = \frac{i}{2^3} = 0.125$ and $\beta = 0.875$;

c) Consistency is 1 when replacing all six 9's by 1.

d) Then, replacing all six 9's by a positive sub unitary number, consistency decreases again. For example, replacing by 0.8 we get $\alpha = \frac{1}{0.8^3} = 1.953125 > 1$, whence $\frac{1}{\alpha} = 0.512$ (consistency) and $\beta = 0.488$ (inconsistency).

9.2. Jean Dezert's Strong Inconsistent Example 12.
The preference matrix is:

$$M_2 = \begin{pmatrix} 1 & 5 & \frac{1}{5} \\ \frac{1}{5} & 1 & 5 \\ 5 & \frac{1}{5} & 1 \end{pmatrix}$$

which is similar to M_1 where we replace all six 9's by 5's.

$\alpha = \frac{1}{5^3} = 0.008$ (consistency) and $\beta = 0.992$ (inconsistency).

The priority vector is $\left[1 \quad 5^2 \quad 5^4\right] = \left[1 \quad 25 \quad 625\right]$ and normalized $\left[\frac{1}{651} \quad \frac{25}{651} \quad \frac{625}{651}\right]$.

M_2 is a little more consistent than M_1 because 0.00800 > 0.00137, but still not enough, so this result is also discarded.

9.3. Generalization of Jean Dezert's Strong Inconsistent Examples.
General Example 13.

Let the preference matrix be:

$$M_t = \begin{pmatrix} 1 & t & \frac{1}{t} \\ \frac{1}{t} & 1 & \mathrm{t} \\ \mathrm{t} & \frac{1}{t} & 1 \end{pmatrix},$$

with $t > 0$, and $c(M_t)$ the consistency of M_t, $i(M_t)$ inconsistency of M_t.

We have for the Fairness Principle:

$\lim_{t\to 1} c(M_t) = 1$ and $\lim_{t\to 1} i(M_t) = 0$;

$\lim_{t\to +\infty} c(M_t) = 0$ and $\lim_{t\to +\infty} i(M_t) = 1$;

$\lim_{t\to 0} c(M_t) = 0$ and $\lim_{t\to 0} i(M_t) = 1$.

Also $\alpha = \frac{1}{t^3}$, the priority vector is $\left[1 \quad t^2 \quad t^4\right]$ which is normalized as

$$\left[\frac{1}{1+t^2+t^4} \quad \frac{t^2}{1+t^2+t^4} \quad \frac{t^4}{1+t^2+t^4}\right].$$

In such situations, when we get strong contradiction of the form x>y>z>x or similarly x<z<x, etc. and the consistency is tiny, we can consider that x=y=z=1/3 (so no criterion is preferable to the other – as in Saaty's AHP), or just x+y+z=1 (which means that one has the total ignorance too: $C1 \cup C2 \cup C3$).

10. Strong Inconsistent Example 14.

Let C = {C1, C2}, and P = {C1 is important twice as much as C2; C2 is important 5 times as much as C1}. Let m(C1)=x, m(C2)=y. Then:

x=2y and y=5x (it is a strong inconsistency since from the first equation we have x>y, while from the second y>x).

Parameterize: $x=2\alpha_1 y$, $y=5\alpha_2 x$, whence we get $2\alpha_1=1/(5\alpha_2)$, or $10\alpha_1\alpha_2=1$.

If we consider the Fairness Principle, then $\alpha_1=\alpha_2=\alpha>0$, and one gets $\alpha = \frac{\sqrt{10}}{10} \approx 31.62\%$ consistency; priority vector is [0.39 0.61], hence y>x. An explanation can be done as in paraconsistent logic (or as in neutrosophic logic): we consider that the preferences were honest, but subjective, therefore it is possible to have two contradictory statements true simultaneously since a criterion C1 can be more important from a point of view than C2, while from another point of view C2 can be more important than C1. In our decision-making problem, not having any more information and having rapidly being required to take a decision, we can prefer C2, since C2 is 5 times more important that C1, while C1 is only 2 times more important than C2, and 5>2.

If it's no hurry, more prudent would be in such dilemma to search for more information on C1 and C2.
If we change Example 14 under the form: x=2y and y=2x (the two coefficients set equal), we get $\alpha = ½$, so the priority vector is [0.5 0.5] and decision-making problem is undecidable.

11. Non-Linear/Linear Equation Mixed System Example 15.

Let C = {C1, C2, C3}, m(C1)=x, m(C2)=y, m(C3)=z.
Let F be:

1. C1 is twice as much important as the product of C2 and C3.
2. C2 is five times as much important as C3.

We get the system: x=2yz (non-linear equation) and y=5z (linear equation).
The general solution vector of this mixed system is: $[10z^2 \quad 5z \quad z]$, where z>0.
A discussion is necessary now.

a) You see for sure that y>z, since 5z>z for z strictly positive. But we don't see anything what the position of x would be?

b) Let's simplify the general solution vector by dividing each vector component by z>0, thus we get: [10z 5 1].
If $z \in (0, 0.1)$, then y>z>x.
If z=0.1, then y>z=x.
If $z \in (0.1, 0.5)$, then y>x>z.
If z=0.5, then y=x>z.
If z>0.5, then x>y>z.

12. Non-Linear/Linear Equation/Inequality Mixed System Example 16.

Since in the previous Example 15 has many variants, assume that a new preference comes in (in addition to the previous two preferences):

3. C1 is less important than C3.

The mixed system becomes now: x=2yz (non-linear equation), y=5z (linear equation), and x<z (linear inequality).
The general solution vector of this mixed system is: $[10z^2 \quad 5z \quad z]$, where z>0 and $10z^2 <$ z. From the last two inequalities we get $z \in (0, 0.1)$. Whence the priorities are: y>z>x.

13. Future Research:

To investigate the connection between α-D MCDM and other methods, such as: the technique for order preference by similarity to ideal solution (TOPSIS) method, the simple additive weighting (SAW) method, Borda-Kendall (BK) method for aggregating

ordinal preferences, and the cross-efficiency evaluation method in data envelopment analysis (DEA).

14. Conclusion.

We have introduced a new method in the multi-criteria decision making, α - Discounting MCDM. In the first part of this method, each preference is transformed into a linear or non-linear equation or inequality, and all together form a system that is resolved – one finds its general solution, from which one extracts the positive solutions. If the system has only the null solution, or it is inconsistent, then one parameterizes the coefficients of the system.

In the second part of the method, one chooses a principle for finding the numerical values of the parameters {we have proposed herein the Fairness Principle, or Expert's Opinion on Discounting, or setting a Consistency (or Inconsistency) Threshold}.

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NEUTROSOPHIC LOGIC AND SET

Neutrosophic Logic -
A Generalization of the Intuitionistic Fuzzy Logic

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Abstract

In this paper one generalizes the intuitionistic fuzzy logic (IFL) and other logics to neutrosophic logic (NL). The differences between IFL and NL (and the corresponding intuitionistic fuzzy set and neutrosophic set) are pointed out.

***Keywords and Phrases*:** Non-Standard Analysis, Paraconsistent Logic, Dialetheism, Paradoxism, Intuitionistic Fuzzy Logic, Neutrosophic Logic.

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1. Introduction

The paper starts with a short paragraph on non-standard analysis because it is necessary in defining non-standard real subsets and especially the non-standard unit interval $]^{-}0, 1^{+}[$, all used by neutrosophic logic. Then a survey of the evolution of logics from Boolean to neutrosophic is presented. Afterwards the neutrosophic logic components are introduced followed by the definition of neutrosophic logic and neutrosophic logic connectors which are based on set operations. Original work consists in the definition of neutrosophic logic and neutrosophic connectors as an extension of intuitionistic fuzzy logic and the comparison between NL and other logics, especially the IFL.

2. A Small Introduction to Non-Standard Analysis

In 1960s Abraham Robinson has developed the *non-standard analysis*, a formalization of analysis and a branch of mathematical logic, that rigorously defines the infinitesimals. Informally, an infinitesimal is an infinitely small number. Formally, x is said to be infinitesimal if and only if for all positive integers n one has $|x| < 1/n$. Let $\varepsilon>0$ be a such infinitesimal number. The *hyper-real number set* is an extension of the real number set, which includes classes of infinite numbers and classes of infinitesimal numbers. Let's consider the non-standard finite numbers $1^{+} = 1+\varepsilon$, where "1" is its standard part and "ε" its non-standard part, and $^{-}0 = 0-\varepsilon$, where "0" is its standard part and "ε" its non-standard part.
Then, we call $]^{-}0, 1^{+}[$ a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. Actually, by "^{-}a" one signifies a monad, i.e. a set of hyper-real numbers in non-standard analysis:

$$(^{-}a)= \{a-x: x\in R^{*}, x \text{ is infinitesimal}\},$$

and similarly "b^{+}" is a monad:

$$(b^{+})= \{b+x: x\in R^{*}, x \text{ is infinitesimal}\}.$$

Generally, the left and right borders of a non-standard interval $]^{-}a, b^{+}[$ are vague, imprecise, themselves being non-standard (sub)sets (^{-}a) and (b^{+}) as defined above.
Combining the two before mentioned definitions one gets, what we would call, a binad of "$^{-}c^{+}$":

$(^{-}c^{+})$= {c-x: x∈R^{*}, x is infinitesimal} ∪ {c+x: x∈R^{*}, x is infinitesimal}, which is a collection of open punctured neighborhoods (balls) of c.
Of course, $^{-}a < a$ and $b^{+} > b$. No order between $^{-}c^{+}$ and c.
Addition of non-standard finite numbers with themselves or with real numbers:
$^{-}a + b = {}^{-}(a + b)$
$a + b^{+} = (a + b)^{+}$
$^{-}a + b^{+} = {}^{-}(a + b)^{+}$
$^{-}a + {}^{-}b = {}^{-}(a + b)$ (the left monads absorb themselves)
$a^{+} + b^{+} = (a + b)^{+}$ (analogously, the right monads absorb themselves).
Similarly for subtraction, multiplication, division, roots, and powers of non-standard finite numbers with themselves or with real numbers.
By extension let inf $]\,{}^{-}a, b^{+}[= {}^{-}a$ and sup $]\,{}^{-}a, b^{+}[= b^{+}$.

3. A Short History

The idea of tripartition (truth, falsehood, indeterminacy) appeared in 1764 when J. H. Lambert investigated the credibility of one witness affected by the contrary testimony of another. He generalized Hooper's rule of combination of evidence (1680s), which was a Non-Bayesian approach to find a probabilistic model. Koopman in 1940s introduced the notions of lower and upper probability, followed by Good, and Dempster (1967) gave a rule of combining two arguments. Shafer (1976) extended it to the Dempster-Shafer Theory of Belief Functions by defining the Belief and Plausibility functions and using the rule of inference of Dempster for combining two evidences proceeding from two different sources. Belief function is a connection between fuzzy reasoning and probability. The Dempster-Shafer Theory of Belief Functions is a generalization of the Bayesian Probability (Bayes 1760s, Laplace 1780s); this uses the mathematical probability in a more general way, and is based on probabilistic combination of evidence in artificial intelligence.
In Lambert "there is a chance p that the witness will be faithful and accurate, a chance q that he will be mendacious, and a chance 1-p-q that he will simply be careless" [apud Shafer (1986)]. Therefore three components: accurate, mendacious, careless, which add up to 1.
Van Fraassen introduced the supervaluation semantics in his attempt to solve the sorites paradoxes, followed by Dummett (1975) and Fine (1975). They all tripartitioned, considering a vague predicate which, having border cases, is undefined for these border cases. Van Fraassen took the vague predicate 'heap' and extended it positively to those objects to which the predicate definitively applies and negatively to those objects to which it definitively doesn't apply. The remaining objects border was called penumbra. A sharp boundary between these two extensions does not exist for a soritical predicate. Inductive reasoning is no longer valid too; if S is a sorites predicate, the proposition "$\exists n(Sa_n \& \neg Sa_{n+1})$" is false. Thus, the predicate Heap (positive extension) = true, Heap (negative extension) = false, Heap (penumbra) = indeterminate.
Narinyani (1980) used the tripartition to define what he called the "indefinite set", and Atanassov (1982) continued on tripartition and gave five generalizations of the fuzzy set, studied their properties and applications to the neural networks in medicine:

a) Intuitionistic Fuzzy Set (IFS):
Given an universe E, an IFS A over E is a set of ordered triples <universe_element, degree_of_membership_to_A(M), degree_of_non-membership_to_A(N)> such that M+N ≤ 1 and M, N ∈ [0, 1]. When M + N = 1 one obtains the fuzzy set, and if M + N < 1 there is an indeterminacy I = 1-M-N.

b) Intuitionistic L-Fuzzy Set (ILFS):
Is similar to IFS, but M and N belong to a fixed lattice L.

c) Interval-valued Intuitionistic Fuzzy Set (IVIFS):
Is similar to IFS, but M and N are subsets of [0, 1] and sup M + sup N ≤ 1.

d) Intuitionistic Fuzzy Set of Second Type (IFS2):

Is similar to IFS, but $M^2 + N^2 \leq 1$. M and N are inside of the upper right quarter of unit circle.

e) Temporal IFS:

Is similar to IFS, but M and N are functions of the time-moment too.

4. Definition of Neutrosophic Components

Let T, I, F be standard or non-standard real subsets of $]^-0, 1^+[$,

with $\sup T = t\_sup$, $\inf T = t\_inf$,
$\sup I = i\_sup$, $\inf I = i\_inf$,
$\sup F = f_{sup}$, $\inf F = f_{inf}$,

and $n_{sup} = t_{sup}+i_{sup}+f_{sup}$,
$n_{inf} = t_{inf}+i_{inf}+f_{inf}$.

The sets T, I, F are not necessarily intervals, but may be any real sub-unitary subsets: discrete or continuous; single-element, finite, or (countably or uncountably) infinite; union or intersection of various subsets; etc.

They may also overlap. The real subsets could represent the relative errors in determining t, i, f (in the case when the subsets T, I, F are reduced to points).

In the next papers, T, I, F, called *neutrosophic components*, will represent the truth value, indeterminacy value, and falsehood value respectively referring to neutrosophy, neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics.

This representation is closer to the human mind reasoning. It characterizes/catches the *imprecision* of knowledge or linguistic inexactitude received by various observers (that's why T, I, F are subsets - not necessarily single-elements), *uncertainty* due to incomplete knowledge or acquisition errors or stochasticity (that's why the subset I exists), and *vagueness* due to lack of clear contours or limits (that's why T, I, F are subsets and I exists; in particular for the appurtenance to the neutrosophic sets).

One has to specify the superior (x_sup) and inferior (x_inf) limits of the subsets because in many problems arises the necessity to compute them.

5. Definition of Neutrosophic Logic

A logic in which each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F, where T, I, F are defined above, is called *Neutrosophic Logic*.

We use a subset of truth (or indeterminacy, or falsity), instead of a number only, because in many cases we are not able to exactly determine the percentages of truth and of falsity but to approximate them: for example a proposition is between 30-40% true and between 60-70% false, even worst: between 30-40% or 45-50% true (according to various analyzers), and 60% or between 66-70% false.

The subsets are not necessary intervals, but any sets (discrete, continuous, open or closed or half-open/half-closed interval, intersections or unions of the previous sets, etc.) in accordance with the given proposition.

A subset may have one element only in special cases of this logic.

Constants: (T, I, F) truth-values, where T, I, F are standard or non-standard subsets of the non-standard interval $]^-0, 1^+[$, where $n_{inf} = \inf T + \inf I + \inf F \geq {}^-0$, and $n_{sup} = \sup T + \sup I + \sup F \leq 3^+$.

Atomic formulas: a, b, c, … .

Arbitrary formulas: A, B, C, … .

Therefore, we finally generalize the intuitionistic fuzzy logic to a transcendental logic, called "neutrosophic logic": where the interval [0, 1] is exceeded, i.e. , the percentages of truth, indeterminacy, and falsity are approximated by non-standard subsets – not by single numbers, and these subsets may overlap and exceed the unit interval in the sense of the non-standard analysis; also the superior sums and inferior sum, n_{sup} = sup T + sup I + sup F $\in$] $^{-}0, 3^{+}$ [, may be as high as 3 or 3^{+}, while n_{inf} = inf T + inf I + inf F $\in$] $^{-}0, 3^{+}$ [, may be as low as 0 or $^{-}0$.

Let's borrow from the modal logic the notion of "world", which is a semantic device of what the world might have been like. Then, one says that the neutrosophic truth-value of a statement A, $NL_t(A) = 1^{+}$ if A is 'true in all possible worlds' (syntagme first used by Leibniz) and all conjunctures, that one may call "absolute truth" (in the modal logic it was named *necessary truth,* Dinulescu-Câmpina (2000) names it 'intangible absolute truth'), whereas $NL_t(A) = 1$ if A is true in at least one world at some conjuncture, we call this "relative truth" because it is related to a 'specific' world and a specific conjuncture (in the modal logic it was named *possible truth*).
Similarly for absolute and relative falsehood and absolute and relative indeterminacy.
The neutrosophic inference ([3]), especially for plausible and paradoxist information, is still a subject of intense research today.

6. Differences between Neutrosophic Logic and Intuitionistic Fuzzy Logic

The differences between IFL and NL (and the corresponding intuitionistic fuzzy set and neutrosophic set) are:

a) Neutrosophic Logic can distinguish between *absolute truth* (truth in all possible worlds, according to Leibniz) and *relative truth* (truth in at least one world), because NL(absolute truth)=1^{+} while NL(relative truth)=1. This has application in philosophy (see the neutrosophy). That's why the unitary standard interval [0, 1] used in IFL has been extended to the unitary non-standard interval]$^{-}0, 1^{+}$[in NL.
Similar distinctions for absolute or relative falsehood, and absolute or relative indeterminacy are allowed in NL.

b) In NL there is no restriction on T, I, F other than they are subsets of]$^{-}0, 1^{+}$[, thus:
$^{-}0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^{+}$.
This non-restriction allows paraconsistent, dialetheist, and incomplete information to be characterized in NL {i.e. the sum of all three components if they are defined as points, or sum of superior limits of all three components if they are defined as subsets can be >1 (for paraconsistent information coming from different sources) or < 1 for incomplete information}, while that information can not be described in IFL because in IFL the components T (truth), I (indeterminacy), F (falsehood) are restricted either to t+i+f=1 or to $t^2 + f^2 \leq 1$, if T, I, F are all reduced to the points t, i, f respectively, or to sup T + sup I + sup F = 1 if T, I, F are subsets of [0, 1].

c) In NL the components T, I, F can also be *non-standard* subsets included in the unitary non-standard interval
]$^{-}0, 1^{+}$[, not only *standard* subsets included in the unitary standard interval [0, 1] as in IFL.

d) NL, like dialetheism, can describe paradoxes, NL(paradox) = (1, I, 1), while IFL can not describe a paradox because the sum of components should be 1 in IFL
([11],[12],[13]).

e) NL has a better and clear name "neutrosophic" (which means the neutral part: i.e. neither true nor false), while IFL's name "intuitionistic" produces confusion with Intuitionistic Logic, which is something different.

7. Operations with Sets

We need to present these set operations in order to be able to introduce the neutrosophic connectors.
Let S_1 and S_2 be two (unidimensional) real standard or non-standard subsets, then one defines:

7.1 Addition of Sets:
$S_1 \oplus S_2 = \{x \mid x=s_1+s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$,

with $\inf S_1 \oplus S_2 = \inf S_1 + \inf S_2$, $\sup S_1 \oplus S_2 = \sup S_1 + \sup S_2$;
and, as some particular cases, we have
$\{a\} \oplus S_2 = \{x \mid x=a+s_2$, where $s_2 \in S_2\}$
with $\inf \{a\} \oplus S_2 = a + \inf S_2$, $\sup \{a\} \oplus S_2 = a + \sup S_2$.

7.2 Subtraction of Sets:
$S_1 \ominus S_2 = \{x \mid x=s_1-s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$.
For real positive subsets (most of the cases will fall in this range) one gets
$\inf S_1 \ominus S_2 = \inf S_1 - \sup S_2$, $\sup S_1 \ominus S_2 = \sup S_1 - \inf S_2$;
and, as some particular cases, we have
$\{a\} \ominus S_2 = \{x \mid x=a-s_2$, where $s_2 \in S_2\}$,
with $\inf \{a\} \ominus S_2 = a - \sup S_2$, $\sup \{a\} \ominus S_2 = a - \inf S_2$;
also $\{1^+\} \ominus S_2 = \{x \mid x=1^+-s_2$, where $s_2 \in S_2\}$,
with $\inf \{1^+\} \ominus S_2 = 1^+ - \sup S_2$, $\sup \{1^+\} \ominus S_2 = 100 - \inf S_2$.

7.3 Multiplication of Sets:
$S_1 \odot S_2 = \{x \mid x=s_1 \cdot s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$.
For real positive subsets (most of the cases will fall in this range) one gets
$\inf S_1 \odot S_2 = \inf S_1 \cdot \inf S_2$, $\sup S_1 \odot S_2 = \sup S_1 \cdot \sup S_2$;
and, as some particular cases, we have
$\{a\} \odot S_2 = \{x \mid x=a \cdot s_2$, where $s_2 \in S_2\}$,
with $\inf \{a\} \odot S_2 = a * \inf S_2$, $\sup \{a\} \odot S_2 = a \cdot \sup S_2$;
also $\{1^+\} \odot S_2 = \{x \mid x=1 \cdot s_2$, where $s_2 \in S_2\}$,
with $\inf \{1^+\} \odot S_2 = 1^+ \cdot \inf S_2$, $\sup \{1^+\} \odot S_2 = 1^+ \cdot \sup S_2$.

7.4 Division of a Set by a Number:
Let $k \in \mathbb{R}^*$, then $S_1 \oslash k = \{x \mid x=s_1/k$, where $s_1 \in S_1\}$.

8. Neutrosophic Logic Connectors

One uses the definitions of neutrosophic probability and neutrosophic set operations.
Similarly, there are many ways to construct such connectives according to each particular problem to solve; here we present the easiest ones:

One notes the neutrosophic logic values of the propositions A_1 and A_2 by
$NL(A_1) = (T_1, I_1, F_1)$ and $NL(A_2) = (T_2, I_2, F_2)$ respectively.
For all neutrosophic logic values below: if, after calculations, one obtains numbers < 0 or > 1, one replaces them by $^-0$ or 1^+ respectively.

8.1. Negation:
$NL(\neg A_1) = (\{1^+\} \ominus T_1, \{1^+\} \ominus I_1, \{1^+\} \ominus F_1)$.

8.2. Conjunction:
$NL(A_1 \wedge A_2) = (T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2)$.
(And, in a similar way, generalized for n propositions.)

8.3 Weak or inclusive disjunction:
$NL(A_1 \vee A_2) = (T_1 \oplus T_2 \ominus T_1 \odot T_2, I_1 \oplus I_2 \ominus I_1 \odot I_2, F_1 \oplus F_2 \ominus F_1 \odot F_2)$.
(And, in a similar way, generalized for n propositions.)

8.4 Strong or exclusive disjunction:

$NL(A_1 \veebar A_2) =$
$(T_1 \odot (\{1\} \ominus T_2) \oplus T_2 \odot (\{1\} \ominus T_1) \ominus T_1 \odot T_2 \odot (\{1\} \ominus T_1) \odot (\{1\} \ominus T_2),$
$I_1 \odot (\{1\} \ominus I_2) \oplus I_2 \odot (\{1\} \ominus I_1) \ominus I_1 \odot I_2 \odot (\{1\} \ominus I_1) \odot (\{1\} \ominus I_2),$
$F_1 \odot (\{1\} \ominus F_2) \oplus F_2 \odot (\{1\} \ominus F_1) \ominus F_1 \odot F_2 \odot (\{1\} \ominus F_1) \odot (\{1\} \ominus F_2)).$
(And, in a similar way, generalized for n propositions.)

8.5 Material conditional (implication):
$NL(A_1 \mapsto A_2) = (\{1^+\} \ominus T_1 \oplus T_1 \odot T_2, \{1^+\} \ominus I_1 \oplus I_1 \odot I_2, \{1^+\} \ominus F_1 \oplus F_1 \odot F_2).$

8.6 Material biconditional (equivalence):
$NL(A_1 \leftrightarrow A_2) = ((\{1^+\} \ominus T_1 \oplus T_1 \odot T_2) \odot (\{1^+\} \ominus T_2 \oplus T_1 \odot T_2),$
$(\{1^+\} \ominus I_1 \oplus I_1 \odot I_2) \odot (\{1^+\} \ominus I_2 \oplus I_1 \odot I_2),$
$(\{1^+\} \ominus F_1 \oplus F_1 \odot F_2) \odot (\{1^+\} \ominus F_2 \oplus F_1 \odot F_2)).$

8.7 Sheffer's connector:
$NL(A_1 \mid A_2) = NL(\neg A_1 \vee \neg A_2) = (\{1^+\} \ominus T_1 \odot T_2, \{1^+\} \ominus I_1 \odot I_2, \{1^+\} \ominus F_1 \odot F_2).$

8.8 Peirce's connector:
$NL(A_1 \downarrow A_2) = NL(\neg A_1 \wedge \neg A_2) =$
$= ((\{1^+\} \ominus T_1) \odot (\{1^+\} \ominus T_2), (\{1^+\} \ominus I_1) \odot (\{1^+\} \ominus I_2), (\{1^+\} \ominus F_1) \odot (\{1^+\} \ominus F_2)).$

9. Generalizations

When all neutrosophic logic set components are reduced to one element, then
$t_{sup} = t_{inf} = t$, $i_{sup} = i_{inf} = i$, $f_{sup} = f_{inf} = f$, and $n_{sup} = n_{inf} = n = t+i+f$, therefore neutrosophic logic generalizes:
- the *intuitionistic logic*, which supports incomplete theories (for $0 < n < 1$ and $i=0$, $0 \le t, i, f \le 1$);
- the *fuzzy logic* (for $n = 1$ and $i = 0$, and $0 \le t, i, f \le 1$);
from "CRC Concise Concise Encyclopedia of Mathematics", by Eric W. Weisstein, 1998, the fuzzy logic is "an extension of two-valued logic such that statements need not to be True or False, but may have a degree of truth between 0 and 1";
- the *intuitionistic fuzzy logic* (for n=1);
- the *Boolean logic* (for $n = 1$ and $i = 0$, with t, f either 0 or 1);
- the *multi-valued logic* (for $0 \le t, i, f \le 1$);
definition of <many-valued logic> from "The Cambridge Dictionary of Phylosophy", general editor Robert Audi, 1995, p. 461: "propositions may take many values beyond simple truth and falsity, values functionally determined by the values of their components"; Lukasiewicz considered three values (1, 1/2, 0). Post considered m values, etc. But they varied in between 0 and 1 only. In the neutrosophic logic a proposition may take values even greater than 1 (in percentage greater than 100%) or less than 0.
- the *paraconsistent logic*, which support conflicting information (for $n > 1$ and $i = 0$, with both $t, f < 1$);
the *dialetheism*, which says that some contradictions are true (for $t = f = 1$ and $i = 0$; some paradoxes can be denoted this way too);
the *faillibilism*, which says that uncertainty belongs to every proposition (for $i > 0$);
Compared with all other logics, the neutrosophic logic and intuitionistic fuzzy logic introduce a percentage of "indeterminacy" - due to unexpected parameters hidden in some propositions, or unknowness, but neutrosophic logic let each component t, i, f be even boiling *over 1* (overflooded), i.e. be 1^+, or freezing *under 0* (underdried), i.e. be $^-0$ in order to be able to make distinction between relative truth and absolute truth, and between relative falsity and absolute falsity in philosophy.

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Neutrosophic Set – A Generalization of the Intuitionistic Fuzzy Set

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***Abstract*:** In this paper one generalizes the intuitionistic fuzzy set (IFS), paraconsistent set, and intuitionistic set to the neutrosophic set (NS). Many examples are presented. Distinctions between NS and IFS are underlined.

***Keywords and Phrases*:** Intuitionistic Fuzzy Set, Paraconsistent Set, Intuitionistic Set, Neutrosophic Set, Non-standard Analysis, Philosophy.
***MSC 2000*:** 03B99, 03E99.

1. Introduction:

One first presents the evolution of sets from fuzzy set to neutrosophic set. Then one introduces the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]^-0, 1^+[$ is the non-standard unit interval, and thus one defines the neutrosophic set. One gives examples from mathematics, physics, philosophy, and applications of the neutrosophic set. Afterwards, one introduces the neutrosophic set operations (complement, intersection, union, difference, Cartesian product, inclusion, and n-ary relationship), some generalizations and comments on them, and finally the distinctions between the neutrosophic set and the intuitionistic fuzzy set.

2. Short History:

The *fuzzy set* (FS) was introduced by L. Zadeh in 1965, where each element had a degree of membership.

The *intuitionistic fuzzy set* (IFS) on a universe X was introduced by K. Atanassov in 1983 as a generalization of FS, where besides the degree of membership $\mu_A(x) \in [0,1]$ of each element $x \in X$ to a set A there was considered a degree of non-membership $\nu_A(x) \in [0,1]$, but such that

$$\forall\, x \in X \;\; \mu_A(x) + \nu_A(x) \leq 1. \tag{2.1}$$

According to Deschrijver & Kerre (2003) the *vague set* defined by Gau and Buehrer (1993) was proven by Bustine & Burillo (1996) to be the same as IFS.

Goguen (1967) defined the *L-fuzzy Set* in X as a mapping $X \to L$ such that $(L^*, \leq_{L^*})$ is a complete lattice, where $L^* = \{(x_1,x_2) \in [0,1]^2,\; x_1 + x_2 \leq 1\}$ and $(x_1,x_2) \leq_{L^*} (y_1,y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$. The *interval-valued fuzzy set* (IVFS) apparently first studied by Sambuc (1975), which were called by Deng (1989) *grey sets*, and IFS are specific kinds of L-fuzzy sets.

According to Cornelis et al. (2003), Gehrke et al. (1996) stated that "Many people believe that assigning an exact number to an expert's opinion is too restrictive, and the assignment of an interval of values is more realistic", which is somehow similar with the imprecise probability theory where instead of a crisp probability one has an interval (upper and lower) probabilities as in Walley (1991).

Atanassov (1999) defined the *interval-valued intuitionistic fuzzy set* (IVIFS) on a universe X as an object A such that:

$$A = \{(x, M_A(X), N_A(x)),\; x \in X\}, \tag{2.2}$$

$$\text{with } M_A: X \to \text{Int}([0,1]) \text{ and } N_A: X \to \text{Int}([0,1]) \tag{2.3}$$

$$\text{and } \forall\, x \in X \; \sup M_A(x) + \sup N_A(x) \leq 1. \tag{2.4}$$

Belnap (1977) defined a four-valued logic, with truth (T), false (F), unknown (U), and contradiction (C). He used a billatice where the four components were inter-related.

In 1995, starting from philosophy (when I fretted to distinguish between *absolute truth* and *relative truth* or between *absolute falsehood* and *relative falsehood* in logics, and respectively between *absolute membership* and *relative membership* or *absolute non-membership* and *relative non-membership* in set theory) I began to use the non-standard analysis. Also, inspired from the sport games (winning, defeating, or tight scores), from votes (pro, contra, null/black votes), from positive/negative/zero numbers, from yes/no/NA, from decision making and control theory (making a decision, not

making, or hesitating), from accepted/rejected/pending, etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics, I combined the non-standard analysis with a tri-component logic/set/probability theory and with philosophy (I was excited by paradoxism in science and arts and letters, as well as by paraconsistency and incompleteness in knowledge). How to deal with all of them at once, is it possible to unity them?
I proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutro-sophy" [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy" and "intuitionistic fuzzy" logic/set, i.e. the *included middle* component (Lupasco-Nicolescu's logic in philosophy), i.e. the neutral/indeterminate/unknown part (besides the "truth"/"membership" and "falsehood"/"non-membership" components that both appear in fuzzy logic/set). See the Proceedings of the First International Conference on Neutrosophic Logic, The University of New Mexico, Gallup Campus, 1-3 December 2001,
at http://www.gallup.unm.edu/~smarandache/FirstNeutConf.htm.

3. Definition of Neutrosophic Set:
Let T, I, F be real standard or non-standard subsets of $]^{-}0, 1^{+}[$,
with sup T = t_sup, inf T = t_inf,
sup I = i_sup, inf I = i_inf,
sup F = f_sup, inf F = f_inf,
and n_sup = t_sup+i_sup+f_sup,
n_inf = t_inf+i_inf+f_inf.
T, I, F are called *neutrosophic components*.
Let U be a universe of discourse, and M a set included in U. An element x from U is noted with respect to the set M as x(T, I, F) and belongs to M in the following way:
it is t% true in the set, i% indeterminate (unknown if it is) in the set, and f% false, where t varies in T, i varies in I, f varies in F.

4. General Examples:
Let A, B, and C be three neutrosophic sets.
One can say, by language abuse, that any element neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1.
Thus: x(0.5,0.2,0.3) belongs to A (which means, with a probability of 50% x is in A, with a probability of 30% x is not in A, and the rest is undecidable); or y(0,0,1) belongs to A (which normally means y is not for sure in A); or z(0,1,0) belongs to A (which means one does know absolutely nothing about z's affiliation with A); here 0.5+0.2+0.3=1; thus A is a NS and an IFS too. More general, y((0.20-0.30), (0.40-0.45)∪[0.50-0.51], {0.20, 0.24, 0.28}) belongs to the set B, which means:
- with a probability in between 20-30% y is in B (one cannot find an exact approximation because of various sources used);
- with a probability of 20% or 24% or 28% y is not in B;
- the indeterminacy related to the appurtenance of y to B is in between 40-45% or between 50-51% (limits included);
The subsets representing the appurtenance, indeterminacy, and falsity may overlap, and n_sup = 0.30+0.51+0.28 > 1 in this case; then B is a NS but is not an IFS; we can call it *paraconsistent set* (from *paraconsistent logic*, which deals with paraconsistent information).
Or, another example, say the element z(0.1, 0.3, 0.4) belongs to the set C, and here 0.1+0.3+0.4<1; then B is a NS but is not an IFS; we can call it *intuitionistic set* (from *intuitionistic logic*, which deals with incomplete information).
Remarkably, in the same NS one can have elements which have paraconsistent information (sum of components >1), others incomplete information (sum of components < 1), others consistent information (in the case when the sum of components = 1), and others interval-valued components (with no restriction on their superior or inferior sums).

5. Physics Examples:

a) For example the Schrödinger's Cat Theory says that the quantum state of a photon can basically be in more than one place in the same time, which translated to the neutrosophic set means that an element (quantum state) belongs and does not belong to a set (one place) in the same time; or an element (quantum state) belongs to two different sets (two different places) in the same time. It is a question of "alternative worlds" theory very well represented by the neutrosophic set theory.

In Schrödinger's Equation on the behavior of electromagnetic waves and "matter waves" in quantum theory, the wave function ψ which describes the superposition of possible states may be simulated by a neutrosophic function, i.e. a function whose values are not unique for each argument from the domain of definition (the vertical line test fails, intersecting the graph in more points).

Don't we better describe, using the attribute "neutrosophic" than "fuzzy" or any others, a quantum particle that neither exists nor non-exists?

b) How to describe a particle ζ in the infinite micro-universe that belongs to two distinct places P_1 and P_2 in the same time? $\zeta \in P_1$ and $\zeta \notin P_1$ as a true contradiction, or $\zeta \in P_1$ and $\zeta \in \neg P_1$.

6. Philosophical Examples:

Or, how to calculate the truth-value of Zen (in Japanese) / Chan (in Chinese) doctrine philosophical proposition: the present is eternal and comprises in itself the past and the future?

In Eastern Philosophy the contradictory utterances form the core of the Taoism and Zen/Chan (which emerged from Buddhism and Taoism) doctrines.

How to judge the truth-value of a metaphor, or of an ambiguous statement, or of a social phenomenon which is positive from a standpoint and negative from another standpoint?

There are many ways to construct them, in terms of the practical problem we need to simulate or approach. Below there are mentioned the easiest ones:

7. Application:

A cloud is a neutrosophic set, because its borders are ambiguous, and each element (water drop) belongs with a neutrosophic probability to the set (e.g. there are a kind of separated water drops, around a compact mass of water drops, that we don't know how to consider them: in or out of the cloud).

Also, we are not sure where the cloud ends nor where it begins, neither if some elements are or are not in the set. That's why the percent of indeterminacy is required and the neutrosophic probability (using subsets - not numbers - as components) should be used for better modeling: it is a more organic, smooth, and especially accurate estimation. Indeterminacy is the zone of ignorance of a proposition's value, between truth and falsehood.

8. Operations with classical Sets

We need to present these set operations in order to be able to introduce the neutrosophic connectors. Let S_1 and S_2 be two (unidimensional) real standard or non-standard subsets included in the non-standard interval $]^-0, \infty)$ then one defines:

8.1 Addition of classical Sets:

$S_1 \oplus S_2 = \{x \mid x=s_1+s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$,

with $\inf S_1 \oplus S_2 = \inf S_1 + \inf S_2$, $\sup S_1 \oplus S_2 = \sup S_1 + \sup S_2$;

and, as some particular cases, we have

$\{a\} \oplus S_2 = \{x \mid x=a+s_2$, where $s_2 \in S_2\}$

with $\inf \{a\} \oplus S_2 = a + \inf S_2$, $\sup \{a\} \oplus S_2 = a + \sup S_2$.

8.2 Subtraction of classical Sets:

$S_1 \ominus S_2 = \{x \mid x=s_1-s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$.

with $\inf S_1 \ominus S_2 = \inf S_1 - \sup S_2$, $\sup S_1 \ominus S_2 = \sup S_1 - \inf S_2$;

and, as some particular cases, we have

$\{a\} \ominus S_2 = \{x \mid x=a-s_2$, where $s_2 \in S_2\}$,

with $\inf \{a\} \ominus S_2 = a - \sup S_2$, $\sup \{a\} \ominus S_2 = a - \inf S_2$;

also $\{1^+\} \ominus S_2 = \{x \mid x=1^+ - s_2$, where $s_2 \in S_2\}$,
with $\inf \{1^+\} \ominus S_2 = 1^+ - \sup S_2$, $\sup \{1^+\} \ominus S_2 = 100 - \inf S_2$.

8.3 Multiplication of classical Sets:

$S_1 \odot S_2 = \{x \mid x=s_1 \cdot s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$.
with $\inf S_1 \odot S_2 = \inf S_1 \cdot \inf S_2$, $\sup S_1 \odot S_2 = \sup S_1 \cdot \sup S_2$;
and, as some particular cases, we have
$\{a\} \odot S_2 = \{x \mid x=a \cdot s_2$, where $s_2 \in S_2\}$,
with $\inf \{a\} \odot S_2 = a * \inf S_2$, $\sup \{a\} \odot S_2 = a \cdot \sup S_2$;
also $\{1^+\} \odot S_2 = \{x \mid x=1 \cdot s_2$, where $s_2 \in S_2\}$,
with $\inf \{1^+\} \odot S_2 = 1^+ \cdot \inf S_2$, $\sup \{1^+\} \odot S_2 = 1^+ \cdot \sup S_2$.

8.4 Division of a classical Set by a Number:

Let $k \in \mathbb{R}^*$, then $S_1 \oslash k = \{x \mid x=s_1/k$, where $s_1 \in S_1\}$.

9. Neutrosophic Set Operations:

One notes, with respect to the sets A and B over the universe U,

$x = x(T_1, I_1, F_1) \in A$ and $x = x(T_2, I_2, F_2) \in B$, by mentioning x's *neutrosophic membership, indeterminacy, and non-membership respectively appurtenance*.

And, similarly, $y = y(T', I', F') \in B$.

If, after calculations, in the below operations one obtains values < 0 or > 1, then one replaces them with $^-0$ or 1^+ respectively.

9.1. Complement of A:

If $x(T_1, I_1, F_1) \in A$,
then $x(\{1^+\} \ominus T_1, \{1^+\} \ominus I_1, \{1^+\} \ominus F_1) \in C(A)$.

9.2. Intersection:

If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$,
then $x(T_1 \odot T_2, I_1 \odot I_2, F_1 \odot F_2) \in A \cap B$.

9.3. Union:

If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$,
then $x(T_1 \oplus T_2 \ominus T_1 \odot T_2, I_1 \oplus I_2 \ominus I_1 \odot I_2, F_1 \oplus F_2 \ominus F_1 \odot F_2) \in A \cup B$.

9.4. Difference:

If $x(T_1, I_1, F_1) \in A$, $x(T_2, I_2, F_2) \in B$,
then $x(T_1 \ominus T_1 \odot T_2, I_1 \ominus I_1 \odot I_2, F_1 \ominus F_1 \odot F_2) \in A \setminus B$,
because $A \setminus B = A \cap C(B)$.

9.5. Cartesian Product:

If $x(T_1, I_1, F_1) \in A$, $y(T', I', F') \in B$,
then $(x(T_1, I_1, F_1), y(T', I', F')) \in A \times B$.

9.6. M is a subset of N:

If $x(T_1, I_1, F_1) \in M \Rightarrow x(T_2, I_2, F_2) \in N$,
where $\inf T_1 \leq \inf T_2$, $\sup T_1 \leq \sup T_2$, and $\inf F_1 \geq \inf F_2$, $\sup F_1 \geq \sup F_2$.

9.7. Neutrosophic n-ary Relation:

Let $A_1, A_2, \ldots, A_n$ be arbitrary non-empty sets.

A Neutrosophic n-ary Relation *R* on $A_1 \times A_2 \times \ldots \times A_n$ is defined as a subset of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$, such that for each ordered n-tuple $(x_1, x_2, \ldots, x_n)(T, I, F)$, T represents the degree of validity, I the degree of indeterminacy, and F the degree of non-validity respectively of the relation *R*.

It is related to the definitions for the *Intuitionistic Fuzzy Relation* independently given by Atanassov (1984, 1989), Toader Buhaescu (1989), Darinka Stoyanova (1993), Humberto Bustince Sola and P. Burillo Lopez (1992-1995).

10. Generalizations and Comments:

From the intuitionistic logic, paraconsistent logic, dialetheism, faillibilism, paradoxes, pseudoparadoxes, and tautologies we transfer the "adjectives" to the sets, i.e. to intuitionistic set (set incompletely known), paraconsistent set, dialetheist set, faillibilist set (each element has a

percenatge of indeterminacy), paradoxist set (an element may belong and may not belong in the same time to the set), pseudoparadoxist set, and tautologic set respectively.

Hence, the neutrosophic set generalizes:

- the *intuitionistic set*, which supports incomplete set theories (for $0 < n < 1$ and $i = 0$, $0 \leq t, i, f \leq 1$) and incomplete known elements belonging to a set;
- the *fuzzy set* (for $n = 1$ and $i = 0$, and $0 \leq t, i, f \leq 1$);
- the *intuitionistic fuzzy set* (for $t+i+f=1$ and $0 \leq i < 1$);
- the *classical set* (for $n = 1$ and $i = 0$, with t, f either 0 or 1);
- the *paraconsistent set* (for $n > 1$ and $i = 0$, with both $t, f < 1$);

there is at least one element x(T,I,F) of a paraconsistent set M which belongs at the same time to M and to its complement set C(M);

- the *faillibilist set* ($i > 0$);
- the *dialethist set*, which says that the intersection of some disjoint sets is not empty (for $t = f = 1$ and $i = 0$; some paradoxist sets can be denoted this way too);

every element x(T,I,F) of a dialethist set M belongs at the same time to M and to its complement set C(M);

- the *paradoxist set*, each element has a part of indeterminacy if it is or not in the set ($i > 1$);
- the *pseudoparadoxist set* ($0 < i < 1$, $t + f > 1$);
- the *tautological set* ($i < 0$).

Compared with all other types of sets, in the neutrosophic set each element has three components which are subsets (not numbers as in fuzzy set) and considers a subset, similarly to intuitionistic fuzzy set, of "indeterminacy" - due to unexpected parameters hidden in some sets, and let the superior limits of the components to even boil *over 1* (overflooded) and the inferior limits of the components to even freeze *under 0* (underdried).

For example: an element in some tautological sets may have $t > 1$, called "overincluded". Similarly, an element in a set may be "overindeterminate" (for $i > 1$, in some paradoxist sets), "overexcluded" (for $f > 1$, in some unconditionally false appurtenances); or "undertrue" (for $t < 0$, in some unconditionally false appurtenances), "underindeterminate" (for $i < 0$, in some unconditionally true or false appurtenances), "underfalse" (for $f < 0$, in some unconditionally true appurtenances).

This is because we should make a distinction between unconditionally true ($t > 1$, and $f < 0$ or $i < 0$) and conditionally true appurtenances ($t \leq 1$, and $f \leq 1$ or $i \leq 1$).

In a *rough set* RS, an element on its boundary-line cannot be classified neither as a member of RS nor of its complement with certainty. In the neutrosophic set a such element may be characterized by x(T, I, F), with corresponding set-values for $T, I, F \subseteq]^{-}0, 1^{+}[$.

Compared to *Belnap's quadruplet logic*, NS and NL do not use restrictions among the components – and that's why the NS/NL have a more general form, while the middle component in NS and NL (the indeterminacy) can be split in more subcomponents if necessarily in various applications.

11. Differences between Neutrosophic Set (NS) and Intuitionistic Fuzzy Set (IFS).

a) Neutrosophic Set can distinguish between *absolute membership* (i.e. membership in all possible worlds; we have extended Leibniz's absolute truth to absolute membership) and *relative membership* (membership in at least one world but not in all), because NS(absolute membership element)$=1^{+}$ while NS(relative membership element)=1. This has application in philosophy (see the neutrosophy). That's why the unitary standard interval [0, 1] used in IFS has been extended to the unitary non-standard interval $]^{-}0, 1^{+}[$ in NS.

Similar distinctions for *absolute or relative non-membership*, and *absolute or relative indeterminant appurtenance* are allowed in NS.

b) In NS there is no restriction on T, I, F other than they are subsets of $]^{-}0, 1^{+}[$, thus: $^{-}0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^{+}$.

The inequalities (2.1) and (2.4) of IFS are relaxed in NS.

This non-restriction allows paraconsistent, dialetheist, and incomplete information to be characterized in NS {i.e. the sum of all three components if they are defined as points, or sum of superior limits of all three components if they are defined as subsets can be >1 (for paraconsistent information coming from different sources), or < 1 for incomplete information}, while that information can not be described in IFS because in IFS the components T (membership), I (indeterminacy), F (non-membership) are restricted either to t+i+f=1 or to $t^2 + f^2 \leq 1$, if T, I, F are all reduced to the points t, i, f respectively, or to sup T + sup I + sup F = 1 if T, I, F are subsets of [0, 1].
Of course, there are cases when paraconsistent and incomplete informations can be normalized to 1, but this procedure is not always suitable.

c) Relation (2.3) from interval-valued intuitionistic fuzzy set is relaxed in NS, i.e. the intervals do not necessarily belong to Int[0,1] but to [0,1], even more general to]-0, 1+[.

d) In NS the components T, I, F can also be *non-standard* subsets included in the unitary non-standard interval $]^-0, 1^+[$, not only *standard* subsets included in the unitary standard interval [0, 1] as in IFS.

e) NS, like dialetheism, can describe paradoxist elements, NS(paradoxist element) = (1, I, 1), while IFL can not describe a paradox because the sum of components should be 1 in IFS.

f) The connectors in IFS are defined with respect to T and F, i.e. membership and non-membership only (hence the Indeterminacy is what's left from 1), while in NS they can be defined with respect to any of them (no restriction).

g) Component "I", indeterminacy, can be split into more subcomponents in order to better catch the vague information we work with, and such, for example, one can get more accurate answers to the *Question-Answering Systems* initiated by Zadeh (2003). {In Belnap's four-valued logic (1977) indeterminacy is split into Uncertainty (U) and Contradiction (C), but they were inter-related.}

h) NS has a better and clear name "neutrosophic" (which means the neutral part: i.e. neither true/membership nor false/nonmembership), while IFS's name "intuitionistic" produces confusion with Intuitionistic Logic, which is something different.

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Single Valued Neutrosophic Sets

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Abstract

Neutrosophic set is a part of neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set is a powerful general formal framework that has been recently proposed. However, neutrosophic set needs to be specified from a technical point of view. To this effect, we define the set-theoretic operators on an instance of neutrosophic set, we call it single valued neutrosophic set (SVNS). We provide various properties of SVNS, which are connected to the operations and relations over SVNS.

Keywords: Neutrosophic set, single valued neutrosophic set, set-theoretic operator

1. Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965[5]. Since then fuzzy sets and fuzzy logic have been applied in many real applications to handle uncertainty. The traditional fuzzy set uses one real value $\mu_A(x) \in [0,1]$ to represent the grade of membership of fuzzy set A defined on universe X. Sometimes $\mu_A(x)$ itself is uncertain and hard to be defined by a crisp value. So the concept of interval valued fuzzy sets was proposed [3] to capture the uncertainty of grade of membership. Interval valued fuzzy set uses an interval value $[\mu_A^L(x), \mu_A^U(x)]$ with $0 \le \mu_A^L(x) \le \mu_A^U(x) \le 1$ to represent the grade of membership of fuzzy set A. In some applications such as expert system, belief system and information fusion, we should consider not only the truth-membership supported by the evident but also the falsity-membership against by the evident. That is beyond the scope of fuzzy sets and interval valued fuzzy sets. In 1986, Atanassov introduced the intuitionistic fuzzy sets [1] which is a generalization of fuzzy sets and provably equivalent to interval valued fuzzy sets. The intuitionistic fuzzy sets consider both truth-membership $t_A(x)$ and falsity-membership $f_A(x)$, with $t_A(x), f_A(x) \in [0,1]$ and $0 \le t_A(x)+f_A(x) \le 1$. Intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information which exists commonly in belief system. In intuitionistic fuzzy sets, indeterminacy is $1-t_A(x)-f_A(x)$ by default. For example, when we ask the opinion of an expert about certain statement, he or she may that the possibility that the statement is true is 0.5 and the statement is false is 0.6 and the degree that he or she is not sure is 0.2.

In neutrosophic set, indeterminacy is quantified explicitly and truth-membership, indeterminacy-membership and falsity-membership are independent. This assumption is very important in a lot of situations such as information fusion when we try to combine the data from different sensors. Neutrosophy was introduced by Smarandache in 1995. "It is a branch of philosophy which studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra" [2]. Neutrosophic set is a power general formal framework which generalizes the concept of the classic set, fuzzy set [5], interval valued fuzzy set [3], intuitionistic fuzzy set [1], etc. A neutrosophic set A defined on universe U. $x = x(T,I,F) \in A$ with T, I and F being the real standard or non-standard subsets of $]0^-,1^+[$. T is the degree of truth-membership function in the set A, I is the indeterminacy-membership function in the set A and F is the falsity-membership function in the set A.

The neutrosophic set generalizes the above mentioned sets from philosophical point of view. From scientific or engineering point of view, the neutrosophic set and set-theoretic operators need to be specified. Otherwise, it will be difficult to apply in the real applications. In this paper, we define the set-theoretic operators on an instance of neutrosophic set called single valued neutrosophic set (svns).

2. Neutrosophic Set

This section gives a brief overview of concepts of neutrosophic set defined in [2]. Here, we use different notations to express the same meaning. Let S_1 and S_2 be two real standard or non-standard subsets, then $S_1 + S_2 = \{x|x = s_1+s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\{1^+\} + S_2 = \{x|x = 1^+ + s_2, s_2 \in S_2\}$. $S_1 - S_2 = \{x|x=s_1-s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$, $\{1^+\} - S_2 = \{x|x = 1^+ - s_2, s_2 \in S_2\}$. $S_1 \times S_2 = \{x|x = s_1 \times s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\}$.

Definition 1 (Neutrosophic Set) Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-,1^+[$. That is

$$T_A: X \rightarrow]0^-,1^+[\quad (1)$$

$$I_A: X \rightarrow]0^-,1^+[\quad (2)$$

$$F_A: X \rightarrow]0^-,1^+[\quad (3)$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup T_A(x)+\sup I_A(x)+\sup F_A(x) \leq 3^+$.

Definition 2 The complement of a neutrosophic set A is denoted by c(A) and is defined by

$$T_{c(A)}(x) = \{1^+\} - T_A(x), \quad (4)$$

$$I_{c(A)}(x) = \{1^+\} - I_A(x), \quad (5)$$

$$F_{c(A)}(x) = \{1^+\} - F_A(x), \quad (6)$$

for all x in X.

Definition 3 (Containment) A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$, if and only if

$$\inf T_A(x) \leq \inf T_B(x), \sup T_A(x) \leq \sup T_B(x) \quad (7)$$

$$\inf F_A(x) \geq \inf F_B(x), \sup F_B(x) \geq \sup F_B(x) \quad (8).$$

Definition 4 (Union) The union of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = T_A(x)+T_B(x)-T_A(x)\times T_B(x), \quad (9)$$

$$I_C(x) = I_A(x)+I_B(x)-I_A(x)\times I_B(x), \quad (10)$$

$$F_C(x) = F_A(x)+F_B(x)-F_A(x)\times F_B(x), \quad (11)$$

for all x in X.

Definition 5 (Intersection) The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = T_A(x) \times T_B(x), \quad (12)$$

$$I_C(x) = I_A(x) \times I_B(x), \quad (13)$$

$$F_C(x) = F_A(x) \times F_B(x), \quad (14)$$

for all x in X.

3. Single Valued Neutrosophic Set

In this section, we present the notion of single valued neutrosophic set (SVNS). SVNS is an instance of neutrosophic set which can be used in real scientific and engineering applications.

Definition 6 (Single Valued Neutrosophic Set) Let X be a space of points (objects), with a generic element in X denoted by x. A single valued neutrosophic set (SVNS) A in X is characterized by truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A. For each point x in X, $T_A(x), I_A(x), F_A(x) \in [0,1]$.

When X is continuous, a SVNS A can be written as

$$A = \int_X \langle T(x), I(x), F(x) \rangle / x, x \in X \quad (15)$$

When X is discrete, a SVNS A can be written as

$$A = \sum_{i=1}^{n} \langle T(xi), I(xi), F(xi) \rangle / xi, xi \in X \quad (16)$$

Consider parameters such as capability, trustworthiness and price of semantic Web services. These parameters are commonly used to define quality of service of semantic Web services. In this section, we will use the evaluation of quality of service of semantic Web services [4] as running example to illustrate every set-theoretic operation on single valued neutrosophic sets.

Example 1 Assume that $X = [x_1,x_2,x_3]$. x_1 is capability, x_2 is trustworthiness and x_3 is price. The values of x_1,x_2 and x_3 are in [0,1]. They are obtained from the questionnaire of some domain experts, their option could be a degree of "good service", a degree of indeterminacy and a degree of "poor service". A is a single valued neutrosophic set of X defined by $A = \langle 0.3,0.4,0.5 \rangle / x_1 + \langle 0.5,0.2,0.3 \rangle / x_2 + \langle 0.7,0.2,0.2 \rangle / x_3$. B is a single valued neutrosophic set of X defined

by B = ⟨0.6,0.1,0.2⟩/x_1 + ⟨0.3,0.2,0.6⟩/x_2 + ⟨0.4,0.1,0.5⟩/x_3.

Definition 7 (Complement) The complement of a single valued neutrosophic set A is denoted by c(A) and is defined by

$$T_{c(A)}(x) = F_A(x), \quad (17)$$
$$I_{c(A)}(x) = 1 - I_A(x), \quad (18)$$
$$F_{c(A)}(x) = T_A(x), \quad (19)$$

for all x in X.

Example 2 Let A be the single valued neutrosophic set defined in Example 1. Then, c(A) = ⟨0.5,0.6,0.3⟩/x_1 + ⟨0.3,0.8,0.5⟩/x_2 + ⟨0.2,0.8,0.7⟩/x_3.

Definition 8 (Containment) A single valued neutrosophic set A is contained in the other single valued neutrosophic set B, A ⊆ B, if and only if

$$T_A(x) \le T_B(x), \quad (20)$$
$$I_A(x) \le I_B(x), \quad (21)$$
$$F_A(x) \ge F_B(x), \quad (22)$$

for all x in X.

Note that by the definition of containment, X is partial order not linear order. For example, let A and B be the single valued neutrosophic sets defined in Example 1. Then, A is not contained in B and B is not contained in A.

Definition 9 Two single valued neutrosophic sets A and B are equal, written as A = B, if and only if A ⊆ B and B ⊆ A.

Theorem 3 $A \subseteq B \leftrightarrow c(B) \subseteq c(A)$
Proof: $A \subseteq B \Leftrightarrow T_A \le T_B, I_A \le I_B, F_A \ge F_B \Leftrightarrow F_B \le F_A, 1 - I_B \le 1 - I_A, T_B \ge T_A \Leftrightarrow c(B) \subseteq c(A)$.

Definition 10 (Union) The union of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A ∪ B, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = \max(T_A(x),T_B(x)), \quad (23)$$
$$I_C(x) = \max(I_A(x),I_B(x)), \quad (24)$$
$$F_C(x) = \min(F_A(x),F_B(x)), \quad (25)$$

for all x in X.

Example 3 Let A and B be the single valued neutrosophic sets defined in Example 1. Then, A ∪ B = ⟨0.6,0.4,0.2⟩/x_1 + ⟨0.5,0.2,0.3⟩/x_2 + ⟨0.7,0.2,0.2⟩/x_3.

Theorem 2 A ∪ B is the smallest single valued neutrosophic set containing both A and B.

Proof: It is straightforward from the definition of the union operator.

Definition 11 (Intersection) The intersection of two single valued neutrosophic sets A and B is a single valued neutrosophic set C, written as C = A ∩ B, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = \min(T_A(x),T_B(x)), \quad (26)$$
$$I_C(x) = \min(I_A(x),I_B(x)), \quad (27)$$
$$F_C(x) = \max(F_A(x),F_B(x)), \quad (28)$$

for all x in X.

Example 4 Let A and B be the single valued neutrosophic sets defined in Example 1. Then, A ∩ B = ⟨0.3,0.1,0.5⟩/x_1 + ⟨0.3,0.2,0.6⟩/x_2 + ⟨0.4,0.1,0.5⟩/x_3.

Theorem 3 A ∩ B is the largest single valued neutrosophic set contained in both A and B.
Proof: It is direct from the definition of intersection operator.

Definition 12 (Difference) The difference of two single valued neutrosophic set C, written as C = A \ B, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x) = \min(T_A(x),F_B(x)), \quad (29)$$
$$I_C(x) = \min(I_A(x),1-I_B(x)), \quad (30)$$
$$F_C(x) = \max(F_A(x),T_B(x)), \quad (31)$$

for all x in X.

Example 5 Let A and B be the single valued neutrosophic sets defined in Example 1. Then A \ B = ⟨0.2,0.4,0.6⟩/x_1 + ⟨0.5,0.2,0.3⟩/x_2 + ⟨0.5,0.2,0.4⟩/x_3.

Now we will define two operators: truth-favorite (Δ) and falsity-favorite(∇) to remove the indeterminacy in the single valued neutrosophic sets and transform it into intuitionistic fuzzy sets or paraconsistent sets. These two operators are unique on single valued neutrosophic sets.

Definition 13 (Truth-favorite) The truth-favorite of a single valued neutrosophic set A is a single valued neutrosophic set B, written as B = ΔA, whose truth-membership and falsity-membership functions are related to those of A by

$$T_B(x) = \min(T_A(x)+I_A(x),1), \quad (32)$$
$$I_B(x) = 0, \quad (33)$$
$$F_B(x) = F_A(x), \quad (34)$$

for all x in X.

Example 6 Let A be the single valued neutrosophic set defined in Example 1. Then $\Delta A = \langle 0.7,0,0.5\rangle/x_1 + \langle 0.7,0,0.3\rangle/x_2 + \langle 0.9,0,0.2\rangle/x_3$.

Definition 14 (Falsity-favorite) The falsity-favorite of a single valued neutrosophic set B, written as $B = \nabla A$, whose truth-membership and falsity-membership functions are related to those of A by

$$T_B(x) = T_A(x), \quad (33)$$

$$I_B(x) = 0, \quad (34)$$

$$F_B(x) = \min(F_A(x)+I_A(x),1), \quad (35)$$

for all x in X.

Example 8 Let A be the single valued neutrosophic set defined in Example 1. Then $\nabla A = \langle 0.3,0,0.9\rangle/x_1 + \langle 0.5,0,0.5\rangle/x_2 + \langle 0.7,0,0.4\rangle/x_3$.

4. Properties of Set-theoretic Operators

In this section, we will give some properties of set-theoretic operators defined on single valued neutrosophic sets as in Section 3.

Property 1 (Commutativity) $A \cup B = B \cup A$, $A \cap B = B \cap A$, $A \times B = B \times A$.

Property 2 (Associativity) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$, $A \times (B \times C) = (A \times B) \times C$.

Property 3 (Distributivity) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Property 4 (Idempotency) $A \cup A = A$, $A \cap A = A$, $\Delta\Delta A = \Delta A$, $\nabla\nabla A = \nabla A$.

Property 5 $A \cap \phi = \phi$, $A \cup X = X$, where $T\phi = I\phi = 0$, $F\phi = 1$ and $T_X = I_X = 1$, $F_X = 0$.

Property 6 $A \cup \phi = A$, $A \cap X = A$, where $T\phi = I\phi = 0$, $F\phi = 1$ and $T_X = I_X = 1$, $F_X = 0$.

Property 7 (Absorption) $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$.

Property 8 (De Morgan's Laws) $c(A \cup B) = c(A) \cap c(B)$, $c(A \cap B) = c(A) \cup c(B)$.

Property 9 (Involution) $c(c(A)) = A$.

Here, we notice that by the definition of complement, union and intersection of single valued neutrosophic sets, single valued neutrosophic sets satisfy the most properties of classic set, fuzzy set and intuitionistic fuzzy set. Same as fuzzy set and intuitionistic fuzzy set, it does not satisfy the principle of middle exclude.

5. Conclusions

In this paper, we have presented an instance of neutrosophic set called single valued neutrosophic set (SVNS). The single valued neutrosophic set is a generalization of classic set, fuzzy set, interval valued fuzzy set, intuitionistic fuzzy set and paraconsistent set. The notion of inclusion, complement, union, intersection, have been defined on single valued neutrosophic sets. Various properties of set-theoretic operators have been provided. In the future, we will create the logic inference system based on single valued neutrosophic sets and apply the theory to solve practical applications in areas such as expert system, information fusion system, question-answering system, bioinformatics and medical informatics, etc.

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Strategy on T, I, F Operators.
A Kernel Infrastructure in Neutrosophic Logic

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Abstract: The paper presents an initial explorations on T, I, F operations based on genetic concept hierarchy and genetic referential hierarchy, as a novel proposal to the indeterminacy issue in neutrosophic logic, in contrast to the T, I, F values inherited from conventional logics in which those values would fail to demonstrate the genetic aspect of a concept and accordingly loose the connection between generality and practicality. Based on the novel definition of logic and on the relativity of T, F concept, it illustrates that T, F are hierarchical operations which inter-consist and inter-complement each other, that "I" relates to a learning behavior profiled by an inspiration from I-ching, and that the neutralization operation, as the means to solve contradictions, will eventually come to the unification of opposites, leading to the fundamental issues in Buddhism and such alike. It also implies that Buddhism and Daoism are not religions.

Keywords: Neutrosophy, Neutrosophic Transdisciplinarity, Indeterminacy, Genetic tree, Referential system, Neutralization, Unified yin-yang

1. The self-contradictory of conventional mathematics

Conventional mathematics has its fatal defects in itself:

- The logic it exploits is nothing more than a misleading concept.

  In (Liu, Smarandache [2]) we have launched a strong argument in the misleading definition of logic. Even the simplest logic as "The earth turns around the sun" and "I'll visit him if it doesn't rain and he is in" can lead to ambiguous or contradictory actions of agent (Liu [1]), as shown in:

  1) Fact: a belief rather than truth
  2) Logic: dependent of situations, not absolute
  3) Logic is negating itself
  4) Logic is only one perspective of learning, not an independent entity
  5) As a part of learning, logic is dynamic
  6) As a part of learning, logic is multilateral
  7) Logic is always partial
  8) Illusion and creativity

  Logic should be, in our opinion, ***a tradeoff operation*** in order to adapt to its environment. Then a specific model becomes such a tradeoff between ideal philosophic description and practical application description, in the hierarchy from philosophic layer down
- The conception in mathematics should be more a unifying operation of tradeoff than rigid definitions or axioms.

  The common silliness in mathematics is that: the more specific, the more contradictory.

  In (Liu [2]) Liu Feng stands that conceptual name actually acts as a tradeoff to unify the diversity of concepts—it comes as the outcome of contradiction:

  - Everything can be named, but never absolutely proper. It is a name, but never a perfect name
  - Name is always subjective, relative to the perception and perspective of observer
  - Name itself implies anti-name. Whenever there is a name, it can never be a perfect name.
  - We are cheated by or trapped in those created by ourselves. First, there is only relative name, no absolute name. Second, name actually acts as a tradeoff to unify the diversity of concepts—whenever there is name, there is contradiction as well.
- Mathematics tries to reach the most complete specifications, but where on earth are they (Liu, Smarandache [1])?
  - There is no absolute completeness in the world; if there is, it is nothing more than

our subjective beliefs, or we were gods.

- There remains an infinite integration procedure of dynamic accumulation of both consistent and inconsistent knowledge, during which the more incomplete knowledge is modified, revised, and adapted as further proposals, which tends to be verified and improved into less incomplete one.
- Accordingly, **if there is complete mathematics, it would be not mathematics at all** (at least not in the conventional sense).

True, conventional mathematics is really negating itself. Let's then try mathematics without mathematics, just as Florentin Smarandache did in his avant-garde movement in literature (Smarandache):

- Let's do literature... without doing literature! Let's write... without actually writing anything.

How? Try mathematics without completeness! Try logic, set, and probability without determinacy, in contradictory dual perspective. Try concept in contradictory definition. This open methodology, I believe, should be the original intention of neutrosophy as a new branch of scientific philosophy.

2. The indeterminacy of nature and the role of neutrosophy

Even a definite logic like "I'll visit him if it doesn't rain and he is in" can lead to ambiguous actions (Liu [1]):

- I have confirmed that it hasn't rain whole day and he really keeps indoor, but when I become confident of it, it is too late to go
  —Pure logic implies indeterminacy itself.
- The truth value of not-raining / raining is (0.3, 0.7), so I decide not to. But just as I start something else, the clouds promises impending shine, so I change my mind.
  —Human behaves in partial way, e.g., in mood.
- Wait until the truth value becomes (1,0), but it never happens.
  —Human cannot behaves in extremity of logic.
- I decided to wait until the truth value of not-raining, raining reaches (0.8, 0.2) and that of he-is-in, he-is-not-in reaches (0.8,0.2), but how can I react to the case when I ring his door, he is out punctually at the moment or he is in the toilet?
  —Human needn't obey the logic he made.

Now let's take a more definite logic: "The succession of spring, summer, autumn and winter on the earth is the absolute truth." Could anyone prove it? No, not even if we exist or not (Gershenson).

- We have just mentioned that fact is merely a belief—it is believed absolute because hard have people doubt the validity; People have long been holding this belief which is also indeterminate.
- Statistics itself exhibits indeterminacy—it only reveals possibility, not determinacy.

Daodejing shows that everything in the world is indeterminate by nature (Liu [2]).

> "Dao, daoable, but not the normal dao; name, namable, but not the normal name." We can say it is dao when referring to the natural law, but it doesn't mean what we say. Whenever we mention it, it is beyond the original sense. We can call something by name, but it doesn't mean what we call. Whenever we call it, it is beyond the original sense too.

Buddhism illustrates that all the appearances in the universe are made up of uncertainty (voidness) (The Diamond Sutra):

- The physical appearances mentioned…are not physical appearances. All appearances are empty and false. If one sees all appearances as no appearances, then one sees the Thus Come One (the truth as the unity of appearance and essence, author's note).
- The view of self, view of others, view of living beings and view of a life ***is not*** the view of self, view of others, view of living beings and view of a life. Therefore ***they are called*** the view (appearance of, author's note) of self, view of others, view of living beings and the view of a life.
- As to speaking Dharma (the truth, law of nature, author's note), no Dharma can be spoken. Therefore it is called 'speaking Dharma'."
- All conditioned (intentioned, author's note) dharmas (the truth, laws of nature) are like a dream, an illusion, a bubble or a shadow, like dew or like a lightning flash. Contemplate them thus.
- A Bodhisattva (a being "gifted" with genuine wisdom, author's note)…should neither

rely on forms, nor sounds, smells, tastes, tangible objects or dharmas…he should not rely on appearances.

As the conclusion, a particular appearance is conditional, or constraint with situations. When we mention the universal appearance, we refer to the essence in infinite layers of depth, it is indeterminate in nature—independent of situations. Or: **determinacy is conditioned while indeterminacy is universal**.

The role of neutrosophy has thus been designated as the bridge between differential appearances and the integral character of individual.

- Law of Equilibrium (Smarandache):

  The more <A> increases, the more <Anti-A> decreases. One has the following relationship:

  $$<A> \cdot <Anti\text{-}A> = k \cdot <Neut\text{-}A>,$$

  where k is a constant depending on <A>, and <Neut-A> is a supporting point for balancing the two extremes.

  If the supporting point is the neutralities' centroid, then the above formula is simplified to:

  $$<A> \cdot <Anti\text{-}A> = k,$$

  where k is a constant depending on <A>.

  Interesting particular cases:

  Industrialization × Spiritualization = constant.

  Science × Religion = constant.

  White × Black = constant.

  Plus × Minus = constant.

  Everything × Nothing = universal constant, or $\infty \times 0$ ($= 0 \times \infty$) = universal constant.

  We are directing towards a mathematization of philosophy, but not in a platonian sense.

3. Mathematics based on T, I, F operators

3.1 Indeterminacy of referential system

The genetic principle of a referential system:

Concept is created only through relativity.

- It is not until "it is" is extracted from "it is not" when concept is born:

  When beauty is abstracted (Merel)

  Then ugliness has been implied;

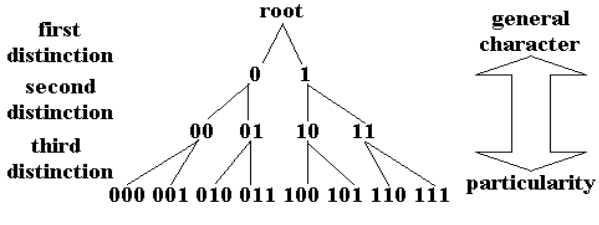
  When good is abstracted

  Then evil has been implied.

  Therefore "it is" and "it is not" inter-depend on each other (T and F inter-depend on each other).

- Accordingly (back to author), an infant builds up his primitive concept through primitive division, e.g., between light and dark, based on which (as the basic distinction) further distinctions can be made as sub-concepts.
- Accordingly, a family tree of concept comes as a cumulative (recursive) effect of distinctions, as illustrated in I-ching as how the trigrams are generated from a primitive abstraction of yin-yang (as 1, 0):

  When yin-yang is abstracted as the first pair of referential objects 1,0, we can begin abstracting those referred to 1 leading to 11, 10, same to 0: 01, 00, to get four referential objects 00 01 10 11, keeping on to have eight 000 001 010 011 100 101 110 111, … this is the way our concept family is built up.



  Concept family comes as a hierarchical structure in the above genetic step, where the nodes closer to the root denote more general classes, and a subtree rooted from a child can be

regarded as sub-concept.

- The hierarchy of attributes of concept should be generated in the same manner: first the distinction of general (or universal) attributes, then the particular ones.
- So is a referential system, more general descriptions of the referential system is closer to the root, and a subsystem is represented by a subtree.

Now that we have generalized the genetic behaviors of concept and referential system based on the basic Daoist principle in I-ching (also Book of Changes), we should note the indeterminacy of concept and the indeterminacy of referential system lying in the following issues:

- The way of split-up of yin-yang (the way of the distinction) is optional or arbitrary, one needn't obey the same way of distinguishing things as another, dependent on time-space factors, individuality, temporal mood, hidden factors, etc.
- Accordingly there can be infinite number of genetic trees to the same root.
- A tree can be attached to an arbitrary parent in an arbitrary manner as a sub-concept, sub-attribute or sub-referential-system.

As the consequence, there is no determinate referential system at all. In other words, concept and referential system are merely relative. This is the bottleneck.

3.2 The need for T, I, F operations

To avoid being trapped in these relative values (persisting in the relativity), we had better regard them as T, I, F operators to explore the relativity to different time-space domains, to different referential systems, and to hidden parameters.

- Conventional values of T, I, F (true, indeterminate, false) represented in percentages depend heavily on the indeterminacy of referential systems. They never seem perfect values in that they are incomplete or relative, so they mean more operators than figures.

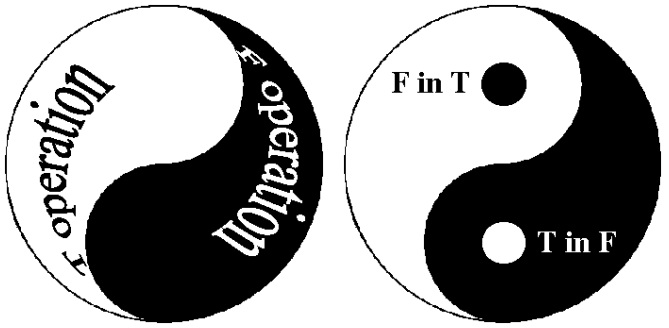
  If T, I, F values can be carried out unconditionally, there is no need to explore indeterminacy. In fact, situation changes from one, supporting T operation, to another, supporting F. Since we cannot predict these situations, the availability of T, I, F values remains doubt.

  Logic in essence is a tradeoff of desired and undesired (Liu, Smarandache [2]), there are no T, I, F values independent of environment. So T, I, F refer more to the tradeoff operation relative to referential systems, etc.

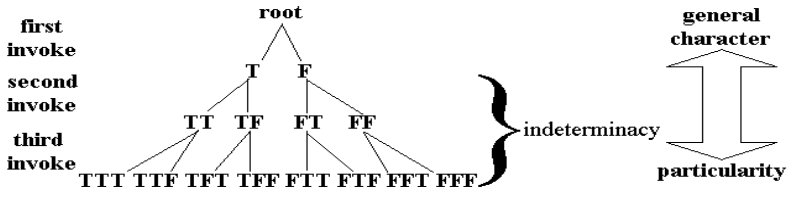
- Based on taiji figure, T and F operations work in pairs, they complement and inhibit each other (Liu, Smarandache [2]), inter-consist each other and transform into each other. This effect is improper to be represented by T, I, F values especially in percentages.

  If a single T or F operation succeeds then it becomes a conventional problem, but strictly conditional.

  In fact there are no absolute T, F operations (or values), a T operation (or value) can be F relative to another referential system (Smarandache). T implies F and vice versa, they inter-consist each other.

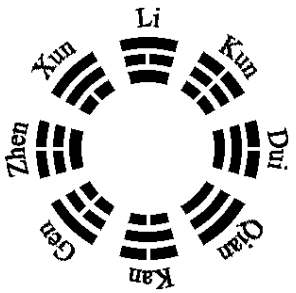


Finally T F operations (also to values) are



neutralized. Note that we are not talking to the trueness or falseness in percentages, but the tradeoff operation in recursive manner—T means feasibility, not truth.

- Indeterminate operation refers more to a learning procedure as has been profiled in (Liu, Smarandache [1]):



The further insight on contradiction compatible learning philosophy inspired from the Later Trigrams of King Wen of I-ching shows that:

When something (controversial) is perceived (in Zhen), it is referred (in Xun) to various knowledge models and, by assembling the fragments perceived from these models, we reach a general pattern to which fragments attach (in Li), as hypothesis, which needs to be nurtured and to grow up (Kun) in a particular environment. When the hypothesis is mature enough, it needs to be represented (in Dui) in diverse situations, and to expand and contradict with older knowledge (in Qian) to make update, renovation, reformation or even revolution in knowledge base, and in this way the new thought is verified, modified and substantialized. When the novel thought takes the principal role (dominant position) in the conflict, we should have a rest (in Kan) to avoid being trapped into depth (it would be too partial of us to persist in any kind of logic, to adapt to the outer changes). Finally the end of cycle (in Gen).

This philosophy shows that contradiction acts as the momentum or impetus to learning evolution. No controversy, no innovation. This is the essentially of neutrosophy (Smarandache).

- T operation may activate a routing in neural network and F inactivate or inhibit it. So neutrosophy should be applicable to neural network as it intended.

4. On the limit of neutralization

When we speak of the most general character of concept, the most general referential system or the most general logic, we are actually talking about the neutralization operation in endless depth, and we will eventually come to the point: is it yin or yang (is it a concept or not, is it T or F)? In fact, it is neither yin nor yang but the unification of yin-yang.

- When we take it as yin (or yang), we unintentionally raise a contradiction (Liu, Smarandache [1]) in this split-up of yin-yang, since “it is” or “it is not” is strictly relative.
- More we uphold the relative opinion, more contradiction underlying. Therefore we can only solve it in the opposite way: neutralization.
- The infinite neutralization process will eventually reach the reunification of yin-yang as if there will not seem any evidence of yin or yang (there doesn’t seem any evidence of the distinction between “it is” and “it is not”)
- Then we will have reached a unification of intentionality and unintentionality—this is the basis for the universal conception.
- Is it possible to everyone? Sure, as long as we stop upholding everything even this doctrine (as if there is no distinction between stop and nonstop).
- In fact, everything is void in nature in the unification perspective, as are illustrated in endless depth in Buddhism. In this referential system most things we normally do worth nothing, what seem worthy to do can seem nothing in our common referential system. We should point out that the “doing nothing” in Daodejing seems rather a distorted translation from Chinese term wuwei.
- This is the starting point of the supreme wisdom with which we can understand ourselves and understand the universe.

5. Concluding remarks

There is no absolutely mathematics in reality, nor to neutrosophic logic discussed here, which is

proposed in a “abnormal” style—indeterminate style.

Determinate mathematics is only applicable in well defined closed (not open to indeterminacy) models which are assumed determinate in a tiny fraction of their domains: time, space, etc., analogous to the differential aspects in calculus (we can assume its determinacy in the smallest fraction of the domain).

If there is a universal mathematics trying to provide the universal resolution, it must be void in form, otherwise it must have stood on a default (particular) referential system. It may be analogous to the integral of opposite parts.

As the neutralization of the opposites, neutrosophy opens a new space of indeterminate mathematics.

What we call sciences are strictly conditional and relative to our common referential system—they can be pseudosciences in other referential systems. What we call religions can be sciences. Buddhism and Daoism may not be religions based on the most perfectly unified field perspective to which our present sciences limit. Through years of practice of Buddhism, it exhibits the supreme relativity in the most unified referential system. It is the limitless wisdom.

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Toward Dialectic Matter Element of Extenics Model

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Abstract: Based on the authors intensive investigation on the oriental dialectics, the paper presents a novel theoretical frame of matter element in the world leading science, extenics dealing with inconsistency or incompatibility, covering the widest range of application area from informatics, system engineering to management and finance. The dialectic matter-element is defined as the integral of all existing and prospecting ones based on all the infinite possible cognitive models. The novel model serves as the origin of constraint matter elements, the unity of both state description and cognitive action (cognition force with respect to neural science), a latent part of extenics, and possibly as essence of matter element. It explains, in a novel perspective, the origin of a name, and uncovers the source of contradiction and even the impetus of cognition.

Keywords: Extenics; matter element; dialectics; cognition; contradiction; identity; reference frame; latent part; imaginary part; extension.

1. Background

The matter-element theory based science, extenics funded by Chinese mathematician Cai Wen, has experienced 28 year's difficulties. Although it becomes the world leading science, we can only say it may seem more a simulation of dialectics than real. Regarding as a source of prospecting dialectic mathematics and methodology, there is no reason to overestimate or to underestimate its value.

Extenics believes in a latent part (imaginary part) description, and then what is the imaginary part of extenics? How to extend extenics itself? As to the scope of mathematics, some mathematicians in the world, American or French we coped with in new generation fuzzy theories or neutrosophic theories and information fusion, have included so called non-mathematics into mathematics, such as "Book of Changes". Yes, mathematics has no scope as long as human does not fix it. To our observation seldom have mathematicians provided anything valuable on the origin issues in this paper.

The point we make out to be a basic logic issue: $A \neq A$, in that there are two aspects of A: symbolic A and the extension of A, where symbolic A refers more to an instant appearance relative to a default scenario, and people refer more to the extension of A that is subject to an evolution chain in which the integral of all the appearances of A seems inexpressible.

To start with we need to induce the background extenics – a traverse discipline of philosophy, mathematics and engineering aiming to solve conflicts, contradictions, inconsistency and incompatibility.

1.1. *The world leading science: extenics.*

"The theoretical frame, which takes basic-element theory, extension set theory and extension logic as pillars, and the special extension methodology are formed. The applied technology in various fields is called extension engineering. Extension theory, extension methodology and extension engineering constitute extenics. What is called contradiction problem is the problem whose goals cannot be realized under the existing conditions. During the study, the researchers of extenics have found there are all kinds of contradiction problems in many engineering fields, such as management, controlling, computer technology, artificial intelligence, machine and electronic engineering, etc. Then, do we have any regular technique to solve contradiction problems? Can we establish a set of method to deal with contradiction problems? This is the start of extenics research. [3]"

"Logical cells and extension models of extenics: Mathematical model can deal with a lot of precise problems, but cannot deal with the problems under certain conditions such as the one in the famous ancient Chinese story "Prince Caochong weighs an elephant", whose goals and conditions are incompatible. The reasons are, when one solves contradiction problems, he has to consider the things themselves and their characters besides the quantity relation; parts of the transformations solving contradiction problems are quantitative and some are qualitative; classical mathematics studies the definite things, but to solve contradiction problems, we have to consider the transformations of things (including quantitative change and qualitative change).Therefore, mathematical model is difficult to describe the process of solving contradiction problems. To deal with all kinds of contradiction problems in real world with formalized method, firstly we should study how to describe the various things. For this purpose, extension theory establishes matter-element R = (N, c, v), affair-element I = (d, b, u) and relation-element Q = (s, a, w) (be called jointly basic-element) to describe matter, affair, and relation. They are the logical cells of extenics. The formalized models describing information, knowledge, intelligence and all kinds of contradiction problems with basic-elements are called extension models. With the extension models, we are able to take advantage of extension theory and extension method to generate many strategies to solve contradiction problems according to the extensibility of basic-element. Extension theory has three pillars: basic-element theory, extension set theory, and extension logic."

"Basic-element theory: Extensibility of basic-element and conjugate nature of matter-element are the core of basic-element theory. And the important feature of extension theory is to represent these natures with formalized symbols. They are the ground on which the strategies solving contradiction problems are generated. Extensibility includes divergence, correlativity, implicative nature and expansiveness. Conjugate nature includes materiality, systematicness, dynamic nature and antithetical nature."

In extenics contradiction comes from "habitual field and the variability of problems". See Ref. 4 for more on theoretical architecture.

1.2. *Matter element theory: the base ground of extenics:*

We consider an object and v the quantity of N about c. These N, c, and v are called three essential factors of a matter element. Here $v = c(N)$ reflects the relationship between the quantity and quality of an object. If an object has multiple characters or more than one of them are needed to be listed out, then a multi-dimensional (for example, n-dimensions) matter element can be defined as:

$$R=\begin{pmatrix} N, & c_1, & v_1 \\ & c_2, & v_2 \\ & \vdots & \vdots \\ & c_n, & v_n \end{pmatrix}.$$

When an object is dynamic or its dynamical character must be studied, then a dynamical matter element could be defined as $R=(N(t),c,v(t))$. The dynamical matter element expresses the variations of the object with time.

1.2.1 *Extensibility of matter elements*

The key point of treating conflict problems is to study the principal natures of matter elements. A researcher must be creative and jump out from customs and try to expand the thought when one is dealing with the conflict problems by using of matter element methods. The extensibility includes characteristics of diffusion, conjugation, interaction, containing and extension.

(i) Divergence natures of matter elements

Nature 1: One object may have multiple characters, it is simplified as single object multiple characters and noted as $N\prec(N,c,v)\prec\{(N,c_1,v_1),(N,c_2,v_2),\cdots(N,c_n,v_n)\}$, $c_i\in C$ and $v_i\in V$. Here the sign $\prec$ stands for extensible.

Example: Let N =a piece of paper, and $c_1=could\ be\ fold\ up$, $c_2=water\ absorb$, $c_3=thickness$. Then we have a matter element

$$R=\begin{pmatrix} N & c_1, & v_1 \\ & c_2, & v_2 \\ & c_3, & v_3 \end{pmatrix}.$$

When a piece of paper is to be used, one can not only think about the normal usage of it, but also the other characters should be calculated, for example, some papers are overlapped and folded up to a box. Another example is that suppose one has been asked to arrange four equilateral triangles with six sticks of match. When an unsuitable condition is added to the equilateral triangles then the problem comes to be a conflict one:

$$R=\begin{pmatrix} Equilateral\ Triangles, & nummber, & 4 \\ & position, & on\ a\ plane \end{pmatrix}.$$

Nature 2: By a single character there may exist many objects, it is simplified as multiple objects single character and noted as $N\prec(N,c,v)\prec\{(N_1,c,v_1),(N_2,c,v_2),\cdots(N_n,c,v_n)\}$, $N_i\in N$ and $v_i\in V$. An example to use this nature is that an American purchaser named Mailce was pointed to buy fire-resistant boards for floor decoration but this kind of materials were sold out in the market. What he did was to substitute the boards with fire-resistant papers. The demand was satisfied and the cost was greatly reduced, too. To describe this question with matter element theory, so we have a normal matter element

$$R_1 = \begin{pmatrix} Fire-resistant\ Boards, & fire-resistant\ function, & a_1 \\ & price, & b_1 \end{pmatrix}$$

and $$R_2 = \begin{pmatrix} Fire-resistant\ papers, & fire-resistant\ function, & a_2 \\ & price, & b_2 \end{pmatrix}.$$

So long as $a_2 \geq a_1$ and $b_2 \leq b_1$, the substitution is reasonable. Later Mailce developed this thought to "Value Engineering Techniques", so we could say that the basis of "Value Engineering Techniques" happened to be the extensibility of matter elements.

Nature 3: For different parameters an object about a character could have different quantities. It is called simply as single object single character with multiple quantities and could be formulated as $N \prec (N, c, v(t)) \prec \{(N, c, v(t_1)), (N, c, v(t_2)), \cdots (N, c, v(t_n))\}$, here $t_i \in T$. An example to support this nature is that by man made satellites more and more channels and signals need to be emitted but too many transmitters would make the satellite overloaded. To overcome this difficulty, a scheme of "divide time to fit multi-channel" was designed. To describe this question with matter element theory, it can be written as: $R = (transmitter\ A,\ \ emit\ parameter,\ \ a) \prec (transmitter\ A,\ \ emit\ parameter,\ \ f(t))$, here

$$f(t) = \begin{cases} a_1, & t \in (n,\ n + \dfrac{1}{17})\ Sec. \\ a_2, & t \in (n + \dfrac{1}{17},\ n + \dfrac{2}{17})\ Sec. \\ \cdots & \cdots \\ a_{17} & t \in (n + \dfrac{16}{17},\ n + 1)\ Sec. \end{cases}$$

and 17 channels with only one transmitter is pre-supposed.

(ii) Conjugation of matter elements

Just like the domain definition for complex numbers, matter elements include real and imaginary parts, too. For a given object N, it could be written as $N = \mathrm{Re}\, N \oplus \mathrm{Im}\, N$, where the $\mathrm{Re}\, N$ is the real part of N and $\mathrm{Im}\, N$ the imaginary part of it. A kind of product as a matter element has two sides of valuation. One is the product itself, the real part of it. The effective of the brand of the product and the reputation of the producer belong to the imaginary part of the matter element. There are many successful examples of using the imaginary parts of matter elements to civil decision makings and even in military directions.

(iii) Interaction of matter elements

When there exist certain interactive dependencies between objects about the quantities of some characters, it will be called interactions of matter elements. For a given matter element R, all the about objects or about characters interactive dependant matter elements are called a interactive dependant network. A varying of a quantity about a character of an object in a net may yields relative varying in quantity of the same character of another object in the same net. This character may be used in market planning. A kind of food for children, its price was 6 yuan/sack. For being more attractive to the customers, the company added small playthings with price of 0.5 yuan/piece in the sacks and the new price for a sack is 10 yuan/sack. And then the sales volume

greatly increased and the company got a big mount of profits. This is an example of taking up transform using the interactive net nature of matter elements.

(iv) Implication of matter elements

If $A@$ and then certainly $B@$, this called A implies B, notified as AB. Here @ stands for existence. The relation between A and B is called Implication.

(v) Extension of matter elements

The extension abilities of matter elements describe the combinations, decompositions and substitutions abilities between different matter elements.

1.2.2. *Matter element transformation*

After introducing the concept of extensibility to matter elements, transformations on objects, characters of objects and quantities of characters could be as specially designed calculations (operations) applied to matter elements. These calculations can handle the transforming of quantities and also the qualities of objects.

The extensibilities of matter elements pointed out the main thought methods of solving practical problems and the methods, policies and knacks to solute problems could be described with a series of matter element transformations. There are four basic transformations of matter elements: substitution, resolution, addition/subtraction and expansion and also four basic calculations for matter elements: and, or, multiplication and inversion. Some researchers studied the philosophical subjects of matter elements and extenics.

2. The Origin of Dialectic Matter Element Issue

To discuss the true face of mater element we need to doubt that his theory might have been based on the same paradox as most sciences do. Although one may congratulate Prof. Cai for the realization of transforming contradiction into consistency, it seems still, to our intuition, a prototype. On the other hand mental cognition is prone to elude us.

It is true that extenics has inherited to some extent fuzzy/neutrosophic logic and fuzzy/neutrosophic set which we are always criticizing, e.g. to matter element R = (N, c, v), we cannot help asking: “How does one know that matter element R is of name N, character c, and value v”.

In his paper “To be or not to be, A multidimensional logic approach” Carlos Gershenson has generalized proofs on:[5]

- Everything is and isn't at a certain degree.
- Nothing can be proved (that it exists or doesn't) (authors: relative to our current intelligence).
- I believe, therefore I am (i.e., I take it true, because I believe so).

“Most of us will believe it exists, though. The same thing happens for existence of psychic powers, aliens or the chupacabras (let's avoid god as long as possible...). You can't prove their existence rationally. But some people believe in them. And you can't say they're absolutely wrong because you can't prove their non-existence. Then, there are some people who believe in them.”

How much do men understand the nature? See the BBS discussion about time, space and matter in China Science and Technology Forum (years ago posts, but new links in Chinese provided):

- Space is not empty (http://blog.china.com/u/060703/2812/200607/8128.html)

“Since 1998 cosmologists have found that the universe expands in extraordinary speed. The driving force is a mysterious hidden energy. According to astronomic observations the hidden matter would make up 90% of the universe and still remains unknown to men … Hidden energy, hidden matter are still black holes in science.”

BBS discussion: “Simply speaking, space is hidden energy, and hidden energy is space. Space is matter. It can be extremely harmful to cut them apart or to refer the hidden energy to some pure antimatter.”

- Time is fixed without change – the absolute miss of relativity (http://blog.china.com/u/060703/2812/200607/8124.html)

"Time is a concept, a measure of the universe vacant space through the concept of space. Time is not matter. It is an invisible and intangible 0 in space – Time does not exist to moving matter… Light speed is capricious ..."

Beside the forum discussion, there are more to see: "Distance and time are 'illusory'; Things can be created from 'nothing'; The origin of the universe. [6]"

As Dr. Odenwald acknowledges, "We don't have a full mathematical theory for describing this state yet, but it was probably 'multi-dimensional'…Nothingness (that gives rise to the present universe) was not nothing, but it was not anything like the kinds of 'something' we know about today. We have no words to describe it and the ones we find in the Oxford English Dictionary are based on the wrong physical insight."

And also the Japanese Emoto's experiment illustrated in the Chinese version of Ref. 6:

"The crystal shape of water varies with human mental and lingual actions – It appears beautiful to kindness and ugly to malice." See Ref. 7 for more in English.

Is human mind highly developed? Not at all in the reincarnation chain.

For valid introduction of scientific research on reincarnation see Ref. 8. An American psychologist holds that "70% of human can clearly recall his past life through hypnotization", and there exists such a kind with inborn recollection. [9]

Modern science relies on instruments, eventually out of eyes, to derive conclusion - To put it bluntly, via sight, hearing, smelling, taste and touch senses in normal state. All achievements of current and prospecting science attribute to a self – I find, I deduce, I devise, I summarize, I set up … However science shows inability to the essence of self. How surprising! How do those rules and theories hold valid? One feels more ignorant to such issues as 'who am I', 'What is the *noumenon*, ontology of mind[10]'.

On the contrary, scientific discoveries in some aspects are approaching or partially proving Albert Einstein's point that space, time and matter are illusions of human cognition.

Therefore the statement "Matter element R is of name N, character c and value v" actually stands for: "Matter element R ***possibly*** has name N, character c and value v - for indication only." To carry out the discussion, I need to trace the origin of the name R.

3. Dialectic Matter Element

Matter element never exists alone. It relies on a cognition system or cognition model existing in a particular cognition background. For convenience we attribute both factors to cognition model which is supposed to imply the background. Prof. Cai calls it condition.

What is space? An unoccupied place – no space exists beyond the occupation, and therefore it is assumed and void. As to time, merely a procedure of matter change, and no time exists beyond this change. How can one calculate time if there were no celestial body? Therefore time relates to a base ground of matter change – it is a mental fabrication. [10]

Then the relativity, or condition in Cai's words, of matter element, for matter element R=(N, c, v). Suppose that under the same condition person 1 sees it as $R_1=(N_1, c_1, v_1)$, but person 2 sees it as $R_2=(N_2, c_2, v_2)$, …, person k sees it as $R_k=(N_k, c_k, v_k)$. Are they same? If exactly so in mental perception, not only in language or name, we thing it would be ridiculous, for the same name is reflected with different things in consciousness even in different instances of time. Now that R_1, R_2, …, R_k are different, where does R=(N, c, v) come from? ***What are the distinctions from other matter elements discovered?*** This is the start of our dialectic model: matter element exists dependently on cognition model and cognition procedure, otherwise it has no meaning.

3.1. *Definition*

Our focus is on the origin of R and seen (in mental reflection) as R_1, R_2, ..., R_k . There is an old saying in China: "Ten thousand time's change does not depart from the original stand" – Let's define the original stand R. If R_1, R_2, ..., R_k are instance, partial or distorted appearances of R, it would be decisive to describe the R. For a man in different clothes at different time, we never regard his name equivalent to an instance in one particular dress, but we do regard in all the possible clothes. In other words, R signifies the identity of all the instant appearances relative to a particular perspective. In philosophy we call it the unity of opposites, or the identity in a contradiction.

Now that every $R_k=(N_k, c_k, v_k)$ is partial, we need to describe the significance of the integral

$$\int \sum \boldsymbol{R_k \cdot d(cognition\ model\ k)} \tag{3.1}$$

as the relatively more general representation of matter element, its limit is defined:

$$\boldsymbol{dialectic(R) = Lim_{n\to\infty} \int \sum_{k=1\ to\ n} R_k \cdot d(cognition\ model\ k)} \tag{3.2}$$

as the most general model, or *dialectic matter element*, which should be recognized as the identity.

However does it converge? Never mind, since it signifies the multidimensional cognition procedure on a single element, and $\boldsymbol{R_k}$ acts as a particular case in cognition model k, one may confirm that he will eventually find the ultimate reality of the multidimensional appearances. How can it come?

Let's consider:

$$\boldsymbol{d(cognition\ model\ k) = d(cognition\ model\ k)/\ d(cognition\ model\ C) \cdot d(cognition\ model\ C)} \tag{3.3}$$

where cognition model C denotes the ultimate common reference frame or reference system, and have

$$\boldsymbol{dialectic(R) = Lim_{n\to\infty} \int \sum_{k=1\ to\ n} R_k \cdot d(cognition\ model\ k)} \tag{3.2}$$

$$\boldsymbol{= Lim_{n\to\infty} \int \sum_{k=1\ to\ n} R'_k \cdot d(cognition\ model\ C)} \tag{3.4}$$

where $\boldsymbol{R'_k = R_k \cdot d(cognition\ model\ k)/\ d(cognition\ model\ C)}$, i.e., matter element R_k in the ultimate common reference frame C's respective.

The point: does there exist such a common reference frame? This can be a problem of unified field theory which science is not able to prove. However based on our persistent investigation on oriental classical culture we believe that the mutual ultimate common reference frame exists in everyone's hidden consciousness, and everyone can realize the genius to see the ultimate truth in the ultimate life.

Meaning: R in broad sense, or the identity in different scenarios.

Although Prof. Cai uses the term "*quanzheng wuyuan*" (literal: holo-character matter element) to describe, we still find it improper to the kernel issues. See the cognition model transformation implied:

What is R - R=(N, c, v) – How do you know – My feeling – Is you feeling valid – Never invalid – How to verify – Myself is the verification – What is yourself - … .

Now to the point one can go no further, nor can science do, especially on something with reincarnation. Never mind. Since all the possible cognition models are included, even of the most genus species in some sense, or the ultimate species in evolution, it sees the essence. For this reason we propose the imaginary part in description as in number theory:

$$\boldsymbol{dialectic(R) = imaginary\ part(R)} \tag{3.5}$$

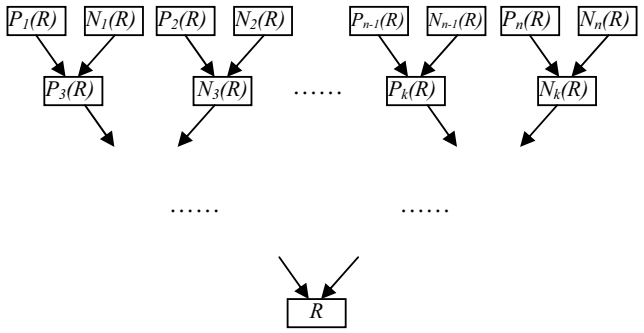
to distinguish from those based on the incompleteness discussed above, and from Prof. Cai's latent part definition lt(R).

As to ***completeness***: Just consider:

$\int$***PositiveImage · d(a positive cognition model) + NegativeImage · d(the negative cognition model)***

Is it a more complete description?

Consider that ***R*** can be the dialect element in some aspect or scenario, but not in another, it is necessary to make it recursive (*P*: positive; *N*: negative) toward the endless chain :



Is it a more complete description now?

We would not use the term ontology which is widely used by scholars, since dialectic matter element can be inexpressible in some sense, but needs a name.

3.2. ***Properties***

It defines existence via innate relations, and indicates the origin of contradiction.

Identity: the similarity of R'_k in view of a particular common cognition model, and thus the dialect sense of the matter element name is derived. This property uncovers the unity of contradiction implied in the name.

As to the term "identity", we originally refer to a property in a Chinese theory "On Contradiction", which means compatibility, consistency in the opposites. We can also let this issue open to the future.

Prof. Cai may express this property with "truth value > 0".

Contradiction: Although there exists identity, and thus exists a name, one cannot exclude the identity of opposites. The driving force of cognition lies just in these opposite cognition models. Extenics aims to contain contradictions and transform them, but might not be clear with such issue that contradiction serves as the source or impetus of evolution. And more in fact, contradiction best indicates characteristic of matter, which is the innate property distinguishing one name from another, and one cognition model from another.

Prof. Cai may express this property with "truth and false value = 0".

Unknown property: identity fails to appear, so makes one feel unknown, or, no one regards the various $\boldsymbol{R_k}$ as the same existence.

Dialectic property: even though one sees identity, contradiction, unknown properties etc. but never sticks to, never asserts anything. He remains peace and quiet undisturbed, and in this way he relies on his deepest instinct to meditate, to touch the hidden forms of existence, and finally toward a kind of freedom, instead of logical means of truth or false. In other words: never arbitrarily assert anything (any ***R***) before the endless purification.

Procedural property: Matter element is not only a data representation, but also a deductive action, such as:

- Confirmative deduction: deduction based on supporting R= (N, c, v)
- Opposing deduction: deduction based on negating R= (N, c, v)
- Contradictory deduction: deduction based on both supporting and negating R= (N, c, v). Extenics is designed to conduct this.
- Unknown behavior: I don't know, but may reserve the question.
- Those above signifies that a dialect reader is ready for both directions, pros and cons, or at a simultaneous parallelism, seemly neutral but not stand – He stands on no background, sticks to no logic, and therefore to be impartial.

- Dialectic behavior: neither to conduct positive deduction, assertion nor negative ones. Seldom intentionally resolve to settle conflicts or contradictions. In stead of sticking to what he understands, one remains calm, undisturbed in meditation, cultivating his innate instinct and finally reaching an insight or even a kind of freedom. Machines however are not apt at this.

Mechanical effect: Multiple deductions may coexist in a single cognition process, and the brain is just in the state of temporary balance under these multiple force actions. Therefore R = (N, c, v) is rather a cognition process or equilibrium property under cognition forces than a description of cognition state which would have no sense apart from a cognition process. Therefore R= (N, c, v) rather refers to behavioral action or force action than a concept or truth-false value based on some criteria. We hope this mechanical issue can reach neural or biological sciences.

4. Significance

This model describes matter existence via identity property, describes matter distinctions via opposition property, and describes deeper relation of matter or characteristic via unity of contradiction. Just because this integral description implies the unity of contradiction based on opposite cognition models, it has the power to depict characteristic of matter.

In the extension perspective, one can develop extension behaviors:

- Positive extension: transformation based on identity property, to discover life-force of element
- Negative extension: transformation based on opposition property, to discover anti-life force of element
- Contradiction extension: transformation based on both identity and opposition properties, to discover characteristic and essence of element
- Unknown state: In case of unclear identity property, propose the question.
- Identity with others: to describe similarity.
- Opposition to others: to describe distinction, thus makes people discriminate things.

Since an element may have the multiple propensities as mentioned above, it would be more dialectic to base our descriptions on force action contrast, rather than on truth-false values which is hard to indicate multiple mechanical actions.

This description of dialectic matter element model has reached the unity of representation and action, of status and procedure, of static and dynamic states, of traditional logic and neural net, of narrow sense and broad sense, and of outside surface and essence.

5. Concluding Remarks

Dialectic processing suggests a revolution in information technology – the extenics oriented knowledge reform and the impact on cyber culture. It would redefine the scope of mathematics, as some pioneer mathematicians did.

As the difference between human and machine, human is precious with his wisdom to know himself. On the contrary, machine can be never more than a logic notebook. In this point of view dialectic cognition should not rely on some intelligent machine – never mount an additional head on your existing one, to make a big fuss beyond mind.

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Self Knowledge and Knowledge Communication

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Abstract: To deal with tough issues in knowledge management, the paper reexamines knowledge structure and has proposed a novel knowledge communication model based on the oriental cultural foundation. Knowledge discovery is not a simple addition or accumulation of information, but needs a feasible structural description. As an oriental approach, the paper constructs the structure on the bases of unity of opposites and their neutralities (as in neutrosophy; neutralities are the included middle of two opposites), which stresses the individualized self knowledge pattern or self knowledge structure. It signifies the dynamics in knowledge management – the principle of attraction between opposite natures and the neutralities in between them. It implies an identity in the opposites which serves as the impartial knowledge (since the opposites tend towards their neutralities), the completeness of knowledge, which is defined in the paper as imaginary part of knowledge.

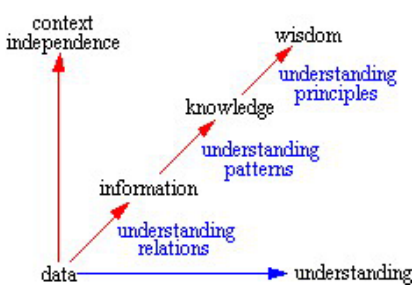
Keywords: knowledge management, knowledge communication, self knowledge, consistency, inconsistency, contradiction, knowledge pattern, imaginary part, latent part, partial knowledge

1. Introduction: the coexistence of consistency and inconsistency

By knowledge management scholars mean an evolved "unique process and methodology by which people engage in deep conversations and explore paradoxes and conflicts to come up with a shared conceptual framework that encapsulates their inquiries, stories, shared vision, intentions, competencies, desired outcomes, possible scenarios and actions"[Kaipa 1999]. However, how can we organize such conversations containing paradoxes and conflicts? We would find consistency and inconsistency be the crucial issue.

Consistency and inconsistency is the main account in information discrimination or knowledge discrimination, and their boundary is rather uniformly defined by some well applied standards. However, does there exist a global knowledge king that rules the world in perfect consistency? How comes the word "consistency"? In fact consistency implies inconsistency, we are afraid. And more: Does computerized knowledge really educate people or fool the world?

"Knowledge management is the hottest subject of the day. The question is: what is this activity called knowledge management…A collection of data is not information. A collection of information is not knowledge. A collection of knowledge is not wisdom. A collection of wisdom is not truth" [Fleming].



"While information entails an understanding of the relations between data, it generally does not provide a foundation for why the data is what it is, nor an indication as to how the data is likely to

change over time. Information has a tendency to be relatively static in time and linear in nature. Information is a relationship between data and, quite simply, is what it is, with great dependence on context for its meaning and with little implication for the future … When a pattern relation exists amidst the data and information, the pattern has the potential to represent knowledge. It only becomes knowledge, however, when one is able to realize and understand the patterns and their implications. The patterns representing knowledge have a tendency to be more self-contextualizing … Wisdom arises when one understands the foundational principles responsible for the patterns representing knowledge being what they are. And wisdom, even more so than knowledge, tends to create its own context." [Bellinger,2004]

As mentioned above, information embeds knowledge which in turn embeds wisdom, or: information embeds a hidden part behind the context, and the hidden part seemingly differs from one another but share a uniform expression, e.g., x theory – although inconsistent in context but consistent in patterns, or inconsistent in patterns but consistent in principle. Sure, human is apt to discriminate information in principle, and build up consistency in this level or even higher, but it still remains a blank in management theory that in which way human reconstructs the consistency, and how people restructure knowledge. Hence the question to knowledge management: should the knowledge structure uniform? Is there any way to the customized structure?

"For example, every person defines it in a particular way based on their affiliations, interests and past experience" [Kaipa 1999]. "Let us explore a societal example that is outside the context of technology."Cults" come about because the cult leader presents a seemingly coherent and "complete" story to the followers, thus making their actions look consistent within a framework. To the "mainstream" population, the same story looks meaningless or absurd because they do not share the same set of principles. Our judgment of the "validity" of such knowledge is determined by the set of values and principles by which we live. Understanding comes when we begin to distinguish our own validation approach and why we do what we do. Each of us has a set of standards (which we have so far been calling values and principles resulting from experiences and biases) that we acquire, mostly unconsciously, from a variety of sources such as family, religion, culture, media, and society. These internalized standards give meaning and understanding to deeper existential questions that each of us ask ourselves at different times in our lives. The more we understand our own dilemmas and inner guidance system that we help navigate our life, the more we consciously and congruently understand the principles by which we live."

In a search of information, the site visitor is sure to look for something to meet his personal background, and the sites tries to recognize his internal map and provide as necessary as possible. However, based on our contemporary observations, the kernel structure of knowledge is more or less standardized, that have assumed that visitors should abide by the relatively uniform standard. Then the terrible outcomes: Should the whole world be instructed by the uniform tutor? Should the whole world abide by the same knowledge pattern and inbreed the future knowledge? As we know in genetics that intermarriage can ruin the future of a family. Then what about the human being in knowledge extensions?

Sure, in cyber age everything is designed or transformed into a computer suitable form, and computer is most apt to deal with consistent problems – Artificial intelligence, or AI, tends to avoid inconsistency. But knowledge management should try to discover controversy, and encourage visitors and learners to hold this inconsistency. The paper tries to stress this issue in theoretical informatics points of view.

2. The role of self knowledge: adaptability

"Peter Drucker addresses the need for increased self knowledge for knowledge workers to build organizations of the 21st Century. Daniel Goleman frames self knowledge in the context of emotional intelligence and as a basic characteristic of a leader. Both are arguing for developing internal navigation systems that shape our perceptual filters, habit patterns, worldviews and perspectives in leading ourselves and others in organizations. The need for personalization or a customization engine is considered to be a critically missing piece in completing Customer Relationship Management (CRM) software packages and E-commerce engines. If the visitor to a website is not finding any meaning or connection to himself or herself,

the ‘stickiness’ of the site is not high and the visitor leaves taking all his buying power with him/her” [Kaipa 1999]. Sure, what happens if the visitor, searches knowledge? Does him/her find the cyber teacher attractive?

And further, more value lies in the keeping of self conceptual structure: to avoid getting drowned in the immense sea of information, as follows.

- Space is not empty (http://blog.china.com/u/060703/2812/200607/8128.html)

“Since 1998 cosmologists have found that the universe expands in extraordinary speed. The driving force is a mysterious hidden energy. According to astronomic observations the hidden matter would make up 90% of the universe and still remains unknown to men … Hidden energy, hidden matter are still black holes in science.”

BBS discussion: “Simply speaking, space is hidden energy, and hidden energy is space. Space is matter. It can be extremely harmful to cut them apart or to refer the hidden energy to some pure antimatter.”

- Time is fixed without change – the absolute miss of relativity (http://blog.china.com/u/060703/2812/200607/8124.html)

“Time is a concept, a measure of the universe vacant space through the concept of space. Time is not matter. It is an invisible and intangible 0 in space – Time does not exist to moving matter… Light speed is capricious ...”

- Beside the forum discussion, there are more to see: “Distance and time are ‘illusory’; Things can be created from ‘nothing’; The origin of the universe.[Zhong]”

As Dr. Odenwald acknowledges, “We don’t have a full mathematical theory for describing this state yet, but it was probably ‘multi-dimensional’…Nothingness (that gives rise to the present universe) was not nothing, but it was not anything like the kinds of ‘something’ we know about today. We have no words to describe it and the ones we find in the Oxford English Dictionary are based on the wrong physical insight.”

- And also the Japanese Emoto’s experiment illustrated in the Chinese version of [Zhong]:

“The crystal shape of water varies with human mental and lingual actions – It appears beautiful to kindness and ugly to malice.” [WellnessGoods.com]

Is human mind highly developed? Not at all in the reincarnation chain.

For valid introduction of scientific research on reincarnation see [Zhong, (Research on Reincarnation)]. An American psychologist holds that “70% of human can clearly recall his past life through hypnotization”, and there exists such a kind with inborn recollection [Suodaji].

- Modern science relies on instruments, eventually out of eyes, to derive conclusion - To put it bluntly, via sight, hearing, smelling, taste and touch senses in normal state. All achievements of current and prospecting science attribute to a self – I find, I deduce, I devise, I summarize, I set up … However science shows inability to the essence of self. How surprising! How do those rules and theories hold valid? One feels more ignorant to such issues as ‘who am I’, ‘What is the noumenon, ontology of mind’ [Cichengluozhu].
- On the contrary, scientific discoveries in some aspects are approaching or partially proving Albert Einstein’s point that space, time and matter are illusions of human cognition.

Therefore authorized knowledge, largely believed true, needs not to tell the truth. It might be only for indication to those who keep his self knowledge on.

3. The attraction of opponents – hypothesis of an imaginary part, or latent part

A well known physics phenomenon shows that opposite natures and their neutralities attract each other. We don’t mean to explore the principle behind physics, but to extend it to a knowledge communication model, i.e., attraction among knowledge groups.

Suppose that knowledge is genetic like kindred clans. Why? A metaphor of consciousness in evolution chain. Therefore someone sees a knowledge piece as male and another sees as female. However they share the identical parents. The reason of such an attraction lies in its source – either the son or the daughter is

only a partial instance of a complete source, the original face of knowledge, and people having one partial part tend to explore the complete part by investigating the opposite nature together with their neutralities. Is that true?

Remember Albert Einstein's point that space, time and matter are illusions of human cognition? Sure, no need to explore the opposite part if our consciousness remains impartial and perfect (i.e. our consciousness remain in the neutral part). The point is, some consciousnesses are prone to positively partial and some others to negatively partial, and of course see the same world in opposite images mentally, since the opposites and their neutralities may all overlap. But the world is neither positively partial nor negatively partial. Do we see it? We are afraid no, since each of us is suffering from such crankiness (a perspective of an imaginary highly developed species). Therefore the true face remains a myth inexpressible in the standard language. It is not the problem of the language but the problem how human interprets the language.

Still why? How can it come? Let's ask Laozi (Lao-tzu); we are not responsible for the accuracy of the translation):

"The Tao that can be expressed [in fact-space] is not the eternal Tao;
The name that can be defined is not the unchanging name.
Non-existence [mindscape] is called the antecedent of heaven and earth;
Existence [fact-space] is the mother of all things.
From eternal non-existence, therefore, we serenely observe
the mysterious beginnings of the Universe;
From eternal existence we clearly see its apparent distinctions.
The two are the same in source and become different when manifested.
This sameness is called profundity. [Minuteness in minuteness]
is the gate whence comes the beginning of all parts of the Universe."

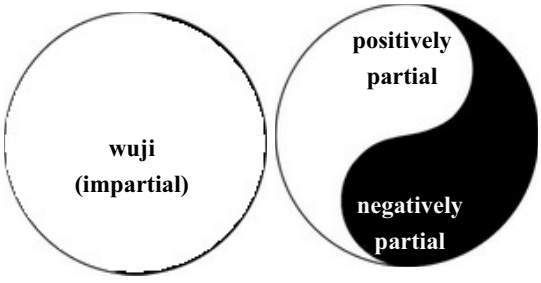
We are not responsible for the accuracy of the translation, but in Chinese Laozi answers our question in a perfect way. Sure, it is the problem of our partial mind. But is there a need to assign a symbol to denote the true face of knowledge? I think so, and even crucial to our knowledge communication model.

Extenics believes in a latent part or imaginary part description, which complements a real part of a well defined frame of matter element [Cai, 1998]:

"Conjugation of matter elements:

Just like the domain definition for complex numbers, matter elements include real and imaginary parts, too. For a given object *N*, it could be written as *N=ReN+ImN*, where the *ReN* is the real part of *N* and *ImN* the imaginary part of it. A kind of product as a matter element has two sides of valuation. One is the product itself, the real part of it. The effective of the brand of the product and the reputation of the producer belong to the imaginary part of the matter element. There are many successful examples of using the imaginary parts of matter elements to civil decision makings and even in military directions." (The matter-element theory based science, extenics funded by Chinese mathematician Cai Wen, has experienced 28 year's difficulties, and become the world leading science dealing with inconsistency or incompatibility, covering the widest range of application area from informatics, system engineering to management and finance.)

Prof. Cai might have made a significant approach to express the hidden content of knowledge, but still undeveloped with this issue into a systematic theory. But however, it signifies the twilight to denote the hidden context inexpressible. To our critics we differ in that *ImN* should have a broader sense.



From those discussed above, we suggest the wuji area as the *ImN*, or:

impartial(knowledge)= imaginary part(knowledge)

as the hidden, latent source of knowledge, that ordinary people see as partial distortion in minds, as denoted by yin and yang in the taiji figure.

Significance: to discover the implications of symbol names, symbol information, literal knowledge and

symbol principle.

4. Communication model – find the consistency in opposite and in their neutralities, to discover the underlying principle.

No inconsistency, no conversation, and in this way people might unconsciously sentence the knowledge to death. Therefore we suggest a heterogeneous structure:

In the top layer is the partial knowledge structure with which traditional communication works. It is crucial that heterogeneous structures complement and compensate each other and initiate new conversations, new paradoxes, new disputes and new issues.

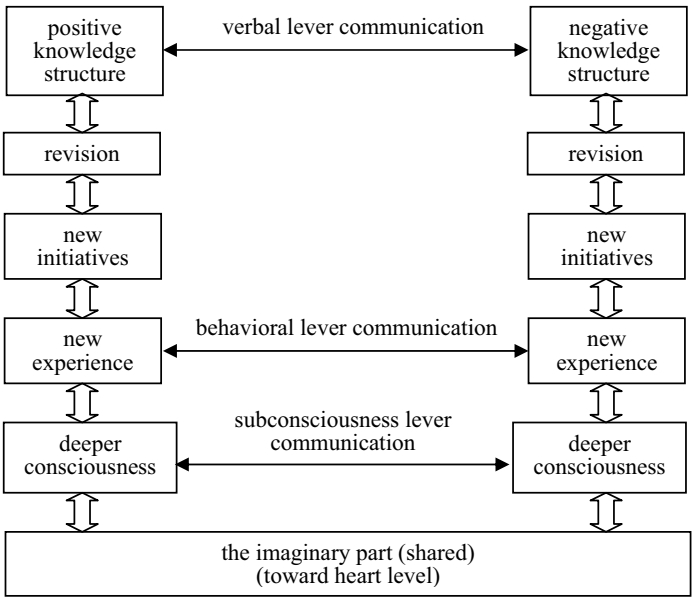
Then each communicator adjusts, modifies his views, understandings, or makes necessary revision to his previous models to initiate a further approach.

Plan the new approach.

Put in practice.

Gain from the new experience that is added to his subconsciousness.

Discovery on his instinct. Suppose that all the "outer" knowledge, experience or wisdom are never beyond his own instinct which he understands little – Do you believe a kind of man living in seclusion knows everything? Therefore this level is shared.



As an example, in reading comprehension, sometimes one needs to constantly shift his stand or view point – possibly on the cons' perspective, in order to start an intensive mental conversation.

5. Concluding remarks: the role of self knowledge pattern in communication

An average information seeker might very probably lose himself/herself in the immense sea of information – because he/she doesn't maintain a good individualized knowledge pattern, and fails in the qualification as an eligible speaker. As the consequence he/she may lose the interest for further paradoxes, disputes or conversation. An acute learner tries to seek underlying truth in the controversies/contradictions but also in the neutralities between them. How can one recognize himself/herself and distinguish one principle from another without contradiction? The value of remaining doubt can be significant – it might lead people to the right path.

A good follower of the mundane world is never good, since he/she never understands the underlying truth hidden in the symbols.

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N-norm and N-conorm in Neutrosophic Logic and Set, and the Neutrosophic Topologies

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Abstract:
In this paper we present the N-norms/N-conorms in neutrosophic logic and set as extensions of T-norms/T-conorms in fuzzy logic and set.
Also, as an extension of the Intuitionistic Fuzzy Topology we present the Neutrosophic Topologies.

1. **Definition of the Neutrosophic Logic/Set**:

Let T, I, F be real standard or non-standard subsets of $]^{-}0, 1^{+}[$,
with sup T = t_sup, inf T = t_inf,
sup I = i_sup, inf I = i_inf,
sup F = f_sup, inf F = f_inf,
and n_sup = t_sup+i_sup+f_sup,
n_inf = t_inf+i_inf+f_inf.

Let U be a universe of discourse, and M a set included in U. An element x from U is noted with respect to the set M as x(T, I, F) and belongs to M in the following way: it is t% true in the set, i% indeterminate (unknown if it is or not) in the set, and f% false, where t varies in T, i varies in I, f varies in F.

Statically T, I, F are subsets, but dynamically T, I, F are functions/operators depending on many known or unknown parameters.

2. In a similar way define the **Neutrosophic Logic**:

A logic in which each proposition x is T% true, I% indeterminate, and F% false, and we write it x(T,I,F), where T, I, F are defined above.

3. As a generalization of T-norm and T-conorm from the Fuzzy Logic and Set, we now introduce the **N-norms and N-conorms** for the Neutrosophic Logic and Set.

We define a *partial relation order* on the neutrosophic set/logic in the following way:
$x(T_1, I_1, F_1) \leq y(T_2, I_2, F_2)$ iff (if and only if) $T_1 \leq T_2$, $I_1 \geq I_2$, $F_1 \geq F_2$ for crisp components.

And, in general, for subunitary set components:
$x(T_1, I_1, F_1) \leq y(T_2, I_2, F_2)$ iff

$$\inf T_1 \leq \inf T_2, \sup T_1 \leq \sup T_2,$$
$$\inf I_1 \geq \inf I_2, \sup I_1 \geq \sup I_2,$$
$$\inf F_1 \geq \inf F_2, \sup F_1 \geq \sup F_2.$$

If we have mixed - crisp and subunitary - components, or only crisp components, we can transform any crisp component, say "a" with $a \in [0,1]$ or $a \in]^{-}0, 1^{+}[$, into a subunitary set [a, a]. So, the definitions for subunitary set components should work in any case.

3.1. N-norms

$N_n: (\,]^{-}0,1^{+}[\times]^{-}0,1^{+}[\times]^{-}0,1^{+}[\,)^2 \rightarrow]^{-}0,1^{+}[\times]^{-}0,1^{+}[\times]^{-}0,1^{+}[$

$N_n\,(x(T_1,I_1,F_1), y(T_2,I_2,F_2)) = (N_nT(x,y), N_nI(x,y), N_nF(x,y))$,
where $N_nT(.,.)$, $N_nI(.,.)$, $N_nF(.,.)$ are the truth/membership, indeterminacy, and respectively falsehood/nonmembership components.

N_n have to satisfy, for any x, y, z in the neutrosophic logic/set M of the universe of discourse U, the following axioms:
a) Boundary Conditions: $N_n(x, \mathbf{0}) = \mathbf{0}$, $N_n(x, \mathbf{1}) = x$.
b) Commutativity: $N_n(x, y) = N_n(y, x)$.
c) Monotonicity: If $x \leq y$, then $N_n(x, z) \leq N_n(y, z)$.
d) Associativity: $N_n(N_n(x, y), z) = N_n(x, N_n(y, z))$.

There are cases when not all these axioms are satisfied, for example the associativity when dealing with the neutrosophic normalization after each neutrosophic operation. But, since we work with approximations, we can call these **N-pseudo-norms**, which still give good results in practice.

N_n represent the *and* operator in neutrosophic logic, and respectively the *intersection* operator in neutrosophic set theory.

Let $J \in \{T, I, F\}$ be a component.
Most known N-norms, as in fuzzy logic and set the T-norms, are:
• The Algebraic Product N-norm: $N_{n-algebraic}J(x, y) = x \cdot y$
• The Bounded N-Norm: $N_{n-bounded}J(x, y) = \max\{0, x + y - 1\}$
• The Default (min) N-norm: $N_{n-min}J(x, y) = \min\{x, y\}$.

A general example of N-norm would be this.
Let $x(T_1, I_1, F_1)$ and $y(T_2, I_2, F_2)$ be in the neutrosophic set/logic M. Then:

$$\mathbf{N_n(x, y) = (T_1 \wedge T_2,\ I_1 \vee I_2,\ F_1 \vee F_2)}$$

where the "$\wedge$" operator, acting on two (standard or non-standard) subunitary sets, is a N-norm (verifying the above N-norms axioms); while the "$\vee$" operator, also acting on two (standard or non-standard) subunitary sets, is a N-conorm (verifying the below N-conorms axioms).

For example, $\wedge$ can be the Algebraic Product T-norm/N-norm, so $T_1 \wedge T_2 = T_1 \cdot T_2$ (herein we have a product of two subunitary sets – using simplified notation); and $\vee$ can be the Algebraic Product T-conorm/N-conorm, so $T_1 \vee T_2 = T_1 + T_2 - T_1 \cdot T_2$ (herein we have a sum, then a product, and afterwards a subtraction of two subunitary sets).

Or $\wedge$ can be any T-norm/N-norm, and $\vee$ any T-conorm/N-conorm from the above and below; for example the easiest way would be to consider the *min* for crisp components (or *inf* for subset components) and respectively *max* for crisp components (or *sup* for subset components).

If we have crisp numbers, we can at the end neutrosophically normalize.

3.2. N-conorms

N_c: $(\ ]^-0,1^+[\times\ ]^-0,1^+[\times\ ]^-0,1^+[\ )^2 \rightarrow\ ]^-0,1^+[\times\ ]^-0,1^+[\times\ ]^-0,1^+[$

$N_c\ (x(T_1,I_1,F_1),\ y(T_2,I_2,F_2)) = (N_cT(x,y),\ N_cI(x,y),\ N_cF(x,y))$,
where $N_nT(.,.)$, $N_nI(.,.)$, $N_nF(.,.)$ are the truth/membership, indeterminacy, and respectively falsehood/nonmembership components.

N_c have to satisfy, for any x, y, z in the neutrosophic logic/set M of universe of discourse U, the following axioms:
a) Boundary Conditions: $N_c(x, \mathbf{1}) = \mathbf{1}$, $N_c(x, \mathbf{0}) = x$.
b) Commutativity: $N_c\ (x, y) = N_c(y, x)$.
c) Monotonicity: if $x \leq y$, then $N_c(x, z) \leq N_c(y, z)$.
d) Associativity: $N_c\ (N_c(x, y), z) = N_c(x, N_c(y, z))$.

There are cases when not all these axioms are satisfied, for example the associativity when dealing with the neutrosophic normalization after each neutrosophic operation. But, since we work with approximations, we can call these **N-pseudo-conorms**, which still give good results in practice.

N_c represent the *or* operator in neutrosophic logic, and respectively the *union* operator in neutrosophic set theory.

Let $J \in \{T, I, F\}$ be a component.
Most known N-conorms, as in fuzzy logic and set the T-conorms, are:
• The Algebraic Product N-conorm: $N_{c-algebraic}J(x, y) = x + y - x \cdot y$
• The Bounded N-conorm: $N_{c-bounded}J(x, y) = \min\{1, x + y\}$
• The Default (max) N-conorm: $N_{c-max}J(x, y) = \max\{x, y\}$.

A general example of N-conorm would be this.
Let $x(T_1, I_1, F_1)$ and $y(T_2, I_2, F_2)$ be in the neutrosophic set/logic M. Then:

$$\mathbf{N_n(x, y) = (T_1 \vee T_2,\ I_1 \wedge I_2,\ F_1 \wedge F_2)}$$

Where – as above - the "$\wedge$" operator, acting on two (standard or non-standard) subunitary sets, is a N-norm (verifying the above N-norms axioms); while the "$\vee$" operator, also acting on two (standard or non-standard) subunitary sets, is a N-conorm (verifying the above N-conorms axioms).

For example, $\wedge$ can be the Algebraic Product T-norm/N-norm, so $T_1 \wedge T_2 = T_1 \cdot T_2$ (herein we have a product of two subunitary sets); and $\vee$ can be the Algebraic Product T-conorm/N-conorm, so $T_1 \vee T_2 = T_1 + T_2 - T_1 \cdot T_2$ (herein we have a sum, then a product, and afterwards a subtraction of two subunitary sets).

Or $\wedge$ can be any T-norm/N-norm, and $\vee$ any T-conorm/N-conorm from the above; for example the easiest way would be to consider the *min* for crisp components (or *inf* for subset components) and respectively *max* for crisp components (or *sup* for subset components).

If we have crisp numbers, we can at the end neutrosophically normalize.

Since the min/max (or inf/sup) operators work the best for subunitary set components, let's present their definitions below. They are extensions from subunitary intervals {defined in [3]} to any subunitary sets. Analogously we can do for all neutrosophic operators defined in [3].

Let $x(T_1, I_1, F_1)$ and $y(T_2, I_2, F_2)$ be in the neutrosophic set/logic M.

Neutrosophic Conjunction/Intersection:

$x \wedge y = (T_\wedge, I_\wedge, F_\wedge)$,

where $\inf T_{\wedge} = \min\{\inf T_1, \inf T_2\}$
$\sup T_{\wedge} = \min\{\sup T_1, \sup T_2\}$
$\inf I_{\wedge} = \max\{\inf I_1, \inf I_2\}$
$\sup I_{\wedge} = \max\{\sup I_1, \sup I_2\}$
$\inf F_{\wedge} = \max\{\inf F_1, \inf F_2\}$
$\sup F_{\wedge} = \max\{\sup F_1, \sup F_2\}$

Neutrosophic Disjunction/Union:

$x \vee y = (T_{\vee}, I_{\vee}, F_{\vee})$,
where $\inf T_{\vee} = \max\{\inf T_1, \inf T_2\}$
$\sup T_{\vee} = \max\{\sup T_1, \sup T_2\}$
$\inf I_{\vee} = \min\{\inf I_1, \inf I_2\}$
$\sup I_{\vee} = \min\{\sup I_1, \sup I_2\}$
$\inf F_{\vee} = \min\{\inf F_1, \inf F_2\}$
$\sup F_{\vee} = \min\{\sup F_1, \sup F_2\}$

Neutrosophic Negation/Complement:

$\mathcal{C}(x) = (T_{\mathcal{C}}, I_{\mathcal{C}}, F_{\mathcal{C}})$,
where $T_{\mathcal{C}} = F_1$
$\inf I_{\mathcal{C}} = 1 - \sup I_1$
$\sup I_{\mathcal{C}} = 1 - \inf I_1$
$F_{\mathcal{C}} = T_1$

Upon the above Neutrosophic Conjunction/Intersection, we can define the

Neutrosophic Containment:

We say that the neutrosophic set A is included in the neutrosophic set B of the universe of discourse U,
iff for any $x(T_A, I_A, F_A) \in A$ with $x(T_B, I_B, F_B) \in B$ we have:
$\inf T_A \leq \inf T_B$; $\sup T_A \leq \sup T_B$;
$\inf I_A \geq \inf I_B$; $\sup I_A \geq \sup I_B$;
$\inf F_A \geq \inf F_B$; $\sup F_A \geq \sup F_B$.

3.3. **Remarks**.

a) The non-standard unit interval $]^{-}0, 1^{+}[$ is merely used for philosophical applications, especially when we want to make a distinction between relative truth (truth in at least one world) and absolute truth (truth in all possible worlds), and similarly for distinction between relative or absolute falsehood, and between relative or absolute indeterminacy.

But, for technical applications of neutrosophic logic and set, the domain of definition and range of the N-norm and N-conorm can be restrained to the normal standard real unit interval [0, 1], which is easier to use, therefore:

$N_n: ([0,1] \times [0,1] \times [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1]$

and

$N_c: ([0,1] \times [0,1] \times [0,1])^2 \rightarrow [0,1] \times [0,1] \times [0,1]$.

b) Since in NL and NS the sum of the components (in the case when T, I, F are crisp numbers, not sets) is not necessary equal to 1 (so the normalization is not required), we can keep the final result un-normalized.

But, if the normalization is needed for special applications, we can normalize at the end by dividing each component by the sum all components.
If we work with intuitionistic logic/set (when the information is incomplete, i.e. the sum of the crisp components is less than 1, i.e. *sub-normalized*), or with paraconsistent logic/set (when the information overlaps and it is contradictory, i.e. the sum of crisp components is greater than 1, i.e. *over-normalized*), we need to define the neutrosophic measure of a proposition/set.
If x(T,I,F) is a NL/NS, and T,I,F are crisp numbers in [0,1], then the **neutrosophic vector norm** of variable/set x is the sum of its components:

$$N_{vector\text{-}norm}(x) = T+I+F.$$

Now, if we apply the N_n and N_c to two propositions/sets which maybe intuitionistic or paraconsistent or normalized (i.e. the sum of components less than 1, bigger than 1, or equal to 1), x and y, what should be the neutrosophic measure of the results $N_n(x,y)$ and $N_c(x,y)$?
Herein again we have more possibilities:

- either the product of neutrosophic measures of x and y:
  $N_{vector\text{-}norm}(N_n(x,y)) = N_{vector\text{-}norm}(x)\cdot N_{vector\text{-}norm}(y)$,
- or their average:
  $N_{vector\text{-}norm}(N_n(x,y)) = (N_{vector\text{-}norm}(x) + N_{vector\text{-}norm}(y))/2$,
- or other function of the initial neutrosophic measures:

$N_{vector\text{-}norm}(N_n(x,y)) = f(N_{vector\text{-}norm}(x), N_{vector\text{-}norm}(y))$, where f(.,.) is a function to be determined according to each application.

Similarly for $N_{vector\text{-}norm}(N_c(x,y))$.
Depending on the adopted neutrosophic vector norm, after applying each neutrosophic operator the result is neutrosophically normalized. We'd like to mention that "**neutrosophically normalizing**" doesn't mean that the sum of the resulting crisp components should be 1 as in fuzzy logic/set or intuitionistic fuzzy logic/set, but the sum of the components should be as above: either equal to the product of neutrosophic vector norms of the initial propositions/sets, or equal to the neutrosophic average of the initial propositions/sets vector norms, etc.
In conclusion, we neutrosophically normalize the resulting crisp components T`,I`,F` by multiplying each neutrosophic component T`,I`,F` with S/(T`+I`+F`), where
$S= N_{vector\text{-}norm}(N_n(x,y))$ for a N-norm or $S= N_{vector\text{-}norm}(N_c(x,y))$ for a N-conorm - as defined above.

c) If T, I, F are subsets of [0, 1] the problem of neutrosophic normalization is more difficult.
   i) If sup(T)+sup(I)+sup(F) < 1, we have an *intuitionistic proposition/set*.
   ii) If inf(T)+inf(I)+inf(F) > 1, we have a *paraconsistent proposition/set*.
   iii) If there exist the crisp numbers $t \in T$, $i \in I$, and $f \in F$ such that t+i+f =1, then we can say that we have a *plausible normalized proposition/set*.
   But in many such cases, besides the normalized particular case showed herein, we also have crisp numbers, say $t_1 \in T$, $i_1 \in I$, and $f_1 \in F$ such that $t_1+i_1+f_1 < 1$ (incomplete information) and $t_2 \in T$, $i_2 \in I$, and $f_2 \in F$ such that $t_2+i_2+f_2 > 1$ (paraconsistent information).

4. Examples of Neutrosophic Operators which are N-norms or N-pseudonorms or, respectively N-conorms or N-pseudoconorms.

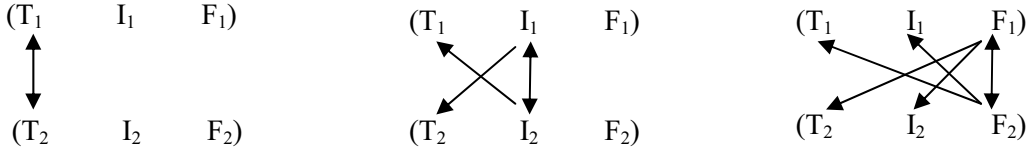
We define a binary **neutrosophic conjunction (intersection)** operator, which is a particular case of a N-norm (neutrosophic norm, a generalization of the fuzzy T-norm):

$$c_N^{TIF}:\left([0,1]\times[0,1]\times[0,1]\right)^2\to[0,1]\times[0,1]\times[0,1]$$
$$c_N^{TIF}(x,y)=\left(T_1T_2,I_1I_2+I_1T_2+T_1I_2,F_1F_2+F_1I_2+F_1T_2+F_2T_1+F_2I_1\right).$$

The neutrosophic conjunction (intersection) operator $x\wedge_N y$ component truth, indeterminacy, and falsehood values result from the multiplication

$$\left(T_1+I_1+F_1\right)\cdot\left(T_2+I_2+F_2\right)$$

since we consider in a prudent way $T\prec I\prec F$, where "$\prec$" is a **neutrosophic relationship** and means "weaker", i.e. the products T_iI_j will go to I, T_iF_j will go to F, and I_iF_j will go to F for all i, j $\in\{1,2\}$, i$\neq$j, while of course the product T_1T_2 will go to T, I_1I_2 will go to I, and F_1F_2 will go to F (or reciprocally we can say that F prevails in front of I which prevails in front of T, and this neutrosophic relationship is transitive):



So, the truth value is T_1T_2, the indeterminacy value is $I_1I_2+I_1T_2+T_1I_2$ and the false value is $F_1F_2+F_1I_2+F_1T_2+F_2T_1+F_2I_1$. The norm of $x\wedge_N y$ is $\left(T_1+I_1+F_1\right)\cdot\left(T_2+I_2+F_2\right)$. Thus, if x and y are normalized, then $x\wedge_N y$ is also normalized. Of course, the reader can redefine the neutrosophic conjunction operator, depending on application, in a different way, for example in a more optimistic way, i.e. $I\prec T\prec F$ or T prevails with respect to I, then we get:

$$c_N^{ITF}(x,y)=\left(T_1T_2+T_1I_2+T_2I_1,I_1I_2,F_1F_2+F_1I_2+F_1T_2+F_2T_1+F_2I_1\right).$$

Or, the reader can consider the order $T\prec F\prec I$, etc.

Let's also define the unary **neutrosophic negation** operator:

$$n_N:[0,1]\times[0,1]\times[0,1]\to[0,1]\times[0,1]\times[0,1]$$
$$n_N\left(T,I,F\right)=\left(F,I,T\right)$$

by interchanging the truth T and falsehood F vector components.

Similarly, we now define a binary **neutrosophic disjunction (or union)** operator, where we consider the neutrosophic relationship $F\prec I\prec T$:

$$d_N^{FIT}:\left([0,1]\times[0,1]\times[0,1]\right)^2\to[0,1]\times[0,1]\times[0,1]$$
$$d_N^{FIT}(x,y)=\left(T_1T_2+T_1I_2+T_2I_1+T_1F_2+T_2F_1,I_1F_2+I_2F_1+I_1I_2,F_1F_2\right)$$

We consider as neutrosophic norm of the neutrosophic variable x, where $NL(x)=T_1+I_1+F_1$, the sum of its components: $T_1+I_1+F_1$, which in many cases is 1, but can also be positive <1 or >1.

Or, the reader can consider the order $F \prec T \prec I$, in a pessimistic way, i.e. focusing on indeterminacy I which prevails in front of the truth T, or other **neutrosophic order** of the neutrosophic components T,I,F depending on the application.
Therefore,

$$d_N^{FTI}(x,y) = \left(T_1T_2 + T_1F_2 + T_2F_1, I_1F_2 + I_2F_1 + I_1I_2 + T_1I_2 + T_2I_1, F_1F_2\right)$$

4.1. Neutrophic Composition k-Law

Now, we define a more general neutrosophic composition law, named k-law, in order to be able to define neutrosophic k-conjunction/intersection and neutrosophic k-disjunction/union for k variables, where $k \geq 2$ is an integer.

Let's consider $k \geq 2$ neutrosophic variables, $x_i\left(T_i, I_i, F_i\right)$, for all $i \in \{1,2,...,k\}$. Let's denote

$$T = \left(T_1,...,T_k\right)$$
$$I = \left(I_1,...,I_k\right)$$
$$F = \left(F_1,...,F_k\right).$$

We now define a neutrosophic composition law o_N in the following way:

$$o_N : \{T, I, F\} \to [0,1]$$

If $z \in \{T, I, F\}$ then $z_{o_N} z = \prod_{i=1}^{k} z_i$.

If $z, w \in \{T, I, F\}$ then

$$z_{o_N} w = w_{o_N} z \quad = \sum_{\substack{r=1 \\ \{i_1,...,i_r,j_{r+1},...,j_k\} \equiv \{1,2,...,k\} \\ (i_1,...,i_r) \in C^r(1,2,...,k) \\ (j_{r+1},...,j_k) \in C^{k-r}(1,2,...,k)}}^{k-1} z_{i_1}...z_{i_r} w_{j_{r+1}}...w_{j_k}$$

where $C^r\left(1,2,...,k\right)$ means the set of combinations of the elements $\{1,2,...,k\}$ taken by r. [Similarly for $C^{k-r}\left(1,2,...,k\right)$.]

In other words, $z_{o_N} w$ is the sum of all possible products of the components of vectors z and w, such that each product has at least a z_i factor and at least a w_j factor, and each product has exactly k factors where each factor is a different vector component of z or of w. Similarly if we multiply three vectors:

$$T_{o_N} I_{o_N} F = \sum_{\substack{u,v,k-u-v=1 \\ \{i_1,...,i_u,j_{u+1},...,j_{u+v},l_{u+v+1},...,l_k\} \equiv \{1,2,...,k\} \\ (i_1,...,i_u) \in C^u(1,2,...,k), (j_{u+1},...,j_{u+v}) \in \\ \in C^v(1,2,...,k), (l_{u+v+1},...,l_k) \in C^{k-u-v}(1,2,...,k)}}^{k-2} T_{i_1...i_u} I_{j_{u+1}...j_{u+v}} F_{l_{u+v+1}}...F_{l_k}$$

Let's see an example for $k = 3$.

$$x_1(T_1, I_1, F_1)$$
$$x_2(T_2, I_2, F_2)$$
$$x_3(T_3, I_3, F_3)$$

$$T_{o_N} T = T_1 T_2 T_3, \quad I_{o_N} I = I_1 I_2 I_3, \quad \mathrm{F}_{o_N} F = F_1 F_2 F_3$$

$$T_{o_N} I = T_1 I_2 I_3 + I_1 T_2 I_3 + I_1 I_2 T_3 + T_1 T_2 I_3 + T_1 I_2 T_3 + I_1 T_2 T_3$$

$$T_{o_N} F = T_1 F_2 F_3 + F_1 T_2 F_3 + F_1 F_2 T_3 + T_1 T_2 F_3 + T_1 F_2 T_3 + F_1 T_2 T_3$$

$$I_{o_N} F = I_1 F_2 F_3 + F_1 I_2 F_3 + F_1 F_2 I_3 + I_1 I_2 F_3 + I_1 F_2 I_3 + F_1 I_2 I_3$$

$$T_{o_N} I_{o_N} F = T_1 I_2 F_3 + T_1 F_2 I_3 + I_1 T_2 F_3 + I_1 F_2 T_3 + F_1 I_2 T_3 + F_1 T_2 I_3$$

For the case when indeterminacy I is not decomposed in subcomponents {as for example $I = P \cup U$ where P =paradox (true and false simultaneously) and U =uncertainty (true or false, not sure which one)}, the previous formulas can be easily written using only three components as:

$$T_{o_N} I_{o_N} F = \sum_{i,j,r \in \mathcal{P}(1,2,3)} T_i I_j F_r$$

where $\mathcal{P}(1,2,3)$ means the set of permutations of $(1,2,3)$ i.e.

$$\{(1,2,3),(1,3,2),(2,1,3),(2,3,1,),(3,1,2),(3,2,1)\}$$

$$z_{o_N} w = \sum_{\substack{\mathrm{i}=1 \\ (i,j,r) \equiv (1,2,3) \\ (j,r) \in \mathcal{P}^2(1,2,3)}}^{3} z_i w_j w_{j_r} + w_i z_j z_r$$

This neurotrophic law is associative and commutative.

4.2. Neutrophic Logic and Set k-Operators

Let's consider the neutrophic logic crispy values of variables x, y, z (so, for k = 3):

$$NL(x) = (T_1, I_1, F_1) \text{ with } 0 \le T_1, I_1, F_1 \le 1$$
$$NL(y) = (T_2, I_2, F_2) \text{ with } 0 \le T_2, I_2, F_2 \le 1$$
$$NL(z) = (T_3, I_3, F_3) \text{ with } 0 \le T_3, I_3, F_3 \le 1$$

In neutrosophic logic it is not necessary to have the sum of components equals to 1, as in intuitionist fuzzy logic, i.e. $T_k + I_k + F_k$ is not necessary 1, for $1 \le k \le 3$

As a particular case, we define the tri-nary conjunction neutrosophic operator:

$$c_{3N}^{TIF} : \left([0,1] \times [0,1] \times [0,1]\right)^3 \to [0,1] \times [0,1] \times [0,1]$$
$$c_{3N}^{TIF}(x, y, z) = \left(T_{o_N} T, I_{o_N} I + I_{o_N} T, F_{o_N} F + F_{o_N} I + F_{o_N} T\right)$$

If all x, y, z are normalized, then $c_{3N}^{TIF}(x, y, z)$ is also normalized.

If x, y, or y are non-normalized, then $\left|c_{3N}^{TIF}(x, y, z)\right| = |x| \cdot |y| \cdot |z|$, where |w| means norm of w.

c_{3N}^{TIF} is a 3-N-norm (neutrosophic norm, i.e. generalization of the fuzzy T-norm).

Again, as a particular case, we define the unary negation neutrosophic operator:

$$n_N : [0,1]\times[0,1]\times[0,1] \to [0,1]\times[0,1]\times[0,1]$$
$$n_N(x) = n_N(T_1, I_1, F_1) = (F_1, I_1, T_1).$$

Let's consider the vectors:

$$\mathrm{T}= \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \quad \mathrm{I}= \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \text{ and } \mathrm{F}= \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}.$$

We note $\mathrm{T}_{\bar{x}}= \begin{pmatrix} F_1 \\ T_2 \\ T_3 \end{pmatrix}$, $\mathrm{T}_{\bar{y}}= \begin{pmatrix} T_1 \\ F_2 \\ T_3 \end{pmatrix}$, $\mathrm{T}_{\bar{z}}= \begin{pmatrix} T_1 \\ T_2 \\ F_3 \end{pmatrix}$, $\mathrm{T}_{\overline{xy}}= \begin{pmatrix} F_1 \\ F_2 \\ T_3 \end{pmatrix}$, etc.

and similarly

$$\mathrm{F}_{\bar{x}}= \begin{pmatrix} T_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad F_{\bar{y}}= \begin{pmatrix} F_1 \\ T_2 \\ F_3 \end{pmatrix}, \quad \mathrm{F}_{\overline{xz}}= \begin{pmatrix} T_1 \\ F_2 \\ T_3 \end{pmatrix}, \text{ etc.}$$

For shorter and easier notations let's denote $z_{o_N} w = zw$ and respectively $z_{o_N} w_{o_N} v = zwv$ for the vector neutrosophic law defined previously.

Then the neutrosophic tri-nary conjunction/intersection of neutrosophic variables x, y, and z is:

$$c_{3N}^{TIF}(x,y,z) = (TT, II + IT, FF + FI + FT + FIT) =$$
$$= (T_1T_2T_3, I_1I_2I_3 + I_1I_2T_3 + I_1T_2I_3 + T_1I_2I_3 + I_1T_2T_3 + T_1I_2T_3 + T_1T_2I_3,$$
$$F_1F_2F_3 + F_1F_2I_3 + F_1I_2F_3 + I_1F_2F_3 + F_1I_2I_3 + I_1F_2I_3 + I_1I_2F_3 +$$
$$+F_1F_2T_3 + F_1T_2F_3 + T_1F_2F_3 + F_1T_2T_3 + T_1F_2T_3 + T_1T_2F_3 +$$
$$+T_1I_2F_3 + T_1F_2I_3 + I_1F_2T_3 + I_1T_2F_3 + F_1I_2T_3 + F_1T_2\,I_3).$$

Similarly, the neutrosophic tri-nary disjunction/union of neutrosophic variables x, y, and z is:

$$d_{3N}^{FIT}(x,y,z) = (TT + TI + TF + TIF, II + IF, FF) =$$

$(T_1T_2T_3 + T_1T_2I_3 + T_1I_2T_3 + I_1T_2T_3 + T_1I_2I_3 + I_1T_2I_3 + I_1I_2T_3 + T_1T_2F_3 + T_1F_2T_3 + F_1T_2T_3 + T_1F_2F_3 + F_1T_2F_3 + F_1F_2T_3 + T_1I_2F_3 + T_1F_2I_3 + I_1F_2T_3 + I_1T_2F_3 + F_1I_2T_3 + F_1T_2I_3, I_1I_2I_3 + I_1I_2F_3 + I_1F_2I_3 + F_1I_2I_3 + I_1F_2F_3 + F_1I_2F_3 + F_1F_2I_3, F_1F_2F_3)$

Surely, other neutrosophic orders can be used for tri-nary conjunctions/intersections and respectively for tri-nary disjunctions/unions among the componenets T, I, F.

5. Neutrosophic Topologies.

A) General Definition of NT:

Let M be a non-empty set.

Let $x(T_A, I_A, F_A) \in A$ with $x(T_B, I_B, F_B) \in B$ be in the neutrosophic set/logic M, where A and B are subsets of M. Then (see Section 2.9.1 about N-norms / N-conorms and examples):

$A \cup B = \{x \in M, x(T_A \vee T_B, I_A \wedge I_B, F_A \wedge F_B)\}$,
$A \cap B = \{x \in M, x(T_A \wedge T_B, I_A \vee I_B, F_A \vee F_B)\}$,
$\mathcal{C}(A) = \{x \in M, x(F_A, I_A, T_A)\}$.

A General Neutrosophic Topology on the non-empty set M is a family η of Neutrosophic Sets in M satisfying the following axioms:

- $\mathbf{0}(0,0,1)$ and $\mathbf{1}(1,0,0) \in \eta$;
- If $A, B \in \eta$, then $A \cap B \in \eta$;
- If the family $\{A_k, k \in K\} \subset \eta$, then $\bigcup_{k \in K} A_k \in \eta$.

B) An alternative version of NT

-We cal also construct a Neutrosophic Topology on NT = $]^-0, 1^+[$, considering the associated family of standard or non-standard subsets included in NT, and the empty set $\emptyset$, called open sets, which is closed under set union and finite intersection.

Let A, B be two such subsets. The union is defined as:

$A \cup B = A+B-A \cdot B$, and the intersection as: $A \cap B = A \cdot B$. The complement of A, $C(A) = \{1^+\}-A$, which is a closed set. {When a non-standard number occurs at an extremity of an internal, one can write "]" instead of "(" and "[" instead of ")".} The interval NT, endowed with this topology, forms a *neutrosophic topological space*.

In this example we have used the Algebraic Product N-norm/N-conorm. But other Neutrosophic Topologies can be defined by using various N-norm/N-conorm operators.

In the above defined topologies, if all x's are paraconsistent or respectively intuitionistic, then one has a Neutrosophic Paraconsistent Topology, respectively Neutrosophic Intuitionistic Topology.

References:

[1] F. Smarandache & J. Dezert, *Advances and Applications of DSmt for Information Fusion*, Am. Res. Press, 2004.

[2] F. Smarandache, *A unifying field in logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics*, 1998, 2001, 2003, 2005.

[3] H. Wang, F. Smarandache, Y.-Q. Zhang, R. Sunderraman, *Interval Neutrosophic Set and Logic: Theory and Applications in Computing*, Hexs, 2005.

[4] L. Zadeh, *Fuzzy Sets*, Information and Control, Vol. 8, 338-353, 1965.

[5] Driankov, Dimiter; Hellendoorn, Hans; and Reinfrank, Michael, *An Introduction to Fuzzy Control.* Springer, Berlin/Heidelberg, 1993.
[6] K. Atanassov, D. Stoyanova, *Remarks on the Intuitionistic Fuzzy Sets. II*, Notes on Intuitionistic Fuzzy Sets, Vol. 1, No. 2, 85 – 86, 1995.
[7] Coker, D., *An Introduction to Intuitionistic Fuzzy Topological Spaces*, Fuzzy Sets and Systems, Vol. 88, 81-89, 1997.

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n-ary Fuzzy Logic and Neutrosophic Logic Operators

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Abstract.

We extend Knuth's 16 Boolean binary logic operators to fuzzy logic and neutrosophic logic binary operators. Then we generalize them to n-ary fuzzy logic and neutrosophic logic operators using the smarandache codification of the Venn diagram and a defined vector neutrosophic law. In such way, new operators in neutrosophic logic/set/probability are built.

Introduction.

For the beginning let's consider the Venn Diagram of two variables x and y, for each possible operator, as in Knuth's table, but we adjust this table to the Fuzzy Logic (FL).

Let's denote the fuzzy logic values of these variables as

$$FL(x) = (t_1, f_1)$$

where

t_1 = truth value of variable x,

f_1 = falsehood value of variable y,

with $0 \le t_1, f_1 \le 1$ and $t_1 + f_1 = 1$;

and similarly for y:

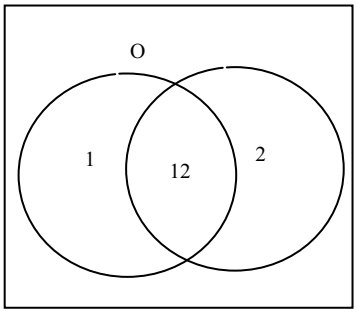
$$FL(y) = (t_2, f_2)$$

with the same $0 \le t_2, f_2 \le 1$ and $t_2 + f_2 = 1$.

We can define all 16 Fuzzy Logical Operators with respect to two FL operators: FL conjunction (FLC) and FL negation (FLN).

Since in FL the falsehood value is equal to 1- truth value, we can deal with only one component: the truth value.

The Venn Diagram for two sets X and Y



has $2^2 = 4$ disjoint parts:

0 = the part that does not belong to any set (the complement or negation)

1 = the part that belongs to 1st set only;

2 = the part that belongs to 2nd set only;

12 = the part that belongs to 1st and 2nd set only;

{called Smarandache's codification [1]}.

Shading none, one, two, three, or four parts in all possible combinations will make $2^4 = 2^{2^2} = 16$ possible binary operators.

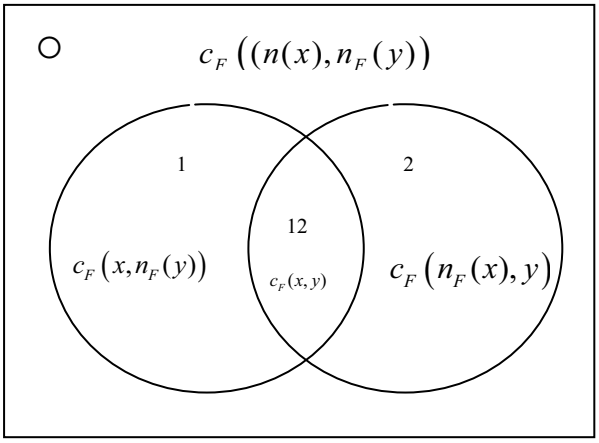
We can start using a $T-norm$ and the negation operator.

Let's take the binary conjunction or intersection (which is a $T-norm$) denoted as $c_F(x, y)$:

$$c_F : \left([0,1]\times[0,1]\right)^2 \to [0,1]\times[0,1]$$

and unary negation operator denoted as $n_F(x)$, with:

$$n_F : [0,1]\times[0,1] \to [0,1]\times[0,1]$$



The fuzzy logic value of each part is:

$P12 = part12 =$ intersection of x and y; so $FL(P12) = c_F(x, y)$.

$P1 = part1 =$ intersection of x and negation of y; $FL(P1) = c_F(x, n_F(y))$.

$P2 = part2 =$ intersection of negation of x and y; $FL(P2) = c_F(n_F(x), y)$.

$P0 = part0 =$ intersection of negation of x and the negation of y; $FL(P0) = c_F(n_F(x), n_F(y))$, and for normalization we set the condition:

$$c_F(x, y) + c_F\left((n(x), y\right) + c_F\left(x, n_F(y)\right) + c_F\left(n_F(x), n_F(y)\right) = (1, 0).$$

Then consider a binary $T-conorm$ (disjunction or union), denoted by $d_F(x, y)$:

$$d_F : \left([0,1]\times[0,1]\right)^2 \to [0,1]\times[0,1]$$

$$d_F(x,y) = (t_1 + t_2, f_1 + f_2 - 1)$$

if x and y are disjoint and $t_1 + t_2 \leq 1$.

This fuzzy disjunction operator of disjoint variables allows us to add the fuzzy truth-values of disjoint parts of a shaded area in the below table. When the truth-value increases, the false value decreases. More general, $d_F^k(x_1, x_2, ..., x_k)$, as a k-ary disjunction (or union), for $k \geq 2$, is defined as:

$$d_F^k : ([0,1] \times [0,1])^k \to [0,1] \times [0,1]$$

$$d_F^k(x_1, x_2, ..., x_k) = (t_1 + t_2 + ... + t_k, f_1 + f_2 + ... + f_k - k + 1)$$

if all x_i are disjoint two by two and $t_1 + t_2 + ... + t_k \leq 1$.

As a particular case let's take as a binary fuzzy conjunction:

$$c_F(x,y) = (t_1 t_2, f_1 + f_2 - f_1 f_2)$$

and as unary fuzzy negation:

$$n_F(x) = (1 - t_1, 1 - f_1) = (f_1, t_1),$$

where

$FL(x) = (t_1, f_1)$, with $t_1 + f_1 = 1$, and $0 \leq t_1, f_1 \leq 1$;

$FL(y) = (t_2, f_2)$, with $t_2 + f_2 = 1$, and $0 \leq t_2, f_2 \leq 1$.

whence:

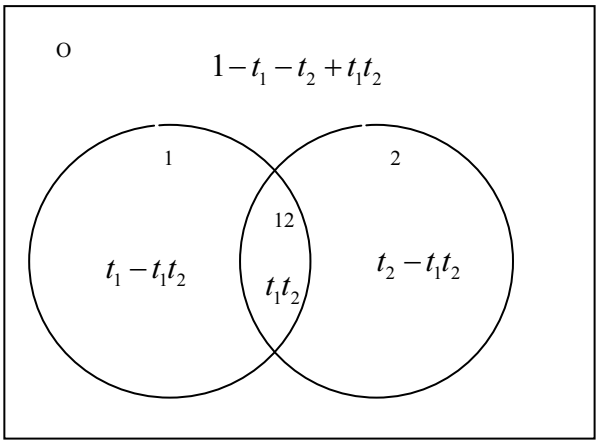
$$FL(P12) = (t_1 t_2, f_1 + f_2 - f_1 f_2)$$

$$FL(P1) = (t_1 f_2, f_1 + t_2 - f_1 t_2)$$

$$FL(P2) = (f_1 t_2, t_1 + f_2 - t_1 f_2)$$

$$FL(P0) = (f_1 f_2, t_1 + t_2 - t_1 t_2)$$

The Venn Diagram for $n = 2$ and considering only the truth values, becomes:



since

$$t_1 f_2 = t_1(1 - t_2) = t_1 - t_1 t_2$$

$$f_1 t_2 = (1 - t_1) t_2 = t_2 - t_1 t_2$$

$$f_1 f_2 = (1-t_1)(1-t_2) = 1 - t_1 - t_2 + t_1 t_2 .$$

We now use:

$$d_F^k\left(P12, P1, P2, P0\right) = \left(\left(t_1 t_2\right) + \left(t_1 - t_1 t_2\right) + \left(t_2 - t_1 t_2\right) + \left(1 - t_1 - t_2 + t_1 t_2\right),\right.$$
$$\left.\left(f_1 + f_2 - f_1 f_2\right) + \left(f_1 + t_2 - f_1 t_2\right) + \left(t_1 + f_2 - t_1 f_2\right) + \left(t_1 + t_2 - t_1 t_2\right) - 3\right) = (1,0) .$$

So, the whole fuzzy space is normalized under $FL(\cdot)$.

For the neurosophic logic, we consider

$$NL(x) = (T_1, I_1, F_1), \text{ with } 0 \le T_1, I_1, F_1 \le 1;$$
$$NL(y) = (T_2, I_2, F_2), \text{ with } 0 \le T_2, I_2, F_2 \le 1;$$

if the sum of components is 1 as in Atanassov's intuitionist fuzzy logic, i.e. $T_i + I_i + F_i = 1$, they are considered *normalized*; otherwise *non-normalized*, i.e. the sum of the components is <1 (*sub-normalized*) or >1 (*over-normalized*).

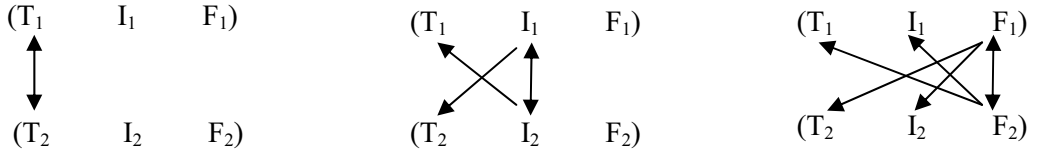
We define a binary neutrosophic conjunction (intersection) operator, which is a particular case of an N-norm (neutrosophic norm, a generalization of the fuzzy t-norm):

$$c_N : \left([0,1]\times[0,1]\times[0,1]\right)^2 \to [0,1]\times[0,1]\times[0,1]$$
$$c_N(x,y) = \left(T_1T_2, I_1I_2 + I_1T_2 + T_1I_2, F_1F_2 + F_1I_2 + F_1T_2 + F_2T_1 + F_2I_1\right).$$

The neutrosophic conjunction (intersection) operator $x \wedge_N y$ component truth, indeterminacy, and falsehood values result from the multiplication

$$\left(T_1 + I_1 + F_1\right)\cdot\left(T_2 + I_2 + F_2\right)$$

since we consider in a prudent way $T \prec I \prec F$, where "$\prec$" means "weaker", i.e. the products T_iI_j will go to I, T_iF_j will go to F, and I_iF_j will go to F (or reciprocally we can say that F prevails in front of I and of T,



So, the truth value is T_1T_2, the indeterminacy value is $I_1I_2 + I_1T_2 + T_1I_2$ and the false value is $F_1F_2 + F_1I_2 + F_1T_2 + F_2T_1 + F_2I_1$. The norm of $x \wedge_N y$ is $\left(T_1 + I_1 + F_1\right)\cdot\left(T_2 + I_2 + F_2\right)$. Thus, if x and y are normalized, then $x \wedge_N y$ is also normalized. Of course, the reader can redefine the neutrosophic conjunction operator, depending on application, in a different way, for example in a more optimistic way, i.e. $I \prec T \prec F$ or T prevails with respect to I, then we get:

$$c_N^{ITF}(x,y) = \left(T_1T_2 + T_1I_2 + T_2I_1, I_1I_2, F_1F_2 + F_1I_2 + F_1T_2 + F_2T_1 + F_2I_1\right).$$

Or, the reader can consider the order $T \prec F \prec I$, etc.

Let's also define the unary neutrosophic negation operator:

$$n_N : [0,1]\times[0,1]\times[0,1] \to [0,1]\times[0,1]\times[0,1]$$
$$n_N\left(T,I,F\right) = \left(F,I,T\right)$$

by interchanging the truth T and falsehood F vector components.
Then:

$$NL(P12)=\left(T_1T_2, I_1I_2+I_1T_2+I_2T_1, F_1F_2+F_1I_2+F_1T_2+F_2T_1+F_2I_1\right)$$
$$NL(P1)=\left(T_1F_2, I_1I_2+I_1F_2+I_2T_1, F_1T_2+F_1I_2+F_1F_2+T_2T_1+T_2I_1\right)$$
$$NL(P2)=\left(F_1T_2, I_1I_2+I_1T_2+I_2F_1, T_1F_2+T_1I_2+T_1T_2+F_2F_1+F_2I_1\right)$$
$$NL(P0)=\left(F_1F_2, I_1I_2+I_1F_2+I_2F_1, T_1T_2+T_1I_2+T_1F_2+T_2F_1+T_2I_1\right)$$

Similarly as in our above fuzzy logic work, we now define a binary $N-conorm$ (disjunction or union), i.e. neutrosophic conform.

$$d_N:\left([0,1]\times[0,1]\times[0,1]\right)^2\to[0,1]\times[0,1]\times[0,1]$$
$$d_N(x,y)=\left(T_1+T_2,\left(I_1+I_2\right)\cdot\frac{\tau-T_1-T_2}{I_1+I_2+F_1+F_2},\left(F_1+F\right)\cdot\frac{\tau-T_1-T_2}{I_1+I_2+F_1+F_2}\right)$$

if x and y are disjoint, and $T_1+T_2\le 1$ where τ is the neutrosophic norm of $x\vee_N y$, i.e.

$$\tau=\left(T_1+I_1+F_1\right)\cdot\left(T_2+I_2+F_2\right).$$

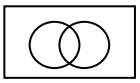
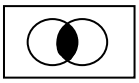
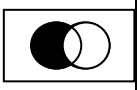
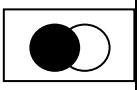
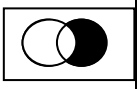
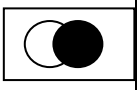
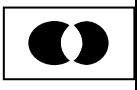
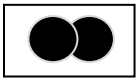
We consider as neutrosophic norm of x, where $NL(x)=T_1+I_1+F_1$, the sum of its components: $T_1+I_1+F_1$, which in many cases is 1, but can also be positive <1 or >1.

When the truth value increases $\left(T_1+T_2\right)$ is the above definition, the indeterminacy and falsehood values decrease proportionally with respect to their sums I_1+I_2 and respectively F_1+F_2.

This neutrosophic disjunction operator of disjoint variables allows us to add neutrosophic truth values of disjoint parts of a shaded area in a Venn Diagram.

Now, we complete Donald E. Knuth's Table of the Sixteen Logical Operators on two variables with Fuzzy Logical operators on two variables with Fuzzy Logic truth values, and Neutrosophic Logic truth/indeterminacy/false values (for the case $T\prec I\prec F$).

Table 1

Fuzzy Logic Truth Values	**Venn Diagram**	**Notations**	**Operator symbol**	**Name(s)**
0		0	$\bot$	Contradiction, falsehood; constant 0
$t_1 t_2$		$xy,\ x \wedge y,\ x \,\&\, y$	$\wedge$	Conjunction; and
$t_1 - t_1 t_2$		$x \wedge \overline{y}, x \not\supset y, [x > y], x - y$	$\overline{\supset}$	Nonimplication; difference, but not
t_1		x	L	Left projection
$t_2 - t_1 t_2$		$\overline{x} \wedge y, x \not\subset y, [x < y], y - x$	$\overline{\subset}$	Converse nonimplication; not…but
t_2		y	R	Right projection
$t_1 + t_2 - 2t_1 t_2$		$x \oplus y, x \not\equiv y, x \hat{} y$	$\oplus$	Exclusive disjunction; nonequivalence; "xor"
$t_1 + t_2 - t_1 t_2$		$x \vee y, x \mid y$	$\vee$	(Inclusive) disjunction; or; and/or
$1 - t_1 - t_2 + t_1 t_2$		$\overline{x} \wedge \overline{y}, \overline{x \vee y}, x \,\overline{\vee}\, y, x \uparrow y$	$\overline{\vee}$	Nondisjunction, joint denial, neither…nor
$1 - t_1 - t_2 + 2t_1 t_2$		$x \equiv y, x \leftrightarrow y, x \Leftrightarrow y$	$\equiv$	Equivalence; if and only if
$1 - t_2$		$\overline{y}, \neg y, !y, \sim y$	$\overline{R}$	Right complementation
$1 - t_2 + t_1 t_2$		$x \vee \overline{y}, x \subset y, x \Leftarrow y,$ $[x \geq y], x^y$	$\subset$	Converse implication if
$1 - t_1$		$\overline{x}, \neg x, !x, \sim x$	$\overline{L}$	Left complementation

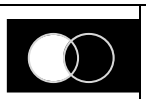
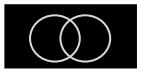
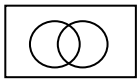
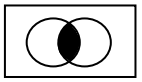
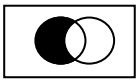
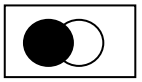
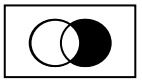
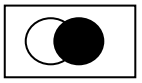
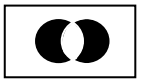
$1-t_1+t_1t_2$		$\overline{x} \vee y, x \supset y, x \Rightarrow y,$ $[x \leq y], y^x$	$\supset$	Implication; only if; if..then
$1-t_1t_2$		$\overline{x} \vee \overline{y}, \overline{x \wedge y}, x \overline{\wedge} y, x \mid y$	$\overline{\wedge}$	Nonconjunction, not both…and; "nand"
1		1	T	Affirmation; validity; tautology; constant 1

Table 2

Venn Diagram	Neutrosophic Logic Values
	$(0,0,1)$
	$(T_1T_2, I_1I_2 + IT, F_1F_2 + FI + FT)$, where $IT = I_1T_2 + I_2T_1$ similarly FI, FT ;
	$\left(T_1F_2, \underbrace{I_1I_2 + IT_{\bar{y}}}_{I_{P1}}, \underbrace{F_{\bar{y}}F_{\bar{y}} + F_{\bar{y}}I + F_{\bar{y}}T}_{F_{P2}}\right)$
	(T_1, I_1, F_1)
	$\left(F_1T_2, \underbrace{I_1I_2 + IT_{\bar{x}}}_{I_{P2}}, \underbrace{F_{\bar{x}}F_{\bar{x}} + F_{\bar{x}}I + F_{\bar{x}}T}_{F_{P2}}\right)$
	(T_2, I_2, F_2)
	$\left(TF, (I_{P1} + I_{P2}) \cdot \frac{\tau - TF}{I_{P1} + I_{P2} + F_{P1} + F_{P2}}, (F_{P1} + F_{P2}) \cdot \frac{\tau - TF}{I_{P1} + I_{P2} + F_{P1} + F_{P2}}\right)$ Where $\tau = (T_1 + I_1 + F_1) \cdot (T_2 + I_2 + F_2)$ which is the neutrosophic norm
	$(T_1T_2 + TI + TF, I_1I_2 + IF, F_1F_2)$
	$(F_1F_2, I_1I_2 + IF, T_1T_2 + TI + TF)$
	$\left((F_{P1} + F_{P2}) \cdot \frac{\tau - TF}{I_{P1} + I_{P2} + F_{P1} + F_{P2}}, (I_{P1} + I_{P2}) \cdot \frac{\tau - TF}{I_{P1} + I_{P2} + F_{P1} + F_{P2}}, TF\right)$
	(F_2, I_2, T_2)
	$(F_{\bar{x}}F_{\bar{x}} + F_{\bar{x}}I + F_{\bar{x}}T, I_1I_2 + IT_{\bar{x}}, F_1T_2)$
	(F_1, I_1, T_1)
	$(F_{\bar{y}}F_{\bar{y}} + F_{\bar{y}}I + F_{\bar{y}}T, I_1I_2 + IT_{\bar{y}}, T_1F_2)$

	$(F_1F_2 + FI + FT, I_1I_2 + IT, T_1T_2)$
	$(1,0,0)$

These 16 neutrosophic binary operators are approximated, since the binary N-conorm gives an approximation because of 'indeterminacy' component.

Tri-nary Fuzzy Logic and Neutrosophic Logic Operators

In a more general way, for $k \geq 2$:

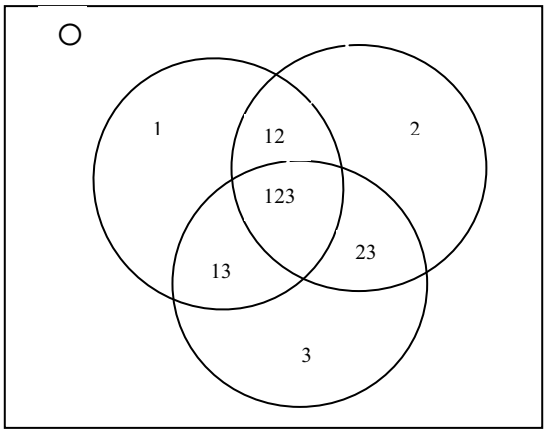
$$d_N^k : \left([0,1]\times[0,1]\times[0,1]\right)^k \to [0,1]\times[0,1]\times[0,1],$$

$$d_N^k(x_1,x_2,...,x_k) = \left(\sum_{i=1}^k T_i, \left(\sum_{i=1}^k I_i\right)\cdot\frac{\tau_k - \sum_{i=1}^k T_i}{\sum_{i=1}^k (I_i+F_i)}, \left(\sum_{i=1}^k F_i\right)\cdot\frac{\tau_k - \sum_{i=1}^k T_i}{\sum_{i=1}^k (I_i+F_i)}\right)$$

if all x_i are disjoint two by two, and $\sum_{i=1}^k T_i \leq 1$.

We can extend Knuth's Table from binary operators to tri-nary operators (and we get $2^{2^3} = 256$ tri-nary operators) and in general to n-ary operators (and we get 2^{2^n} n-ary operators).

Let's present the tri-nary Venn Diagram, with 3 variables x, y, z



using the name Smarandache codification.

This has $2^3 = 8$ disjoint parts, and if we shade none, one, two, …, or eight of them and consider all possible combinations we get $2^8 = 256$ tri-nary operators in the above tri-nary Venn Diagram.

For n=3 we have:

$$P123 = c_F(x, y, z)$$
$$P12 = c_F\left(x, y, n_F(z)\right)$$
$$P13 = c_F\left(x, n_F(y), z\right)$$
$$P23 = c_F\left(n_F(x), y, z\right)$$
$$P1 = c_F\left(x, n_F(y), n_F(z)\right)$$
$$P2 = c_F\left(n_F(x), y, n_F(z)\right)$$
$$P3 = c_F\left(n_F(x), n_F(y), z\right)$$
$$P0 = c_F\left(n_F(x), n_F(y), n_F(z)\right)$$

Let

$$FL(x) = (t_1, f_1), \text{ with } t_1 + f_1 = 1, \quad 0 \le t_1, f_1 \le 1,$$
$$FL(y) = (t_2, f_2), \text{ with } t_2 + f_2 = 1, \quad 0 \le t_2, f_2 \le 1,$$
$$FL(z) = (t_3, f_3), \text{ with } t_3 + f_3 = 1, \quad 0 \le t_3, f_3 \le 1.$$

We consider the particular case defined by tri-nary conjunction fuzzy operator:

$$c_F : \left([0,1]\times[0,1]\right)^3 \to [0,1]\times[0,1]$$
$$c_F(x, y, z) = \left(t_1t_2t_3, f_1 + f_2 + f_3 - f_1f_2 - f_2f_3 - f_3f_1 + f_1f_2f_3\right)$$

because

$$\left((t_1, f_1)\wedge_F (t_2, f_2)\right)\wedge_F (t_3, f_3) = \left(t_1t_2, f_1 + f_2 - f_1f_2\right)\wedge_F (t_3, f_3) =$$
$$= \left(t_1t_2t_3, f_1 + f_2 + f_3 - f_1f_2 - f_2f_3 - f_3f_1 + f_1f_2f_3\right)$$

and the unary negation operator:

$$n_F : \left([0,1]\times[0,1]\right) \to [0,1]\times[0,1]$$
$$n_F(x) = (1 - t_1, 1 - f_1) = (f_1, t_1).$$

We define the function:

$$L_1 : [0,1]\times[0,1]\times[0,1] \to [0,1]$$
$$L_1(\alpha, \beta, \gamma) = \alpha \cdot \beta \cdot \gamma$$

and the function

$$L_2 : [0,1]\times[0,1]\times[0,1] \to [0,1]$$
$$L_2(\alpha, \beta, \gamma) = \alpha + \beta + \gamma - \alpha\beta - \beta\gamma - \gamma\alpha + \alpha\beta\gamma$$

then:

$$FL(P123) = \left(L_1(t_1,t_2,t_3), L_2(f_1,f_2,f_3)\right)$$
$$FL(P12) = \left(L_1(t_1,t_2,f_3), L_2(f_1,f_2,t_3)\right)$$
$$FL(P13) = \left(L_1(t_1,f_2,t_3), L_2(f_1,t_2,f_3)\right)$$
$$FL(P23) = \left(L_1(f_1,t_2,t_3), L_2(t_1,f_2,f_3)\right)$$
$$FL(P1) = \left(L_1(t_1,f_2,f_3), L_2(f_1,t_2,t_3)\right)$$
$$FL(P2) = \left(L_1(f_1,t_2,f_3), L_2(t_1,f_2,t_3)\right)$$
$$FL(P3) = \left(L_1(f_1,f_2,t_3), L_2(t_1,t_2,f_3)\right)$$
$$FL(P0) = \left(L_1(f_1,f_2,f_3), L_2(t_1,t_2,t_3)\right)$$

We thus get the fuzzy truth-values as follows:

$$FL_t(P123) = t_1t_2t_3$$
$$FL_t(P12) = t_1t_2(1-t_3) = t_1t_2 - t_1t_2t_3$$
$$FL_t(P13) = t_1(1-t_2)t_3 = t_1t_3 - t_1t_2t_3$$
$$FL_t(P23) = (1-t_1)t_2t_3 = t_2t_3 - t_1t_2t_3$$
$$FL_t(P1) = t_1(1-t_2)(1-t_3) = t_1 - t_1t_2 - t_1t_3 + t_1t_2t_3$$
$$FL_t(P2) = (1-t_1)t_2(1-t_3) = t_2 - t_1t_2 - t_2t_3 + t_1t_2t_3$$
$$FL_t(P3) = (1-t_1)(1-t_2)t_3 = t_3 - t_1t_3 - t_2t_3 + t_1t_2t_3$$
$$FL_t(P0) = (1-t_1)(1-t_2)(1-t_3) = 1 - t_1 - t_2 - t_3 + t_1t_2 + t_1t_3 + t_2t_3 - t_1t_2t_3.$$

We, then, consider the same disjunction or union operator $d_F(x,y) = t_1 + t_2, f_1 + f_2 - 1$, if x and y are disjoint, and $t_1 + t_2 \leq 1$ allowing us to add the fuzzy truth values of each part of a shaded area.

Neutrophic Composition Law

Let's consider $k \geq 2$ neutrophic variables, $x_i(T_i, I_i, F_i)$, for all $i \in \{1,2,...,k\}$. Let denote

$$T = (T_1,...,T_k)$$
$$I = (I_1,...,I_k)$$
$$F = (F_1,...,F_k).$$

We now define a neutrosophic composition law o_N in the following way:

$$o_N : \{T,I,F\} \to [0,1]$$

If $z \in \{T,I,F\}$ then $z_{o_N} z = \prod_{i=1}^{k} z_i$.

If $z, w \in \{T,I,F\}$ then

$$z_{o_N} w = w_{o_N} z \quad = \sum_{\substack{r=1 \\ \{i_1,...,i_r,j_{r+1},...,j_k\}\equiv\{1,2,...,k\} \\ (i_1,...,i_r)\in C^r(1,2,...,k) \\ (j_{r+1},...,j_k)\in C^{k-r}(1,2,...,k)}}^{k-1} z_{i_1}...z_{i_r} w_{j_{r+1}}...w_{j_k}$$

where $C^r(1,2,...,k)$ means the set of combinations of the elements $\{1,2,...,k\}$ taken by r. [Similarly for $C^{k-r}(1,2,...,k)$].

In other words, $z_{o_N} w$ is the sum of all possible products of the components of vectors z and w, such that each product has at least a z_i factor and at least w_j factor, and each product has exactly k factors where each factor is a different vector component of z or of w. Similarly if we multiply three vectors:

$$T_{o_N} I_{o_N} F = \sum_{\substack{u,v,k-u-v=1 \\ \{i_1,...,i_u,j_{u+1},...,j_{u+v},l_{u+v+1},...,l_k\}\equiv\{1,2,...,k\} \\ (i_1,...,i_u)\in C^u(1,2,...,k),(j_{u+1},...,j_{u+v})\in \\ \in C^v(1,2,...,k),(l_{u+v+1},...,l_k)\in C^{k-u-v}(1,2,...,k)}}^{k-2} T_{i_1...i_u} I_{j_{u+1}...j_{u+v}} F_{l_{u+v+1}}...F_{l_k}$$

Let's see an example for $k=3$.

$$x_1(T_1, I_1, F_1)$$
$$x_2(T_2, I_2, F_2)$$
$$x_3(T_3, I_3, F_3)$$
$$T_{o_N} T = T_1T_2T_3, \quad I_{o_N} I = I_1I_2I_3, \quad \mathrm{F}_{o_N} F = F_1F_2F_3$$
$$T_{o_N} I = T_1I_2I_3 + I_1T_2I_3 + I_1I_2T_3 + T_1T_2I_3 + T_1I_2T_3 + I_1T_2T_3$$
$$T_{o_N} F = T_1F_2F_3 + F_1T_2F_3 + F_1F_2T_3 + T_1T_2F_3 + T_1F_2T_3 + F_1T_2T_3$$
$$I_{o_N} F = I_1F_2F_3 + F_1I_2F_3 + F_1F_2I_3 + I_1I_2F_3 + I_1F_2I_3 + F_1I_2I_3$$
$$T_{o_N} I_{o_N} F = T_1I_2F_3 + T_1F_2I_3 + I_1T_2F_3 + I_1F_2T_3 + F_1I_2T_3 + F_1T_2I_3$$

For the case when indeterminacy I is not decomposed in subcomponents {as for example $I = P \cup U$ where P =paradox (true and false simultaneously) and U =uncertainty (true or false, not sure which one)}, the previous formulas can be easily written using only three components as:

$$T_{o_N} I_{o_N} F = \sum_{i,j,r\in\mathcal{P}(1,2,3)} T_i I_j F_r$$

where $\mathcal{P}(1,2,3)$ means the set of permutations of $(1,2,3)$ i.e.

$$\{(1,2,3),(1,3,2),(2,1,3),(2,3,1,),(3,1,2),(3,2,1)\}$$

$$z_{o_N} w = \sum_{\substack{\mathrm{i}=1 \\ (i,j,r)\equiv(1,2,3) \\ (j,r)\in\mathcal{P}^2(1,2,3)}}^{3} z_i w_j w_{j_r} + w_i z_j z_r$$

This neurotrophic law is associative and commutative.

Neutrophic Logic Operators

Let's consider the neutrophic logic cricy values of variables x, y, z (so, for $n = 3$) $NL(x) = (T_1, I_1, F_1)$ with $0 \le T_1, I_1, F_1 \le 1$

$$NL(y) = (T_2, I_2, F_2) \text{ with } 0 \le T_2, I_2, F_2 \le 1$$

$$NL(z) = (T_3, I_3, F_3) \text{ with } 0 \le T_3, I_3, F_3 \le 1$$

In neutrosophic logic it is not necessary to have the sum of components equals to 1, as in intuitionist fuzzy logic, i.e. $T_k + I_k + F_k$ is not necessary 1, for $1 \le k \le 3$

As a particular case, we define the tri-nary conjunction neutrosophic operator:

$$c_N : ([0,1] \times [0,1] \times [0,1])^3 \to [0,1] \times [0,1] \times [0,1]$$

$$c_N(x, y) = \left(T_{o_N} T, I_{o_N} I + I_{o_N} T, F_{o_N} F + F_{o_N} I + F_{o_N} T\right)$$

If x or y are normalized, then $c_N(x, y)$ is also normalized.

If x or y are non-normalized then $|c_N(x, y)| = |x| \cdot |y|$ where $|\cdot|$ means norm.

c_N is an N-norm (neutrosophic norm, i.e. generalization of the fuzzy t-norm).

Again, as a particular case, we define the unary negation neutrosophic operator:

$$n_N : [0,1] \times [0,1] \times [0,1] \to [0,1] \times [0,1] \times [0,1]$$

$$n_N(x) = n_N(T_1, I_1, F_1) = (F_1, I_1, T_1).$$

We take the same Venn Diagram for $n = 3$.

So,

$$NL(x) = (T_1, I_1, F_1)$$

$$NL(y) = (T_2, I_2, F_2)$$

$$NL(z) = (T_3, I_3, F_3).$$

Vectors

$$\mathrm{T} = \begin{pmatrix} T_1 \\ T_2 \\ T_3 \end{pmatrix}, \quad \mathrm{I} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \text{ and } \mathrm{F} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}.$$

We note $\mathrm{T}_{\bar{x}} = \begin{pmatrix} F_1 \\ T_2 \\ T_3 \end{pmatrix}$, $\mathrm{T}_{\bar{y}} = \begin{pmatrix} T_1 \\ F_2 \\ T_3 \end{pmatrix}$, $\mathrm{T}_{\bar{z}} = \begin{pmatrix} T_1 \\ T_2 \\ F_3 \end{pmatrix}$, $\mathrm{T}_{\overline{xy}} = \begin{pmatrix} F_1 \\ F_2 \\ T_3 \end{pmatrix}$, etc.

and similarly

$$\mathrm{F}_{\bar{x}} = \begin{pmatrix} T_1 \\ F_2 \\ F_3 \end{pmatrix}, \quad F_{\bar{y}} = \begin{pmatrix} F_1 \\ T_2 \\ F_3 \end{pmatrix}, \quad \mathrm{F}_{\overline{xz}} = \begin{pmatrix} T_1 \\ F_2 \\ T_3 \end{pmatrix}, \text{ etc.}$$

For shorter and easier notations let's denote $z_{o_N} w = zw$ and respectively $z_{o_N} w_{o_N} v = zwv$ for the vector neutrosophic law defined previously.

Then

$$NL(P123) = c_N(x, y) = (TT, II + IT, FF + FI + FT + FIT) =$$

$$= \big(T_1 T_2 T_3, I_1 I_2 I_3 + I_1 I_2 T_3 + I_1 T_2 I_3 + T_1 I_2 I_3 + I_1 T_2 T_3 + T_1 I_2 T_3 + T_1 T_2 I_3,$$
$$F_1 F_2 F_3 + F_1 F_2 I_3 + F_1 I_2 F_3 + I_1 F_2 F_3 + F_1 I_2 I_3 + I_1 F_2 I_3 + I_1 I_2 F_3 +$$
$$+ F_1 F_2 T_3 + F_1 T_2 F_3 + T_1 F_2 F_3 + F_1 T_2 T_3 + T_1 F_2 T_3 + T_1 T_2 F_3 +$$
$$+ T_1 I_2 F_3 + T_1 F_2 I_3 + I_1 F_2 T_3 + I_1 T_2 F_3 + F_1 I_2 T_3 + F_1 T_2 I_3 \big)$$

$$NL(P12) = c_N\left(x, y, n_N(z)\right) = \left(T_{\bar{z}} T_{\bar{z}}, II + IT_{\bar{z}}, F_{\bar{z}} F_{\bar{z}} + F_{\bar{z}} I + F_{\bar{z}} T_{\bar{z}} + F_{\bar{z}} IT_{\bar{z}}\right)$$

$$NL(P13) = c_N\left(x, n_N(y), z\right) = \left(T_{\bar{y}} T_{\bar{y}}, II + IT_{\bar{y}}, F_{\bar{y}} F_{\bar{y}} + F_{\bar{y}} I + F_{\bar{y}} T_{\bar{y}} + F_{\bar{y}} IT_{\bar{y}}\right)$$

$$NL(P23) = c_N\left(n_N(x), y, z\right) = \left(T_{\bar{x}} T_{\bar{x}}, II + IT_{\bar{x}}, F_{\bar{x}} F_{\bar{x}} + F_{\bar{x}} I + F_{\bar{x}} T_{\bar{x}} + F_{\bar{x}} IT_{\bar{y}}\right)$$

$$NL(P1) = c_N\left(x, n_N(y), n_N(z)\right) = \left(T_{\overline{yz}} T_{\overline{yz}}, II + IT_{\overline{yz}}, F_{\overline{yz}} F_{\overline{yz}} + F_{\overline{yz}} I + F_{\overline{yz}} T_{\overline{yz}} + F_{\overline{yz}} IT_{\overline{yz}}\right)$$

$$NL(P2) = c_N\left(n_N(x), y, n_N(z)\right) = \left(T_{\overline{xz}} T_{\overline{xz}}, II + IT_{\overline{xz}}, F_{\overline{xz}} F_{\overline{xz}} + F_{\overline{xz}} I + F_{\overline{xz}} T_{\overline{xz}} + F_{\overline{xz}} IT_{\overline{xz}}\right)$$

$$NL(P0) = c_N\left(n_N(x), n_N(y), n_N(z)\right) = \left(T_{\overline{xyz}} T_{\overline{xyz}}, II + IT_{\overline{xyz}}, F_{\overline{xyz}} F_{\overline{xyz}} + F_{\overline{xyz}} I + F_{\overline{xyz}} T_{\overline{xyz}} + F_{\overline{xyz}} IT_{\overline{xyz}}\right) =$$
$$= \left(FF, II + IF, TT + TI + TF + TIF\right).$$

n-ary Fuzzy Logic and Neutrosophic Logic Operators

We can generalize for any integer $n \geq 2$.

The Venn Diagram has 2^{2^n} disjoint parts. Each part has the form $Pi_1 ... i_k j_{k+1} ... j_n$, where $0 \leq k \leq n$, and of course $0 \leq n-k \leq n$; $\{i_1, ..., i_k\}$ is a combination of k elements of the set $\{1, 2, ..., n\}$, while $\{j_{k+1}, ..., j_n\}$ the $n-k$ elements left, i.e. $\{j_{k+1}, ..., j_k\} = \{1, 2, ..., n\} \setminus \{i_1, ..., i_k\}$. $\{i_1, ..., i_k\}$ are replaced by the corresponding numbers from $\{1, 2, ..., n\}$, while $\{j_{k+1}, ..., j_n\}$ are replaced by blanks.

For example, when $n = 3$,

$$Pi_1 i_2 j_3 = P13 \text{ if } \{i_1, i_2\} = \{1, 3\},$$
$$Pi_1 j_2 j_3 = P1 \text{ if } \{i_1\} = \{1\}.$$

Hence, for fuzzy logic we have:

$$Pi_1 ... i_k j_{k+1} ... j_n = c_F\left(x_{i_1}, ..., x_{i_k}, n_F\left(x_{j_{k+1}}\right), ..., n_F\left(x_{j_n}\right)\right)$$

whence

$$FL\left(Pi_1 ... i_k j_{k+1} ... j_n\right) = \left(\left(\prod_{r=1}^{k} t_{i_r}\right)\left(\prod_{s=k+1}^{n}\left(1 - t_{j_s}\right)\right), \varphi\left(f_1 f_2, ..., f_n\right)\right)$$

where $\varphi : [0,1]^n \to [0,1]$,

$$\varphi\left(\alpha_1, \alpha_2, ..., \alpha_n\right) = S_1 - S_2 + S_3 + ... + (-1)^{n+1} S_n = \sum_{l=1}^{n} (-1)^{l+1} S_l$$

where

$$S_1 = \sum_{i=1}^{n} \alpha_i$$

$$S_2 = \sum_{1 \le i < j \le n} \alpha_i \alpha_j$$

.....

$$S_l = \sum_{1 \le i_1 < i_2 < \ldots < i_l \le n} \alpha_{i_1} \alpha_{i_2} .. \alpha_{i_l}$$

.....

$$S_n = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n$$

And for neutrosophic logic we have:

$$Pi_1 \ldots i_k j_{k+1} \ldots j_n = c_N \left(x_{i_1}, \ldots, x_{i_k}, n_N \left(x_{j_{k+1}} \right), \ldots, n_N \left(x_{j_n} \right) \right)$$

whence:

$$NL\left(Pi_1 \ldots i_k j_{k+1} \ldots j_n \right) = \left(T_{12\ldots n}, I_{12\ldots n}, F_{12\ldots n} \right),$$

where

$$T_{12\ldots n} = T_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} T_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} = \left(\prod_{r=1}^{k} T_{i_r} \right) \cdot \prod_{s=k+1}^{n} F_{j_s} \ .$$

$$I_{12\ldots n} = II + IT_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}},$$

$$F_{12\ldots n} = F_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} F_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} + F_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} I + F_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} T_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} + F_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}} IT_{\bar{x}_{j_{k+1}} \ldots \bar{x}_{j_n}}$$

Conclusion:

A generalization of Knuth's Boolean binary operations is presented in this paper, i.e. we present n-ary Fuzzy Logic Operators and Neutrosophic Logic Operators based on Smarandache's codification of the Venn Diagram and on a defined vector neutrosophic law which helps in calculating fuzzy and neutrosophic operators.

Better neutrosophic operators than in [2] are proposed herein.

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A Neutrosophic Description Logic

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Abstract. Description Logics (DLs) are appropriate, widely used, logics for managing structured knowledge. They allow reasoning about individuals and concepts, *i.e.* set of individuals with common properties. Typically, DLs are limited to dealing with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership: we may say that an individual is an instance of a concept only to a certain degree, depending on the individual's properties. The DLs that deal with such fuzzy concepts are called fuzzy DLs. In order to deal with fuzzy, incomplete, indeterminate and inconsistent concepts, we need to extend the capabilities of fuzzy DLs further.
In this paper we will present an extension of fuzzy $\mathcal{ALC}$, combining Smarandache's neutrosophic logic with a classical DL. In particular, concepts become neutrosophic (here neutrosophic means fuzzy, incomplete, indeterminate and inconsistent), thus, reasoning about such neutrosophic concepts is supported. We will define its syntax, its semantics, describe its properties and present a constraint propagation calculus for reasoning in it.

Keywords: Description logic, fuzzy description logic, fuzzy $\mathcal{ALC}$, neutrosophic description logic.

1 Introduction

The modelling and reasoning with uncertainty and imprecision is an important research topic in the Artificial Intelligence community. Almost all the real world

knowledge is imperfect. A lot of works have been carried out to extend existing knowledge-based systems to deal with such imperfect information, resulting in a number of concepts being investigated, a number of problems being identified and a number of solutions being developed [2, 6, 8, 9].

Description Logics (DLs) have been utilized in building a large amount of knowledge-based systems. DLs are a logical reconstruction of the so-called frame-based knowledge representation languages, with the aim of providing a simple well-established Tarski-style declarative semantics to capture the meaning of the most popular features of structured representation of knowledge. A main point is that DLs are considered as to be attractive logics in knowledge based applications as they are a good compromise between expressive power and computational complexity.

Nowadays, a whole family of knowledge representation systems has been build using DLs, which differ with respect to their expressiveness, their complexity and the completeness of their algorithms, and they have been used for building a variety of applications [10, 3, 1, 7].

The classical DLs can only deal with crisp, well defined concepts. That is, concepts for which the problem whether an individual is an instance of it is a yes/no question. More often than not, the concepts encountered in the real world do not have a precisely defined criteria of membership. There are many works attempted to extend the DLs using fuzzy set theory [12–14, 5, 15, 17]. These fuzzy DLs can only deal with fuzzy concepts but not incomplete, indeterminate, and inconsistent concepts (neutrosophic concepts). For example, "Good Person" is a neutrosophic concepts, in the sense that by different subjective opinions, the truth-membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.6, which is inconsistent, or the truth- membership degree of tom is good person is 0.6, and the falsity-membership degree of tom is good person is 0.3, which is incomplete.

The set and logic that can model and reason with fuzzy, incomplete, indeterminate, and inconsistent information are called neutrosophic set and neutrosophic logic, respectively [11, 16]. In Smarandache's neutrosophic set theory,a neutrosophic set A defined on universe of discourse X, associates each element x in X with three membership functions: truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$, where $T_A(x), I_A(x), F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$, and $T_A(x), I_A(x), F_A(x)$ are independent. For simplicity, in this paper, we will extend Straccia's fuzzy DLs [12, 14] with neutrosophic logic, called neutrosophic DLs, where we only use two components $T_A(x)$ and $F_A(x)$, with $T_A(x) \in [0, 1], F_A(x) \in [0, 1], 0 \leq T_A(x) + F_A(x) \leq 2$. The neutrosophic DLs is based on the DL $\mathcal{ALC}$, a significant and expressive representative of the various DLs. This allows us to adapt it easily to the different DLs presented in the literature. Another important point is that we will show that the additional expressive power has no impact from a computational complexity point of view. The neutrosophic $\mathcal{ALC}$ is a strict generalization of fuzzy $\mathcal{ALC}$, in the sense that every fuzzy concept and fuzzy terminological axiom can be represented by a

corresponding neutrosophic concept and neutrosophic terminological axiom, but not *vice versa*.

The rest of paper is organized as follows. In the following section we first introduce Straccia's $\mathcal{ALC}$. In section 3 we extend $\mathcal{ALC}$ to the neutrosophic case and discuss some properties in Section 4, while in Section 5 we will present a constraint propagation calculus for reasoning in it. Section 6 concludes and proposes future work.

2 A Quick Look to Fuzzy $\mathcal{ALC}$

We assume three alphabets of symbols, called *atomic concepts* (denoted by A), *atomic roles* (denoted by R) and *individuals* (denoted by a and b). [1]

A *concept* (denoted by C or D) of the language $\mathcal{ALC}$ is built out of atomic concepts according to the following syntax rules:

$$
\begin{aligned}
C, D \longrightarrow \quad & \top \mid \text{(top concept)} \\
& \bot \mid \text{(bottom concept)} \\
& A \mid \text{(atomic concept)} \\
& C \sqcap D \mid \text{(concept conjunction)} \\
& C \sqcup D \mid \text{(concept disjunction)} \\
& \neg C \mid \text{(concept negation)} \\
& \forall R.C \mid \text{(universal quantification)} \\
& \exists R.C \text{ (existential quantification)}
\end{aligned}
$$

Fuzzy DL extends classical DL under the framework of Zadeh's fuzzy sets [18].A *fuzzy set* S with respect to an universe U is characterized by a *membership function* $\mu_S : U \rightarrow [0, 1]$, assigning an S-membership degree, $\mu_S(u)$, to each element u in U. In fuzzy DL, (i) a concept C, rather than being interpreted as a classical set, will be interpreted as a fuzzy set and, thus, concepts become *fuzzy*; and, consequently, (ii) the statement "a is C", *i.e.* $C(a)$, will have a truth-value in $[0, 1]$ given by the degree of membership of being the individual a a member of the fuzzy set C.

2.1 Fuzzy Interpretation

A *fuzzy interpretation* is now a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, .^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is, as for the crisp case, the *domain*, whereas $.^{\mathcal{I}}$ is an *interpretation function* mapping

1. individual as for the crisp case, *i.e.* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$;
2. a concept C into a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \rightarrow [0, 1]$;
3. a role R into a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \rightarrow [0, 1]$.

[1] Through this work we assume that every metavariable has an optional subscript or superscript.

If C is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as the *membership degree function* of the fuzzy concept (set) C w.r.t. $\mathcal{I}$, *i.e.* if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object d an element of the fuzzy concept C under the interpretation $\mathcal{I}$. Similarly for roles. Additionally, the interpretation function $.^{\mathcal{I}}$ has to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$,

$$\begin{array}{ll}
\top^{\mathcal{I}}(d) & = 1 \\
\bot^{\mathcal{I}}(d) & = 0 \\
(C \sqcap D)^{\mathcal{I}}(d) & = \min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(C \sqcup D)^{\mathcal{I}}(d) & = \max\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\} \\
(\neg C)^{\mathcal{I}}(d) & = 1 - C^{\mathcal{I}}(d) \\
(\forall R.C)^{\mathcal{I}}(d) & = \inf_{d' \in \Delta^{\mathcal{I}}}\{\max\{1 - R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\} \\
(\exists R.C)^{\mathcal{I}}(d) & = \sup_{d' \in \Delta^{\mathcal{I}}}\{\min\{R^{\mathcal{I}}(d, d'), C^{\mathcal{I}}(d')\}\}.
\end{array}$$

We will say that two concepts C and D are said to be *equivalent* (denoted by $C \cong D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $\mathcal{I}$. As for the crisp non fuzzy case, dual relationships between concepts hold: *e.g.* $\top \cong \neg\bot$, $(C \sqcap D) \cong \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong \neg(\exists R.\neg C)$.

2.2 Fuzzy Assertion

A *fuzzy assertion* (denoted by ψ) is an expression having one of the following forms $\langle \alpha \geq n \rangle$ or $\langle \alpha \leq m \rangle$, where α is an $\mathcal{ALC}$ assertion, $n \in (0, 1]$ and $m \in [0, 1)$. From a semantics point of view, a fuzzy assertion $\langle \alpha \leq n \rangle$ constrains the truth-value of α to be less or equal to n (similarly for $\geq$). Consequently, *e.g.* $\langle$ (**Video** $\sqcap \exists$**About.Basket**)(**v1**) $\geq 0.8\rangle$ states that video **v1** is likely about basket. Formally, an interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle C(a) \geq n \rangle$ (resp. $\langle R(a, b) \geq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$). Similarly, an interpretation $\mathcal{I}$ *satisfies* a fuzzy assertion $\langle C(a) \leq n \rangle$ (resp. $\langle R(a, b) \leq n \rangle$) iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq n$ (resp. $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq n$). Two fuzzy assertion ψ_1 and ψ_2 are said to be *equivalent* (denoted by $\psi_1 \cong \psi_2$) iff they are satisfied by the same set of interpretations. An *atomic fuzzy assertion* is a fuzzy assertion involving an atomic assertion (assertion of the form $A(a)$ or $R(a, b)$).

2.3 Fuzzy Terminological Axiom

From a syntax point of view, a *fuzzy terminological axiom* (denoted by $\tilde{\tau}$ is either a fuzzy concept specialization or a fuzzy concept definition. A *fuzzy concept specialization* is an expression of the form $A \prec C$, where A is an atomic concept and C is a concept. On the other hand, a *fuzzy concept definition* is an expression of the form $A :\approx C$, where A is an atomic concept and C is a concept. From a semantics point of view, a fuzzy interpretation $\mathcal{I}$ *satisfies* a fuzzy concept specialization $A \prec C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, A^{\mathcal{I}}(d) \leq C^{\mathcal{I}}(d), \tag{1}$$

whereas $\mathcal{I}$ *satisfies* a fuzzy concept definition $A :\approx C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, A^{\mathcal{I}}(d) = C^{\mathcal{I}}(d). \tag{2}$$

2.4 Fuzzy Knowlege Base, Fuzzy Entailment and Fuzzy Subsumption

A *fuzzy knowledge base* is a finite set of fuzzy assertions and fuzzy terminological axioms. Σ_A denotes the set of fuzzy assertions in Σ, Σ_T denotes the set of fuzzy terminological axioms in Σ (the terminology), if $\Sigma_T = \emptyset$ then Σ is *purely assertional*, and we will assume that a terminology Σ_T is such that no concept A appears more than once on the left hand side of a fuzzy terminological axiom $\tilde{\tau} \in \Sigma_T$ and that no cyclic definitions are present in Σ_T.

An interpretation $\mathcal{I}$ *satisfies* (is *a model of*) a set of fuzzy Σ iff $\mathcal{I}$ satisfies each element of Σ. A fuzzy KB Σ *fuzzy entails* a fuzzy assertion ψ (denoted by $\Sigma \models_f \psi$) iff every model of Σ also satisfies ψ.

Furthermore, let Σ_T be a terminology and let C, D be two concepts. We will say that D *fuzzy subsumes* C w.r.t. Σ_T (denoted by $C \preceq_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of Σ_T, $\forall d \in \Delta^{\mathcal{I}}, C^{\mathcal{I}}(d) \leq D^{\mathcal{I}}(d)$ holds.

3 A Neutrosophic DL

Our neutrosophic extension directly relates to Smarandache's work on neutrosophic sets [11, 16]. A neutrosophic set S defined on universe of discourse U, associates each element u in U with three membership functions: truth-membership function $T_S(u)$, indeterminacy-membership function $I_S(u)$, and falsity-membership function $F_S(u)$, where $T_S(u), I_S(u), F_S(u)$ are real standard or non-standard subsets of $]^-0, 1^+[$, and $T_S(u), I_S(u), F_S(u)$ are independent. For simplicity, here we only use two components $T_S(u)$ and $F_S(u)$, with $T_S(u) \in [0, 1], F_S(u) \in [0, 1], 0 \leq T_S(u) + F_S(u) \leq 2$. It is easy to extend our method to include indeterminacy-membership function. $T_S(u)$ gives us an estimation of degree of u belonging to U and $F_S(u)$ gives us an estimation of degree of u not belonging to U. $T_S(u) + F_S(u)$ can be 1 (just as in classical fuzzy sets theory). But it is not necessary. If $T_S(u) + F_S(u) < 1$, for all u in U, we say the set S is incomplete, if $T_S(u) + F_S(u) > 1$, for all u in U, we say the set S is inconsistent. According to Wang [16], the truth-membership function and falsity-membership function has to satisfy three restrictions: for all $u \in U$ and for all neutrosophic sets S_1, S_2 with respect to U

$$T_{S_1 \cap S_2}(u) = \min\{T_{S_1}(u), T_{S_2}(u)\}, F_{S_1 \cap S_2}(u) = \max\{F_{S_1}(u), F_{S_2}(u)\}$$

$$T_{S_1 \cup S_2}(u) = \max\{T_{S_1}(u), T_{S_2}(u)\}, F_{S_1 \cup S_2}(u) = \min\{F_{S_1}(u), F_{S_2}(u)\}$$

$$T_{\overline{S_1}}(u) = F_{S_1}(u), F_{\overline{S_1}}(u) = T_{S_1}(u),$$

where $\overline{S_1}$ is the complement of S_1 in U. Wang [16] gives the definition of N-norm and N-conorm of neutrosophic sets, min and max is only one of the choices. In general case, they may be the simplest and the best.

When we switch to neutrosophic logic, the notion of degree of truth-membership $T_S(u)$ of an element $u \in U$ w.r.t. the neutrosophic set S over U is regarded as the *truth-value* of the statement "u is S", and the notion of degree of falsity-membership $F_S(u)$ of an element $u \in U$ w.r.t. the neutrosophic set S over U is regarded as the *falsity-value* of the statement "u is S". Accordingly, in our neutrosophic DL, (*i*) a concept C, rather than being interpreted as a fuzzy set, will be interpreted as a neutrosophic set and, thus, concepts become *imprecise* (fuzzy, incomplete, and inconsistent); and, consequently, (*ii*) the statement "a is C", *i.e.* $C(a)$ will have a truth-value in $[0,1]$ given by the degree of truth-membership of being the individual a a member of the neutrosophic set C and a falsity-value in $[0,1]$ given by the degree of falsity-membership of being the individual a not a member of the neutrosophic set C.

3.1 Neutrosophic Interpretation

A em neutrosophic interpretation is now a tuple $\mathcal{I} = (\Delta^{\mathcal{I}}, (\cdot)^{\mathcal{I}}, |\cdot|^t, |\cdot|^f)$, where $\Delta^{\mathcal{I}}$ is, as for the fuzzy case, the *domain*, and

1. $(\cdot)^{\mathcal{I}}$ is an *interpretation function* mapping
   (a) individuals as for the fuzzy case, *i.e.* $a^{\mathcal{I}} \neq b^{\mathcal{I}}$, if $a \neq b$;
   (b) a concept C into a membership function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1] \times [0,1]$;
   (c) a role R into a membership function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1] \times [0,1]$.
2. $|\cdot|^t$ and $|\cdot|^f$ are *neutrosophic valuation*, *i.e.* $|\cdot|^t$ and $|\cdot|^f$ map
   (a) every atomic concept into a function from $\Delta^{\mathcal{I}}$ to $[0,1]$;
   (b) every atomic role into a function from $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to $[0,1]$.

If C is a concept then $C^{\mathcal{I}}$ will naturally be interpreted as a pair of membership functions $\langle |C|^t, |C|^f \rangle$ of the neutrosophic concept (set) C w.r.t. $\mathcal{I}$, *i.e.* if $d \in \Delta^{\mathcal{I}}$ is an object of the domain $\Delta^{\mathcal{I}}$ then $C^{\mathcal{I}}(d)$ gives us the degree of being the object d an element of the neutrosophic concept C and the degree of being the object d not an element of the neutrosophic concept C under the interpretation $\mathcal{I}$. Similarly for roles. Additionally, the interpretation function $(\cdot)^{\mathcal{I}}$ has to satisfy the following equations: for all $d \in \Delta^{\mathcal{I}}$,

$$
\begin{aligned}
\top^{\mathcal{I}}(d) &= \langle 1, 0 \rangle \\
\bot^{\mathcal{I}}(d) &= \langle 0, 1 \rangle \\
(C \sqcap D)^{\mathcal{I}}(d) &= \langle \min\{|C|^t(d), |D|^t(d)\}, \max\{|C|^f(d), |D|^f(d)\} \rangle \\
(C \sqcup D)^{\mathcal{I}}(d) &= \langle \max\{|C|^t(d), |D|^t(d)\}, \min\{|C|^f(d), |D|^f(d)\} \rangle \\
(\neg C)^{\mathcal{I}}(d) &= \langle |C|^f(d), |C|^t \rangle \\
(\forall R.C)^{\mathcal{I}}(d) &= \langle \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{|R|^f(d,d'), |C|^t(d')\}\}, \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{|R|^t(d,d'), |C|^f(d')\}\} \rangle \\
(\exists R.C)^{\mathcal{I}}(d) &= \langle \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{|R|^t(d,d'), |C|^t(d')\}\}, \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{|R|^f(d,d'), |C|^f(d')\}\} \rangle
\end{aligned}
$$

Note that the semantics of $\forall R.C$

$$
(\forall R.C)^{\mathcal{I}}(d) = \langle \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{|R|^f(d,d'), |C|^t(d')\}\}, \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{|R|^t(d,d'), |C|^f(d')\}\} \rangle \tag{3}
$$

is the result of viewing $\forall R.C$ as the open first order formula $\forall y.\neg F_R(x,y) \vee F_C(y)$, where the universal quantifier $\forall$ is viewed as a conjunction over the elements of the domain. Similarly, the semantics of $\exists R.C$

$$(\exists R.C)^{\mathcal{I}}(d) = \langle \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{|R|^t(d,d'), |C|^t(d')\}\}, \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{|R|^f(d,d'), |C|^f(d')\}\}\rangle \tag{4}$$

is the result of viewing $\exists R.C$ as the open first order formula $\exists y.F_R(x,y) \wedge F_C(y)$ and the existential quantifier $\exists$ is viewed as a disjunction over the elements of the domain. Moreover, $|\cdot|^t$ and $|\cdot|^f$ are extended to complex concepts as follows: for all $d \in \Delta^{\mathcal{I}}$

$$\begin{array}{ll}
|C \sqcap D|^t(d) & = \min\{|C|^t(d), |D|^t(d)\} \\
|C \sqcap D|^f(d) & = \max\{|C|^f(d), |D|^f(d)\} \\
 & \\
|C \sqcup D|^t(d) & = \max\{|C|^t(d), |D|^t(d)\} \\
|C \sqcup D|^f(d) & = \min\{|C|^f(d), |D|^f(d)\} \\
 & \\
|\neg C|^t(d) & = |C|^f(d) \\
|\neg C|^f(d) & = |C|^t(d) \\
 & \\
|\forall R.C|^t(d) & = \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{|R(d,d')|^f, |C|^t(d)\}\} \\
|\forall R.C|^f(d) & = \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{|R(d,d')|^t, |C|^f(d)\}\} \\
 & \\
|\exists R.C|^t(d) & = \sup_{d' \in \Delta^{\mathcal{I}}} \{\min\{|R(d,d')|^t, |C|^t(d)\}\} \\
|\exists R.C|^f(d) & = \inf_{d' \in \Delta^{\mathcal{I}}} \{\max\{|R(d,d')|^f, |C|^f(d)\}\}
\end{array}$$

We will say that two concepts C and D are said to be *equivalent* (denoted by $C \cong^n D$) when $C^{\mathcal{I}} = D^{\mathcal{I}}$ for all interpretation $\mathcal{I}$. As for the fuzzy case, dual relationships between concepts hold: *e.g.* $\top \cong^n \neg\bot, (C \sqcap D) \cong^n \neg(\neg C \sqcup \neg D)$ and $(\forall R.C) \cong^n \neg(\exists R.\neg C)$.

3.2 Neutrosophic Assertion

A *neutrosophic assertion* (denoted by φ) is an expression having one of the following form $\langle \alpha :\geq n, \leq m\rangle$ or $\langle \alpha :\leq n, \geq m\rangle$, where α is an $\mathcal{ALC}$ assertion, $n \in [0,1]$ and $m \in [0,1]$. From a semantics point of view, a neutrosophic assertion $\langle \alpha :\geq n, \leq m\rangle$ constrains the truth-value of α to be greater or equal to n and falsity-value of α to be less or equal to m (similarly for $\langle \alpha :\leq n, \geq m\rangle$). Consequently, *e.g.* $\langle(\mathbf{Poll} \sqcap \exists \mathbf{Support.War\_x})(\mathbf{p1}) :\geq 0.8, \leq 0.1\rangle$ states that poll **p1** is close to support War_x. Formally, an interpretation $\mathcal{I}$ *satisfies* a neutrosophic assertion $\langle \alpha :\geq n, \leq m\rangle$ (resp. $\langle R(a,b) :\geq n, \leq m\rangle$) iff $|C|^t(a^{\mathcal{I}}) \geq n$ and $|C|^f(a^{\mathcal{I}}) \leq m$ (resp. $|R|^t(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$ and $|R|^f(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq m$). Similarly, an interpretation $\mathcal{I}$ *satisfies* a neutrosophic assertion $\langle \alpha :\leq n, \geq m\rangle$ (resp. $\langle R(a,b) :\leq n, \geq m\rangle$) iff $|C|^t(a^{\mathcal{I}}) \leq n$ and $|C|^f(a^{\mathcal{I}}) \geq m$ (resp. $|R|^t(a^{\mathcal{I}}, b^{\mathcal{I}}) \leq n$

and $|R|^f(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq m$). Two fuzzy assertion φ_1 and φ_2 are said to be *equivalent* (denoted by $\varphi_1 \cong^n \varphi_2$) iff they are satisfied by the same set of interpretations. Notice that $\langle \neg C(a) :\geq n, \leq m\rangle \cong^n \langle C(a) :\leq m, \geq n\rangle$ and $\langle \neg C(a) :\leq n, \geq m\rangle \cong^n \langle C(a) :\geq m, \leq n\rangle$. An *atomic neutrosophic assertion* is a neutrosophic assertion involving an atomic assertion.

3.3 Neutrosophic Terminological Axiom

Neutrosophic terminological axioms we will consider are a natural extension of fuzzy terminological axioms to the neutrosophic case. From a syntax point of view, a *neutrosophic terminological axiom* (denoted by $\hat{\tau}$) is either a neutrosophic concept specialization or a neutrosophic concept definition. A *neutrosophic concept specialization* is an expression of the form $A \prec^n C$, where A is an atomic concept and C is a concept. On the other hand, a *neutrosophic concept definition* is an expression of the form $A :\approx^n C$, where A is an atomic concept and C is a concept. From a semantics point of view, we consider the natural extension of fuzzy set to the neutrosophic case [11, 16]. A neutrosophic interpretation $\mathcal{I}$ *satisfies* a neutrosophic concept specialization $A \prec^n C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, |A|^t(d) \leq |C|^t(d), |A|^f(d) \geq |C|^f(d), \tag{5}$$

whereas $\mathcal{I}$ *satisfies* a neutrosophic concept definition $A :\approx^n C$ iff

$$\forall d \in \Delta^{\mathcal{I}}, |A|^t(d) = |C|^t(d), |A|^f(d) = |C|^f(d). \tag{6}$$

3.4 Neutrosophic Knowledge Base, Neutrosophic Entailment and Neutrosophic Subsumption

A *neutrosophic knowledge base* is a finite set of neutrosophic assertions and neutrosophic terminological axioms. As for the fuzzy case, with Σ_A we will denote the set of neutrosophic assertions in Σ, with Σ_T we will denote the set of neutrosophic terminological axioms in Σ (the terminology), if $\Sigma_T = \emptyset$ then Σ is *purely assertional*, and we will assume that a terminology Σ_T is such that no concept A appears more than once on the left hand side of a neutrosophic terminological axiom $\hat{\tau} \in \Sigma_T$ and that no cyclic definitions are present in Σ_T.

An interpretation $\mathcal{I}$ *satisfies* (is *a model of*) a neutrosophic Σ iff $\mathcal{I}$ satisfies each element of Σ. A neutrosophic KB Σ *neutrosophically entails* a neutrosophic assertion φ (denoted by $\Sigma \models^n \varphi$) iff every model of Σ also satisfies φ.

Furthermore, let Σ_T be a terminology and let C, D be two concepts. We will say that D *neutrosophically subsumes* C w.r.t. Σ_T (denoted by $C \preceq^n_{\Sigma_T} D$) iff for every model $\mathcal{I}$ of Σ_T, $\forall d \in \Delta^{\mathcal{I}}, |C|^t(d) \leq |D|^t(d)$ and $|C|^f(d) \geq |D|^f(d)$ holds.

Finally, given a neutrosophic KB Σ and an assertion α, we define the *greatest lower bound* of α w.r.t. Σ (denoted by $glb(\Sigma, \alpha)$) to be $\langle \sup\{n : \Sigma \models^n \langle \alpha :\geq n, \leq m\rangle\}, \inf\{m : \Sigma \models^n \langle \alpha :\geq n, \leq m\rangle\}\rangle$. Similarly, we define the *least upper bound* of α with respect to Σ (denoted by $lub(\Sigma, \alpha)$) to be $\langle \inf\{n : \Sigma \models^n \langle \alpha :\leq n, \geq m\rangle\}, \sup\{m : \Sigma \models^n \langle \alpha :\leq n, \geq m\rangle\}\rangle$ ($\sup \emptyset = 0, \inf \emptyset = 1$). Determing the *lub* and the *glb* is called the *Best Truth-Value Bound* (BTVB) problem.

4 Some Properties

In this section, we discuss some properties of our neutrosophic $\mathcal{ALC}$.

4.1 Concept Equivalence

The first ones are straightforward: $\neg\top \approx^n \bot, C \sqcap \top \approx^n C, C \sqcup \top \approx^n \top, C \sqcap \bot \approx^n \bot, C \sqcup \bot \approx^n C, \neg\neg C \approx^n C, \neg(C \sqcap D) \approx^n \neg C \sqcup \neg D, \neg(C \sqcup D) \approx^n \neg C \sqcap \neg D, C_1 \sqcap (C_2 \sqcup C_3) \approx^n (C_1 \sqcap C_2) \sqcup (C_1 \sqcap C_3)$ and $C_1 \sqcup (C_2 \sqcap C_3) \approx^n (C_1 \sqcup C_2) \sqcap (C_1 \sqcup C_3)$. For concepts involving roles, we have $\forall R.C \approx^n \neg\exists R.\neg C, \forall R.\top \approx^n \top, \exists R.\bot \approx^n \bot$ and $(\forall R.C) \sqcap (\forall R.D) \approx^n \forall R.(C \sqcap D)$. Please note that we do not have $C \sqcap \neq C \approx^n \bot$, nor we have $C \sqcup \neg C \approx^n \top$ and, thus, $(\exists R.C) \sqcap (\forall R.\neg C) \approx^n \bot$ and $(\exists R.C) \sqcup (\forall R.\neg C) \approx^n \top$ do not hold.

4.2 Entailment Relation

Of course, $\Sigma \models^n \langle \alpha :\geq n, \leq m \rangle$ iff $glb(\Sigma, \alpha) = \langle f, g \rangle$ with $f \geq n$ and $g \leq m$, and similarly $\Sigma \models^n \langle \alpha :\leq n, \geq m \rangle$ iff $lub(\Sigma, \alpha) = \langle f, g \rangle$ with $f \leq n$ and $g \geq m$. Concerning roles, note that $\Sigma \models^n \langle R(a,b) :\geq n, \leq m \rangle$ iff $\langle R(a,b) :\geq f, \leq g \rangle \in \Sigma$ with $f \geq n$ and $g \leq m$. Therefore,

$$glb(\Sigma, R(a,b)) = \langle \max\{n : \langle R(a,b) :\geq n, \leq m \rangle \ \in \Sigma\}, \min\{m : \langle R(a,b) :\geq n, \leq m \rangle \ \in \Sigma\} \rangle \quad (7)$$

while the same is not true for the $\langle R(a,b) :\leq n, \geq m$ case. While $\langle R(a,b) :\leq f, \geq g \rangle \in \Sigma$ and $f \leq n, g \geq m$ imply $\Sigma \models^n \langle R(a,b) :\leq n, \geq m \rangle$, the converse is false (*e.g.* $\{\langle \forall R.A(a) :\geq 1, \leq 0 \rangle, \langle A(b) :\leq 0, \geq 1 \rangle\} \models^n \langle R(a,b) :\leq 0, \geq 1 \rangle$).

Furthermore, from $\Sigma \models^n \langle C(a) :\leq n, \geq m \rangle$ iff $\Sigma \models^n \langle \neg C(a) :\geq m, \leq n \rangle$, it follows $lub(\Sigma, C(a)) = \langle f, g \rangle$ iff $glb(\Sigma, \neg C(a)) = \langle g, f \rangle$. Therefore, lub can be determined through glb (and vice versa). The same reduction to glb does not hold for $lub(\Sigma, R(a,b))$ as $\neg R(a,b)$ is not an expression of our language.

Modus ponens on concepts is supported: if $n > g$ and $m < f$ then $\{\langle C(a) :\geq n, \leq m \rangle, \langle (\neg C \sqcup D)(a) :\geq f, \leq g \rangle\} \models^n \rangle D(a) :\geq f, \leq g \rangle$ holds.

Modus ponens on roles is supported: if $n > g$ and $m < f$ then $\{\langle R(a,b) :\geq n, \leq m \rangle, \langle \forall R.D(a) :\geq f, \leq g \rangle\} \models^n \langle D(b) :\geq f, \leq g \rangle$ and $\{\langle \exists R.C(a) :\geq n, \leq m \rangle, \langle \forall R.D(a) :\geq f, \leq g \rangle\} \models^n \langle \exists R.(C \sqcap D)(a) :\geq \min\{n, f\}, \leq \max\{m, g\} \rangle$ hold. Moreover, $\{\langle \forall R.C(a) :\geq n, \leq m \rangle, \langle \forall R.D(a) :\geq f, \leq g \rangle\} \models^n \langle \forall (R.(C \sqcap D))(a) :\geq \min\{n, f\}, \leq \max\{m, g\} \rangle$ holds.

Modus ponens on specialization is supported. The following degree bounds propagation through a taxonomy is supported. If $C \preceq^n_\Sigma D$ then (*i*) $\Sigma \cup \{\langle C(a) :\geq n, \leq m \rangle\} \models^n \langle D(a) :\geq n, \leq m \rangle\}$; and (*ii*) $\Sigma \cup \{\langle D(a) :\leq n, \geq m \rangle\} \models^n \langle C(a) :\leq n, \geq m \rangle$ hold.

4.3 Soundness and Completeness of the Semantics

Our neutrosophic semantics is *sound* and *complete* w.r.t. fuzzy semantics. First we must note that the neutrosophic $\mathcal{ALC}$ is a strict generalization of fuzzy $\mathcal{ALC}$, in the sense that every fuzzy concept and fuzzy terminological axiom can be represented by a corresponding neutrosophic concept and neutrosophic terminological axiom, but not *vice versa*. It is easy to verify that,

Proposition 1. *A classical fuzzy $\mathcal{ALC}$ can be simulated by a neutrosophic $\mathcal{ALC}$, in the way that a fuzzy assertion $\langle \alpha \geq n \rangle$ represented by a neutrosophic assertion $\langle \alpha :\geq n, \leq 1-n \rangle$, a fuzzy assertion $\langle \alpha \leq n \rangle$ represented by a neutrosophic assertion $\langle \alpha :\leq n, \geq 1-n \rangle$ and a fuzzy terminological axiom $\tilde{\tau}$ represented by a neutrosophic terminological axiom $\hat{\tau}$ in the sense that if $\mathcal{I}$ is a fuzzy interpretation then $|C|^t(a) = C^{\mathcal{I}}(a)$ and $|C|^f(a) = 1 - C^{\mathcal{I}}(a)$.* ⊣

Let us consider the following transformations $\sharp(\cdot)$ and $\star(\cdot)$ of neutrosophic assertions into fuzzy assertions,

$$\begin{aligned}
\sharp\langle \alpha :\geq n, \leq m \rangle &\mapsto \langle \alpha \geq n \rangle, \\
\star\langle \alpha :\geq n, \leq m \rangle &\mapsto \langle \alpha \leq m \rangle, \\
\sharp\langle \alpha :\leq n, \geq m \rangle &\mapsto \langle \alpha \leq n \rangle, \\
\star\langle \alpha :\leq n, \geq m \rangle &\mapsto \langle \alpha \geq m \rangle,
\end{aligned}$$

We extend $\sharp(\cdot)$ and $\star(\cdot)$ to neutrosophic terminological axioms as follows: $\sharp\hat{\tau} = \tilde{\tau}$ and $\star\hat{\tau} = \tilde{\tau}$. Finally, $\sharp\Sigma = \{\sharp\varphi : \varphi \in \Sigma_A\} \cup \{\sharp\hat{\tau} : \hat{\tau} \in \Sigma_T\}$ and $\star\Sigma = \{\star\varphi : \varphi \in \Sigma_A\} \cup \{\star\hat{\tau} : \hat{\tau} \in \Sigma_T\}$.

Proposition 2. *Let Σ be a neutrosophic KB and let φ be a neutrosophic assertion ($\langle \alpha :\geq n, \leq m \rangle$ or $\langle \alpha :\leq n, \geq m \rangle$). Then $\Sigma \models^n \varphi$ iff $\sharp\Sigma \models \sharp\varphi$ and $\star\Sigma \models \star\varphi$.* ⊣

Proof. ($\Rightarrow$): Let φ be $\langle \alpha :\geq n, \leq m \rangle$. Consider a fuzzy interpretation $\mathcal{I}$ satisfying $\sharp\Sigma$ and $\mathcal{I}'$ satisfying $\star\Sigma$. $\langle \mathcal{I}, \mathcal{I}' \rangle$ is also a neutrosophic interpretation such that $a^{\mathcal{I}} = a^{\mathcal{I}'}$, $C^{\mathcal{I}}(a) = |C|^t(a)$ and $C^{\mathcal{I}'}(a) = |C|^f(a)$, $R^{\mathcal{I}}(d, d') = |R|^t(d, d')$ and $R^{\mathcal{I}'}(d, d') = |R|^f(d, d')$ hold. By induction on the structure of a concept C it can be shown that $\mathcal{I}$ ($\mathcal{I}'$) satisfies $C(a)$ iff $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$ ($C^{\mathcal{I}'}(a^{\mathcal{I}'} \geq n)$ for fuzzy assertion $\langle C(a) \geq n \rangle$ and $C^{\mathcal{I}}(a^{\mathcal{I}}) \leq n$ ($C^{\mathcal{I}'}(a^{\mathcal{I}'})$ for fuzzy assertion $\langle C(a) \leq n \rangle$. Similarly for roles. By the definition of $\sharp(\cdot)$ and $\star(\cdot)$, therefore $\langle \mathcal{I}, \mathcal{I}' \rangle$ is a neutrosophic interpretation satisfying Σ. By hypothesis, $\langle \mathcal{I}, \mathcal{I}' \rangle$ satisfies $\langle \alpha :\geq n, \leq m \rangle$. Therefore, $\mathcal{I}$ satisfies $\sharp\varphi$ and $\mathcal{I}'$ satisfies $\sharp\varphi$. The proof is similar for $\varphi = \langle \alpha :\leq n, \geq m \rangle$.

($\Leftarrow$): Let φ be $\langle \alpha :\geq n, \leq m \rangle$. Consider a neutrosophic $\mathcal{I}$ satisfying Σ. $\mathcal{I}$ can be regarded as two fuzzy interpretations $\mathcal{I}'$ and $\mathcal{I}''$ such that $a^{\mathcal{I}} = a^{\mathcal{I}'} = a^{\mathcal{I}''}$, $C^{\mathcal{I}'}(d) = |C|^t(d)$ and $C^{\mathcal{I}''}(d) = |C|^f(d)$, $R^{\mathcal{I}'}(d, d') = |R|^t(d, d')$ and $R^{\mathcal{I}''}(d, d') = |R|^f(d, d')$hold. By induction on the structure of a concept C it can be shown that $\mathcal{I}$ satisfies $C(a)$ iff $|C|^t(a^{\mathcal{I}}) \geq n, |C|^f(a^{\mathcal{I}}) \leq m$ for neutrosophic assertion

$\langle C(a) :\geq n, \leq m\rangle$ and $|C|^t(a^{\mathcal{I}}) \leq n, |C|^f(a^{\mathcal{I}}) \geq m$ for neutrosophic assertion $\langle C(a) :\leq n, \geq m\rangle$. Similarly for roles. By the definition of $\sharp(\cdot)$ and $\star(\cdot)$, therefore, $\mathcal{I}^{'}$ is a fuzzy interpretation satisfying $\sharp\Sigma$ and $\mathcal{I}^{"}$ satisfying $\star\Sigma$. By hypothesis, $\mathcal{I}^{'}$ satisfies $\sharp\varphi$ and $\mathcal{I}^{"}$ satisfies $\star\varphi$. And according to the definition of $\sharp(\cdot)$ and $\star(\cdot)$, $\mathcal{I}$ satisfies $\langle\alpha :\geq n, \leq m\rangle$. The proof is similar for $\varphi = \langle\alpha :\leq n, \geq m\rangle$. □

4.4 Subsumption

As for the fuzzy case, subsumption between two concepts C and D w.r.t. a terminology Σ_T, *i.e.* $C \preceq^n_{\Sigma_T} D$, can be reduced to the case of an empty terminology, *i.e.* $C' \preceq^n_{\emptyset} D'$.

Example 1. Suppose we have two polls *p1* and *p2* about two wars war_x and war_y, separately. By the result of *p1*, it establishes that, to some degree n people in the country support the war_x and to some degree m people in the country do not support the war_x, whereas by the result of *p2*, it establishes that, to some degree f people in the country support the war_y and to some degree g people in the country do not support the war_y. Please note that, truth-degree and falsity-degree give a quantitative description of the *supportness* of a poll w.r.t. a war, *i.e.* the supportness is handled as a neutrosophic concept. So, let us consider

$$\Sigma = \{\langle p1 : \exists Support.war\_x :\geq 0.6, \leq 0.5\rangle, \langle p2 : \exists Support.war\_y :\geq 0.8, \leq 0.1\rangle, \\ war\_x \prec^n War, war\_y \prec^n War\}$$

where the axioms specify that both war_x and war_y are a War. According to the expansion process, Σ will be replaced by

$$\Sigma^{'} = \{\langle p1 : \exists Support.war\_x :\geq 0.6, \leq 0.5\rangle, \langle p2 : \exists Support.war\_y :\geq 0.8, \leq 0.1\rangle, \\ war\_x :\approx^n War \sqcap war\_x^*, war\_y :\approx^n War \sqcap war\_y^*\},$$

which will be simplified to

$$\Sigma^{"} = \{\langle p1 : \exists Support.(War \sqcap war\_x^*) :\geq 0.6, \leq 0.5\rangle, \\ \langle p2 : \exists Support.(War \sqcap war\_y^*) :\geq 0.8, \leq 0.1\rangle\}.$$

Now, if we are looking for supportness of polls of War, then from Σ we may infer that $\Sigma \models^n \langle p1 : \exists Support.War :\geq 0.6, \leq 0.5\rangle$ and $\Sigma \models^n \langle p2 : \exists Support.War :\geq 0.8, \leq 0.1\rangle$. Furthermore, it is easily verified that $\Sigma^{"} \models^n \langle p1 : \exists Support.War :\geq 0.6, \leq 0.5\rangle$ and $\Sigma^{"} \models^n \langle p2 : \exists Support.War :\geq 0.8, \leq 0.1\rangle$ hold as well. Indeed, for any neutrosophic assertion φ, $\Sigma \models^n \varphi$ iff $\Sigma^{"} \models^n \varphi$ holds. □

5 Decision Algorithms in Neutrosophic $\mathcal{ALC}$

Deciding whether $\Sigma \models^n \langle\alpha :\geq n, \leq m\rangle$ or $\Sigma \models^n \langle\alpha :\leq n, \geq m\rangle$ requires a calculus. Without loss of generality we will consider purely assertional neutrosophic KBs only.

We will develop a calculus in the style of the constraint propagation method, as this method is usually proposed in the context of DLs[4] and fuzzy DLs[12, 14]. We first address the entailment problem, then the subsumption problem and finally the BTVB problem. Both the subsumption problem and the BTVB problem will be reduced to the entailment problem.

5.1 A Decision Procedure for the Entailment Problem

Consider a new alphabet of $\mathcal{ALC}$ *variables*. An interpretation is extended to variables by mapping these into elements of the interpretation domain. An $\mathcal{ALC}$ *object* (denoted by ω) is either an individual or a variable.[2]

A *constraint* (denoted by α is an expression of the form $C(\omega)$ or $R(\omega, \omega^{'})$, where $\omega, \omega^{'}$ are objects, C is an $\mathcal{ALC}$ concept and R is a role. A *neutrosophic constraint* (denoted by φ) is an expression having one of the following four forms: $\langle \alpha :\geq n, \leq m\rangle, \langle \alpha :\leq n, \geq m\rangle, \langle \alpha :> n, < m\rangle, \langle \alpha :< n, > m\rangle$. Note that neutrosophic assertions are neutrosophic constraints.

The definitions of satisfiability of a constraint, a neutrosophic constraint, a set of constraints, a set of neutrosophic constraints, atomic constraint and atomic neutrosophic constraint are obvious.

It is quite easily verified that the neutrosophic entailment problem can be reduced to the unsatisfiability problem of a set of neutrosophic constraints:

$$\Sigma \models^n \langle \alpha :\geq n, \leq m\rangle \text{ iff } \Sigma \cup \{\langle \alpha :< n, > m\rangle\} \text{ not satisfiable} \tag{8}$$

$$\Sigma \models^n \langle \alpha :\leq n, \geq m\rangle \text{ iff } \Sigma \cup \{\langle \alpha :> n, < m\rangle\} \text{ not satisfiable} \tag{9}$$

Our calculus, determining whether a finite set S of neutrosophic constraints is satisfiable or not, is based on a set of constraint propagation rules transforming a set S of neutrosophic constraints into "simpler" satisfiability preserving sets S_i until either all S_i contain a clash (indicating that from all the S_i no model of S can be build) or some S_i is completed and clash-free, that is, no rule can be further applied to S_i and S_i contains no clash (indicating that from S_i a model of S can be build).

A set of neutrosophic constraints S contains a *clash* iff it contains either one of the constraints in Table 1 or S contains a conjugated pair of neutrosophic constraints. Each entry in Table 2 says us under which condition the row-column pair of neutrosophic constraints is a *conjugated pair*. Given a neutrosophic constraint φ, with φ^c we indicate a conjugate of φ (if there exists one). Notice that a conjugate of a neutrosophic constraint may be not unique, as there could be infinitely many. For instance, both $\langle C(a) :< 0.6, > 0.3\rangle$ and $\langle C(a) :\leq 0.7, \geq 0.4\rangle$ are conjugates of $\langle C(a) :\geq 0.8, \leq 0.1\rangle$.

Concerning the rules, for each connective $\sqcap, \sqcup, \neg, \forall, \exists$ there is a rule for each relation $\langle \geq, \leq\rangle, \langle >, <\rangle, \langle \leq, \geq\rangle, \langle <, >\rangle$, *i.e.* there are 20 rules. The rules have the form:

$$\Phi \rightarrow \Psi \text{ if } \Gamma \tag{10}$$

[2] In the following, if there is no ambiguity, $\mathcal{ALC}$ variables and $\mathcal{ALC}$ objects are called variables and objects, respectively.

$\langle \bot(\omega) :\geq n, \leq m \rangle$, where $n > 0$ or $m < 1$
$\langle \top(\omega) :\leq n, \geq m \rangle$, where $n < 1$ or $m > 0$
$\langle \bot(\omega) :> n, < m \rangle$, $\langle \top(\omega) :< n, > m \rangle$
$\langle C(\omega) :< 0, > m \rangle$, $\langle C(\omega) :> 1, < m \rangle, \langle C(\omega) :< n, > 1 \rangle, \langle C(\omega) :> n, < 0 \rangle$

Table 1. Clashes

	$\langle \alpha :< f, > g \rangle$	$\langle \alpha :\leq f, \geq g$
$\langle \alpha :\geq n, \leq m$	$n \geq f$ or $m \leq g$	$n > f$ or $m < g$
$\langle \alpha :> n, < m$	$n \geq f$ or $m \leq g$	$n \geq f$ or $m \leq g$

Table 2. Conjugated Pairs

where Φ and Ψ are sequences of neutrosophic constraints and Γ is a condition. A rule fires only if *the condition Γ holds, if the current set S of neutrosophic constraints contains neutrosophic constraints matching the precondition Φ and the consequence Ψ is not already in S*. After firing, the constraints from Ψ are added to S. The rules are the following:

$$\begin{aligned}
&(\neg_{\langle \geq, \leq \rangle}) \ \langle \neg C(\omega) :\geq n, \leq m \rangle \rightarrow \langle C(\omega) :\leq m, \geq n \rangle \\
&(\neg_{\langle >, < \rangle}) \ \langle \neg C(\omega) :> n, < m \rangle \rightarrow \langle C(\omega) :< m, > n \rangle \\
&(\neg_{\langle \leq, \geq \rangle}) \ \langle \neg C(\omega) :\leq n, \geq m \rangle \rightarrow \langle C(\omega) :\geq m, \leq n \rangle \\
&(\neg_{\langle <, > \rangle}) \ \langle \neg C(\omega) :< n, > m \rangle \rightarrow \langle C(\omega) :> m, < n \rangle
\end{aligned} \tag{11}$$

$$\begin{aligned}
&(\sqcap_{\langle \geq, \leq \rangle}) \ \langle (C \sqcap D)(\omega) :\geq n, \leq m \rangle \rightarrow \langle C(\omega) :\geq n, \leq m \rangle, \langle D(\omega) :\geq n, \leq m \rangle \\
&(\sqcap_{\langle >, < \rangle}) \ \langle (C \sqcap D)(\omega) :> n, < m \rangle \rightarrow \langle C(\omega) :> n, < m \rangle, \langle D(\omega) :> n, < m \rangle \\
&(\sqcap_{\langle \leq, \geq \rangle}) \ \langle (C \sqcap D)(\omega) :\leq n, \geq m \rangle \rightarrow \langle C(\omega) :\leq n, \geq m \rangle, \langle D(\omega) :\geq n, \leq m \rangle | \\
&\qquad \langle C(\omega) :\geq n, \leq m \rangle, \langle D(\omega) :\leq n, \geq m \rangle | \\
&\qquad \langle C(\omega) :\leq n, \geq 0 \rangle, \langle C(\omega) :\geq 0, \leq m \rangle, \langle D(\omega) :\geq n, \leq 1 \rangle, \langle D(\omega) :\leq 1, \geq m \rangle | \\
&\qquad \langle C(\omega) :\geq n, \leq 1 \rangle, \langle C(\omega) :\leq 1, \geq m \rangle, \langle D(\omega) :\geq 0, \leq m \rangle, \langle D(\omega) :\leq n, \geq 0 \rangle \\
&(\sqcap_{\langle <, > \rangle}) \ \langle (C \sqcap D)(\omega) :< n, > m \rangle \rightarrow \langle C(\omega) :< n, > m \rangle, \langle D(\omega) :\geq n, \leq m \rangle | \\
&\qquad \langle C(\omega) :\geq n, \leq m \rangle, \langle D(\omega) :< n, > m \rangle | \\
&\qquad \langle C(\omega) :< n, > 0 \rangle, \langle C(\omega) \geq 0, \leq m \rangle, \langle D(\omega) :\geq n, \leq 1 \rangle, \langle D(\omega) :< 1, > m \rangle | \\
&\qquad \langle C(\omega) :\geq n, \leq 1 \rangle, \langle C(\omega) :< 1, > m \rangle, \langle D(\omega) :< n, > 0 \rangle, \langle D(\omega) :\geq 0, \leq m \rangle
\end{aligned}$$

$$\begin{aligned}
&(\sqcup_{\langle \geq, \leq \rangle}) \ \langle (C \sqcup D)(\omega) :\geq n, \leq m \rangle \rightarrow \langle C(\omega) :\geq n, \leq m \rangle, \langle D(\omega) :\leq n, \geq m \rangle | \\
&\qquad \langle C(\omega) :\leq n, \geq m \rangle, \langle D(\omega) :\geq n, \leq m \rangle | \\
&\qquad \langle C(\omega) :\geq n, \leq 1 \rangle, \rangle C(\omega) :\leq 1, \geq m \rangle, \langle D(\omega) :\leq n, \geq 0 \rangle, \langle D(\omega) :\geq 0, \leq m \rangle | \\
&\qquad \langle C(\omega) :\geq 0, \leq m \rangle, \langle C(\omega) :\leq n, \geq 0 \rangle, \langle D(\omega) :\geq n, \leq 1 \rangle, \langle D(\omega) :\leq 1, \geq m \rangle \\
&(\sqcup_{\langle >, < \rangle}) \ \langle (C \sqcup D)(\omega) :> n, < m \rangle \rightarrow \langle C(\omega) :> n, < m \rangle, \langle D(\omega) :\leq n, \geq m \rangle |
\end{aligned}$$

$\langle C(\omega) :\leq n, \geq m\rangle, \langle D(\omega) :> n, < m\rangle|$

$\langle C(\omega) :> n, < 1\rangle, \langle C(\omega) :\leq 1, \geq m\rangle, \langle D(\omega) :\leq n, \geq 0\rangle, \langle D(\omega) :> 0, < m\rangle|$

$\langle C(\omega) :\leq n, \geq 0\rangle, \langle C(\omega) :> 0, < m\rangle, \langle D(\omega) :> n, < 1\rangle, \langle D(\omega) :\leq 1, \geq m\rangle$

$(\sqcup_{\langle\leq,\geq\rangle})$ $\langle (C \sqcup D)(\omega) :\leq n, \geq m\rangle \rightarrow \langle C(\omega) :\leq n, \geq m\rangle, \langle D(\omega) :\leq n, \geq m\rangle$

$(\sqcup_{\langle<,>\rangle})$ $\langle (C \sqcup D)(\omega) :< n, > m\rangle \rightarrow \langle C(\omega) :< n, > m\rangle, \langle D(\omega) :< n, > m\rangle$

$(\forall_{\langle\geq,\leq\rangle})$ $\langle (\forall R.C)(\omega_1) :\geq n, \leq m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :\geq n, \leq m\rangle$
if $f > m$ and $g < n$

$(\forall_{\langle>,<\rangle})$ $\langle (\forall R.C)(\omega_1) :> n, < m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :> n, < m\rangle$
if $f \geq m$ and $g \leq n$

$(\exists_{\langle\leq,\geq\rangle})$ $\langle (\exists R.C)(\omega_1) :\leq n, \geq m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :\leq n, \geq m\rangle$
if $f > n$ and $g < m$

$(\exists_{\langle<,>\rangle})$ $\langle (\exists R.C)(\omega_1) :< n, > m\rangle, \langle R(\omega_1, \omega_2) :\geq f, \leq g\rangle \rightarrow \langle C(\omega_2) :< n, > m\rangle$
if $f \geq n$ and $g \leq m$

$(\exists_{\geq,\leq})$ $\langle (\exists R.C)(\omega) :\geq n, \leq m\rangle \rightarrow \langle R(\omega, x) :\geq n, \leq m\rangle, \langle C(x) :\geq n, \leq m\rangle$
if x is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :\geq n, \leq m\rangle$ and $\langle C(\omega^{'}) :\geq n, \leq m\rangle$ are already in the constraint set

$(\exists_{>,<})$ $\langle (\exists R.C)(\omega) :> n, < m\rangle \rightarrow \langle R(\omega, x) :> n, < m\rangle, \langle C(x) :> n, < m\rangle$
if x is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :> n, < m\rangle$ and $\langle C(\omega^{'}) :> n, < m\rangle$ are already in the constraint set

$(\forall_{\leq,\geq})$ $\langle (\forall R.C)(\omega) :\leq n, \geq m\rangle \rightarrow \langle R(\omega, x) :\geq m, \leq n\rangle, \langle C(x) :\leq n, \geq m\rangle$
if x is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :\geq m, \leq n\rangle$ and $\langle C(\omega^{'}) :\leq n, \geq m\rangle$ are already in the constraint set

$(\forall_{<,>})$ $\langle (\forall R.C)(\omega) :< n, > m\rangle \rightarrow \langle R(\omega, x) :> m, < n\rangle, \langle C(x) :< n, > m\rangle$
if x is new variable and there is no $\omega^{'}$ such that both
$\langle R(\omega, \omega^{'}) :> m, < n\rangle$ and $\langle C(\omega^{'}) :< n, > m\rangle$ are already in the constraint set

A set of neutrosophic constraints S is said to be *complete* if no rule is applicable to it. Any complete set of neutrosophic constraints S_2 obtained from a set of neutrosophic constraints S_1 by applying the above rules (11) is called a *completion* of S_1. Due to the rules $(\sqcup_{\geq,\leq}), (\sqcup_{>,<}), (\sqcap_{\leq,\geq})$ and $(\sqcap_{<,>})$, more than one completion can be obtained. These rules are called *nondeterministic rules*. All other rules are called *deterministic rules*.

It is easily verified that the above calculus has the *termination property, i.e.* any completion of a finite set of neutrosophic constraints S can be obtained after a finite number of rule applications.

Example 2. Consider Example 1 and let us prove that $\Sigma^{"} \models^n \langle (\exists Support.War)(p1) \geq 0.6, \leq 0.5\rangle$. We prove the above relation by verifying that all completions of

$S = \Sigma^{"} \cup \{\langle(\exists Support.War)(p1) :< 0.6, > 0.5\rangle\}$ contain a clash. In fact, we have the following sequence.

(1) $\langle(\exists Support.(War \sqcap war\_x^*))(p1) :\geq 0.6, \leq 0.5\rangle$	Hypothesis:S
(2) $\langle(\exists Support.(War \sqcap war\_y^*))(p2) :\geq 0.8, \leq 0.1\rangle$	
(3) $\langle(\exists Support.War)(p1) :< 0.6, > 0.5\rangle$	
(4) $\langle Support(p1, x) :\geq 0.6, \leq 0.5\rangle, \langle(War \sqcap war\_x^*)(x) :\geq 0.6, \leq 0.5\rangle$	$(\exists_{\geq,\leq}) : (1)$
(5) $\langle War(x) :< 0.6, > 0.5\rangle$	$(\exists_{<,>}) : (3), (4)$
(6) $\langle War(x) :\geq 0.6, \leq 0.5\rangle, \langle war\_x^*(x) :\geq 0.6, \leq 0.5\rangle$	$(\sqcap_{\geq,\leq}) : (4)$
(7) clash	(5), (6)

□

Proposition 3. *A finite set of neutrosophic constraints S is satisfiable iff there exists a clash free completion of S.* ⊣

From a computational complexity point of view, the neutrosophic entailment problem can be proven to be a PSPACE-complete problem, as is the classical entailment problem and fuzzy entailment problem.

Proposition 4. *Let Σ be a neutrosophic KB and let φ be a neutrosophic assertion. Determining whether $\Sigma \models^n \varphi$ is a PSPACE-complete problem.* ⊣

Proof. By the Proposition 1, $\Sigma \models^n \varphi$ iff $\sharp\Sigma \models \sharp\varphi$ and $\star\Sigma \models \star\varphi$. From the PSPACE-completeness of the entailment problem in fuzzy $\mathcal{ALC}$[14], PSPACE-completeness of the neutrosophic entailment problems follows. □

This result establishes an important property about our neutrosophic DLs. In effect, it says that no additional computational cost has to be paid for the major expressive power.

5.2 A Decision Procedure for the Subsumption Problem

In this section we address the subsumption problem, *i.e.* deciding whether $C \preceq^n_{\Sigma_T} D$, where C and D are two concepts and Σ_T is a neutrosophic terminology. As we have seen (see Example 1), $C \preceq^n_{\Sigma_T} D$ can be reduced to the case of an empty terminology by applying the KB expansion process. So, without loss of generality, we can limit our attention to the case $C \preceq^n_{\emptyset} D$.

It can easily be shown that

Proposition 5. *Let C and D be two concepts. It follows that $C \preceq^n_{\emptyset} D$ iff for all $n, m, \langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$, where a is a new individual.* ⊣

Proof. ($\Rightarrow$) Assume that $C \preceq^n_\emptyset D$ holds. Suppose to the contrary that $\exists n, m$ such that $\langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ does not hold. Therefore, there is an interpretation $\mathcal{I}$ and an n, m such that $|C|^t(a^\mathcal{I}) \geq n$ and $|D|^t(a^\mathcal{I}) < n$ or $|C|^f(a^\mathcal{I}) \leq m$ and $|D|^f(a^\mathcal{I}) > m$. But, from the hypothesis $n \leq |C|^t(a^\mathcal{I}) \leq |D|^t(a^\mathcal{I}) < n$ or $m \geq |C|^f(a^\mathcal{I}) \geq |D|^f(a^\mathcal{I}) > m$ follow. Absurd.
($\Leftarrow$) Assume that for all $n, m, \langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ holds. Suppose to the contrary that $C \preceq^n_\emptyset D$ does not hold. Therefore, there is an interpretation $\mathcal{I}$ and $d \in \Delta^\mathcal{I}$ such that $|C|^t(d) > |D|^t(d) \geq 0$ or $|C|^f(d) < |D|^f(d) \leq 1$. Let us extent $\mathcal{I}$ to a such that $a^\mathcal{I} = d$ and consider $\overline{n} = |C|^t(d)$ and $\overline{m} = |C|^f(d)$. Of course, $\mathcal{I}$ satisfies $\langle C(a) :\geq \overline{n}, \leq \overline{m}\rangle$. Therefore, from the hypothesis it follows that $\mathcal{I}$ satisfies $\langle D(a) :\geq \overline{n}, \leq \overline{m}\rangle$, *i.e.* $|D|^t(d) \geq \overline{n} = |C|^t(d) > |D|^t(d)$ or $|D|^f(d) \leq \overline{m} = |C|^f(d) < |D|^f(d)$. Absurd. □

How can we check whether for all $n, m, \langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ holds? The following proposition shows that

Proposition 6. *Let C and D be two concepts, $n_1, m_1 \in \{0, 0.25, 0.5, 0.75, 1\}$ and let a be an individual. It follows that for all $n, m\langle C(a) :\geq n, \leq m\rangle \models^n \langle D(a) :\geq n, \leq m\rangle$ iff $\langle C(a) :\geq n_1, \leq m_1\rangle \models^n \langle D(a) :\geq n_1, \leq m_1\rangle$ holds.* ⊣

As a consequence, the subsumption problem can be reduced to the entailment problem for which we have a decision algorithm.

5.3 A Decision Procedure for the BTVB Problem

We address now the problem of determining $glb(\Sigma, \alpha)$ and $lub(\Sigma, \alpha)$. This is important, as computing , *e.g.* $glb(\Sigma, \alpha)$, is in fact the way to answer a query of type "to which degree is α (at least) true and (at most) false, given the (imprecise) facts in Σ?".

Without loss of generality, we will assume that all concepts are in NNF (*Negation Normal Form*).

Proposition 7. *Let Σ be a set of neutrosophic assertions in NNF and let α be an assertion. Then $glb(\Sigma, \alpha) \in N^\Sigma$ and $lub(\Sigma, \alpha) \in M^\Sigma$, where*

$$N^\Sigma = \{\langle n, m\rangle : \langle \alpha :\geq n, \leq m'\rangle \in \Sigma, \langle \alpha :\geq n', \leq m\rangle \in \Sigma\}$$

$$M^\Sigma = \{\langle n, m\rangle : \langle \alpha :\leq n, \geq m'\rangle \in \Sigma, \langle \alpha :\leq n', \geq m\rangle \in \Sigma\}$$

⊣

The algorithm computing $glb(\Sigma, \alpha)$ and $lub(\Sigma, \alpha)$ are described in Table 3.

6 Conclusions and Future Work

In this paper, we have presented a quite general neutrosophic extension of the fuzzy DL $\mathcal{ALC}$, a significant and expressive representative of the various DLs.

Algorithm $glb(\Sigma, \alpha)$
Set $Min := \langle 0, 1\rangle$ and $Max := \langle 1, 0\rangle$.
1. Pick $\langle n, m\rangle \in M^{\Sigma}$ such that *first element of Min* $< n <$ *first element of Max* and *second element of Max* $< m <$ *second element of Min*. If there is no such $\langle n, m\rangle$, then set $glb(\Sigma, \alpha) := Min$ and exit.
2. If $\Sigma \models^n \langle \alpha :\geq n, \leq m\rangle$ then set $Min = \langle n, m\rangle$, else set $Max = \langle n, m\rangle$. Go to Step 1.

Algorithm $lub(\Sigma, \alpha)$
Set $Min := \langle 1, 0\rangle$ and $Max := \langle 0, 1\rangle$.
1. Pick $\langle n, m\rangle \in N^{\Sigma}$ such that *first element of Max* $< n <$ *first element of Min* and *second element of Min* $< m <$ *second element of Max*. If there is no such $\langle n, m\rangle$, then set $lub(\Sigma, \alpha) := Min$ and exit.
2. If $\Sigma \models^n \langle \alpha :\leq n, \geq m\rangle$ then set $Min = \langle n, m\rangle$, else set $Max = \langle n, m\rangle$. Go to Step 1.

Table 3. Algorithms $glb(\Sigma, \alpha)$ and $lub(\Sigma, \alpha)$

Our neutrosophic DL enables us to reason in presence of imprecise (fuzzy, incomplete, and inconsistent) $\mathcal{ALC}$ concepts, *i.e.* neutrosophic $\mathcal{ALC}$ concepts. From a semantics point of view, neutrosophic concepts are interpreted as neutrosophic sets, *i.e.* given a concept C and an individual a, $C(a)$ is interpreted as the truth-value and falsity-value of the sentence "a is C". From a syntax point of view, we allow to specify lower and upper bounds of the truth-value and falsity-value of $C(a)$. Complete algorithms for reasoning in it have been presented, that is, we have devised algorithms for solving the entailment problem, the subsumption problem as well as the best truth-value bound problem.

An important point concerns computational complexity. The complexity result shows that the additional expressive power has no impact from a computational complexity point of view.

This work can be used as a basis both for extending existing DL and fuzzy DL based systems and for further research. In this latter case, there are several open points. For instance, it is not clear yet how to reason both in case of neutrosophic specialization of the general form $C \prec^n D$ and in the case cycles are allowed in a neutrosophic KB. Another interesting topic for further research concerns the semantics of neutrosophic connectives. Of course several other choices for the semantics of the connectives $\sqcap, \sqcup, \neg, \exists, \forall$ can be considered.

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Neutrosophic Relational Data Model

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Abstract

In this paper, we present a generalization of the relational data model based on interval neutrosophic set [1]. Our data model is capable of manipulating incomplete as well as inconsistent information. Fuzzy relation or intuitionistic fuzzy relation can only handle incomplete information. Associated with each relation are two membership functions one is called truth-membership function *T* which keeps track of the extent to which we believe the tuple is in the relation, another is called falsity-membership function *F* which keeps track of the

extent to which we believe that it is not in the relation. A neutrosophic relation is inconsistent if there exists one tuple α such that $T(\alpha) + F(\alpha) > 1$. In order to handle inconsistent situation, we propose an operator called "split" to transform inconsistent neutrosophic relations into pseudo-consistent neutrosophic relations and do the set-theoretic and relation-theoretic operations on them and finally use another operator called "combine" to transform the result back to neutrosophic relation. For this data model, we define algebraic operators that are generalizations of the usual operators such as intersection, union, selection, join on fuzzy relations. Our data model can underlie any database and knowledge-base management system that deals with incomplete and inconsistent information.

Keyword: Interval neutrosophic set, fuzzy relation, inconsistent information, incomplete information, neutrosophic relation.

1. Introduction

Relational data model was proposed by Ted Codd's pioneering paper [2]. Since then, relational database systems have been extensively studied and a lot of commercial relational database systems are currently available [3, 4]. This data model usually takes care of only well-defined and unambiguous data. However, imperfect information is ubiquitous – almost all the information that we have about the real world is not certain, complete and precise [5]. Imperfect information can be classified as: incompleteness, imprecision, uncertainty, and inconsistency. Incompleteness arises from the absence of a value, imprecision from the existence of a value which cannot be measured with suitable precision, uncertainty from the fact that a person has given a subjective opinion about the truth of a fact which he/she does not know for certain, and inconsistency from the fact that there are two or more conflicting values for a variable.

In order to represent and manipulate various forms of incomplete information in relational databases, several extensions of the classical relational model have been proposed [6, 7, 8, 9, 10, 11]. In some of these extensions, a variety of "null values" have been introduced to model unknown or not-applicable data values. Attempts have also been made to generalize operators of relational algebra to manipulate such extended data models [6, 8, 11, 12, 13]. The fuzzy set theory and fuzzy logic proposed by Zadeh [14] provide a requisite mathematical framework for dealing with incomplete and imprecise information. Later on, the concept of interval-valued fuzzy sets was proposed to capture the fuzziness of grade of membership itself [15]. In 1986, Atanassov introduced the intuitionistic fuzzy set [16] which is a generalization of fuzzy set and provably equivalent to interval-valued fuzzy set. The intuitionistic fuzzy sets consider both truth-membership T and falsity-membership F with $T(a), F(a) \in [0,1]$ and $T(a) + F(a) \leq 1$. Because of the restriction, the fuzzy set, interval-valued fuzzy set, and intuitionistic fuzzy set

cannot handle inconsistent information. Some authors [17, 18, 19, 20, 21, 22, 23] have studied relational databases in the light of fuzzy set theory with an objective to accommodate a wider range of real-world requirements and to provide closer man-machine interactions. Probability, possibility, and Dempster-Shafer theory have been proposed to deal with uncertainty. Possibility theory [24] is built upon the idea of a fuzzy restriction. That means a variable could only take its value from some fuzzy set of values and any value within that set is a possible value for the variable. Because values have different degrees of membership in the set, they are possible to different degrees. Prade and Testemale [25] initially suggested using possibility theory to deal with incomplete and uncertain information in database. Their work is extended in [26] to cover multivalued attributes. Wong [27] proposes a method that quantifies the uncertainty in a database using probabilities. His method maybe is the simplest one which attached a probability to every member of a relation, and to use these values to provide the probability that a particular value is the correct answer to a particular query. Carvallo and Pittarelli [28] also use probability theory to model uncertainty in relational databases systems. Their method augmented projection and join operations with probability measures.

However, unlike incomplete, imprecise, and uncertain information, inconsistent information has not enjoyed enough research attention. In fact, inconsistent information exists in a lot of applications. For example, in data warehousing application, inconsistency will appear when trying to integrate the data from many different sources. Another example is that in the expert system, there exist facts which are inconsistent with each other. Generally, two basic approaches have been followed in solving the inconsistency problem in knowledge base: belief revision and paraconsistent logic. The goal of the first approach is to make an inconsistent theory consistent, either by revising it or by representing it by a consistent semantics. On the other hand, the paraconsistent approach allows reasoning in the presence of inconsistency, and contradictory information can be derived or introduced without trivialization [29]. Bagai and Sunderraman [30, 31] proposed a paraconsistent realational data model to deal with incomplete and inconsistent information. The data model has been applied to compute the well-founded and fitting model of logic programming [32, 33]. This data model is based on paraconsistent logics which were studied in detail by de Costa [34] and Belnap [35].

In this paper, we present a new relational data model – neutrosophic relational data model (NRDM). Our model is based on the neutrosophic set theory which is an extension of intuitionistic fuzzy set theory [36] and is capable of manipulating incomplete as well as inconsistent information. We use both truth-membership function grade α and falsity-membership function grade β to denote the status of a tuple of a certain relation with $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 2$. NRDM is the generalization of fuzzy relational data model (FRDM). That is , when α + β = 1, neutroshophic relation is the ordinary fuzzy relation. This model is

distinct with paraconsistent relational data model (PRDM), in fact it can be easily shown that PRDM is a special case of NRDM. That is, when α, β = 0 or 1, neutrosophic relation is just paraconsistent relation. We can use Figure 1 to express the relationship among FRDM, PRDM, and NRDM.

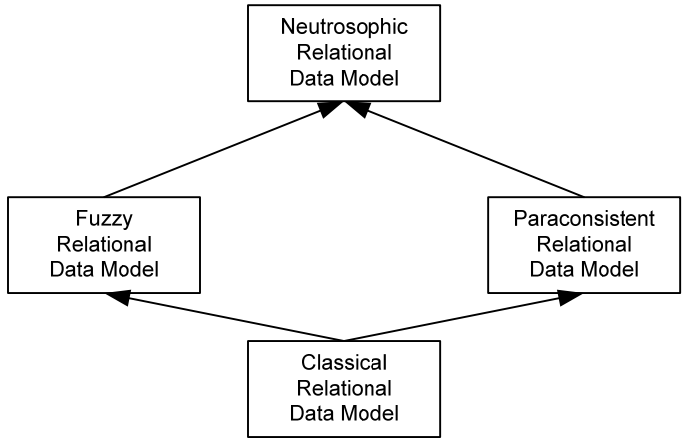

Figure 1. Relationship among RDM, FRDM, PRDM, and NRDM

We introduce neutrosophic relations, which are the fundamental mathematical structures underlying our model. These structures are strictly more general than classical fuzzy relations and intuititionistic fuzzy relations (interval-valued fuzzy relations), in that for any fuzzy relation or intuitionistic fuzzy relation there is a neutrosophic relation with the same information content, but not *vice versa*. The claim is also true for the relationship between neutrosophic relations and paraconsistent relations. We define algebraic operators over neutrosophic relations that extend the standard operators such as selection, join, union over fuzzy relations.

There are many potential applications of our new data model. Here are some examples:

a) Web mining. Essentially the data and documents on the Web are heterogeneous, inconsistency is unavoidable. Using the presentation and reasoning method of our data model, it is easier to capture imperfect information on the Web which will provide more potentially valued-added information.

b) Bioinformatics. There is a proliferation of data sources. Each research group and each new experimental technique seems to generate yet another source of valuable data. But these data can be incomplete and imprecise, and even inconsistent. We could not simply throw away one data in favor of other data. So how to represent and extract useful information from these data will be a challenge problem.

c) Decision Support System. In decision support system, we need to combine the database with the knowledge base. There will be a lot of uncertain and inconsistent information, so we need an efficient data model to capture these information and reasoning with these information.

The paper is organized as follow. Section 2 deals with some of the basic definitions and concepts of fuzzy relations and operations. Section 3 introduces neutrosophic relations and two notions of generalizing the fuzzy relational operators such as union, join, projection for these relations. Section 4 presents some actual generalized algebraic operators for the neutrosophic relations. These operators can be used for specifying queries for database systems built on such relations. Section 5 gives an illustrative application of these operators. Finally, section 6 contains some concluding remarks and directions for future work.

2. Fuzzy Relations and Operations

In this section, we present the essential concepts of a fuzzy relational database. Fuzzy relations associate a value between 0 and 1 with every tuple representing the degree of membership of the tuple in the relation. We also present several useful query operators on fuzzy relations.

Let a *relation scheme* (or just *scheme*) $\sum$ be a finite set of *attribute names*, where for any attribute name $A \in \sum$, *dom*(*A*) is a non-empty *domain* of values for *A*. *A tuple* on $\sum$ is any map $t : \sum \to \bigcup_{A \in \sum} dom(A)$, such that $t(A) \in dom(A)$, for each $A \in \sum$. Let $\tau(\sum)$ denote the set of all tuples on $\sum$.

Definition 1 A *fuzzy relation* on scheme $\sum$ is any map $R : \tau(\sum) \to [0,1]$. We let $F(\sum)$ be the set of all fuzzy relations on $\sum$.

If $\sum$ and Δ are relation schemes such that $\Delta \subseteq \sum$, then for any tuple $t \in \tau(\Delta)$, we let t^{Σ} denote the set $\{t' \in \tau(\sum) \mid t'(A) = t(A), \text{for all } A \in \Delta\}$ of all extensions of t. We extend this notion for any $T \subseteq \tau(\Delta)$ by defining $T^{\Sigma} = \bigcup_{t \in T} t^{\Sigma}$.

2.1 Set-theoretic operations on Fuzzy relations

Definition 2 Union: Let R and S be fuzzy relations on scheme $\sum$. Then, $R \cup S$ is a fuzzy relation on scheme $\sum$ given by

$$(R \cup S)(t) = \max\{R(t), S(t)\}, \text{for any } t \in \tau(\sum).$$

Definition 3 Complement: Let R be a fuzzy relation on scheme $\sum$. Then, $-R$ is a fuzzy relation on scheme $\sum$ given by

$$(-R)(t) = 1 - R(t), \text{for any } t \in \tau(\sum).$$

Definition 4 Intersection: Let R and S be fuzzy relations on scheme $\sum$. Then $R \cap S$ is a fuzzy relation on scheme $\sum$ given by

$$(R \cap S)(t) = \min\{R(t), S(t)\}, \text{for any } t \in \tau(\sum).$$

Definition 5 Difference: Let R and S be fuzzy relations on scheme $\sum$. Then, $R - S$ is a fuzzy relation on scheme $\sum$ given by

$$(R - S)(t) = \min\{R(t), 1 - S(t)\}, \text{for any } t \in \tau(\sum).$$

2.2 Relation-theoretic operations on Fuzzy relations

Definition 6 Let R and S be fuzzy relations on schemes $\sum$ and Δ, respectively. Then, the *natural join* (or just *join*) of R and S, denoted $R \infty S$ is a fuzzy relation on scheme $\sum \cup \Delta$, given by

$$(R \infty S)(t) = \min\{R(\pi_{\sum}(t)), S(\pi_{\Delta}(t))\}, \text{for any } t \in \tau(\sum \cup \Delta).$$

Definition 7 Let R be a fuzzy relation on scheme $\sum$ and let $\Delta \subseteq \sum$. Then, the *projection* of R onto Δ, denoted by $\prod_{\Delta}(R)$ is a fuzzy relation on scheme Δ given by

$$(\prod_{\Delta}(R))(t) = \max\{R(u) \mid u \in t^{\sum}\}, \text{for any } t \in \tau(\Delta).$$

Definition 8 Let R be a fuzzy relation on scheme $\sum$, and let F be any logic formula involving attribute names in $\sum$, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the selection of R by F, denoted $\dot{\sigma}_F(R)$, is a fuzzy relation on scheme $\sum$, given by

$$(\dot{\sigma}_F(R))(t) = \begin{cases} R(t) & \text{if } t \in \sigma_F(\tau(\sum)) \\ 0 & \text{otherwise} \end{cases}$$

where σ_F is the usual selection of tuples satisfying F.

3. Neutrosophic Relations

In this section, we generalize fuzzy relations in such a manner that we are now able to assign a measure of belief and a measure of doubt to each tuple. We shall refer to these generalized fuzzy relations as *neutrosophic relations*. So, a tuple in a neutrosophic relation is assigned a measure $\langle \alpha, \beta \rangle, 0 \leq \alpha, \beta \leq 1$. α will be referred to as the *belief* factor and β will be referred to as the *doubt* factor. The interpretation of this measure is that we believe with confidence α and doubt with confidence β that the tuple is in the relation. The belief and doubt confidence factors for a tuple need not add to exactly 1. This allows for incompleteness and inconsistency to be represented. If the belief and doubt factors add up to less than 1, we have incomplete information regarding the tuple's status in the relation and if the belief and doubt factors add up to more than 1, we have inconsistent information regarding the tuple's status in the relation.

In contrast to fuzzy relations where the grade of membership of a tuple is fixed, neutrosophic relations bound the grade of membership of a tuple to a subinterval $[\alpha, 1-\beta]$ for the case $\alpha + \beta \leq 1$.

The operators on fuzzy relations can also be generalized for neutrosophic relations. However, any such generalization of operators should maintain the belief system intuition behind neutrosophic relations.

This section also develops two different notions of operator generalizations.

We now formalize the notion of a neutrosophic relation.

Recall that $\tau(\sum)$ denotes the set of all tuples on any scheme $\sum$.

Definition 9 A *neutrosophic relation* R on scheme $\sum$ is any subset of

$$\tau(\sum) \times [0,1] \times [0,1]$$

For any $t \in \tau(\sum)$, we shall denote an element of R as $\langle t, R(t)^+, R(t)^- \rangle$, where $R(t)^+$ is the belief factor assigned to t by R and $R(t)^-$ is the doubt factor assigned to t by R. Let $V(\sum)$ be the set of all neutrosophic relations on $\sum$.

Definition 10 A neutrosophic relation R on scheme $\sum$ is *consistent* if $R(t)^+ + R(t)^- \leq 1$, for all $t \in \tau(\sum)$. Let $C(\sum)$ be the set of all consistent neutrosophic relations on $\sum$. R is said to be *complete* if $R(t)^+ + R(t)^- \geq 1$, for all $t \in \tau(\sum)$. If R is both consistent and complete, *i.e.*

$R(t)^+ + R(t)^- = 1$, for all $t \in \tau(\sum)$, then it is a *total* neutrosophic relation, and let $T(\sum)$ be the set of all total neutrosophic relations on $\sum$.

Definition 11 R is said to be *pseudo-consistent* if $\max\{b_i \mid (\exists t \in \tau(\sum))(\exists d_i)(\langle t, b_i, d_i \rangle \in R)\} + \max\{d_i \mid (\exists t \in \tau(\sum))(\exists b_i)(\langle t, b_i, d_i \rangle \in R\} > 1$, where for these $\langle t, b_i, d_i \rangle, b_i + d_i = 1$. Let $P(\sum)$ be the set of all pseudo-consistent neutrosophic relations on $\sum$.

Example 1 Neutrosophic relation $R = \{\langle a,0.3,0.7\rangle, \langle a,0.4,0.6\rangle, \langle b,0.2,0.5\rangle, \langle c,0.4,0.3\rangle\}$ is pseudo-consistent. Because for $t = a, \max\{0.3,0.4\} + \max\{0.7,0.6\} = 1.1 > 1$.

It should be observed that total neutrosophic relations are essentially fuzzy relations where the uncertainty in the grade of membership is eliminated. We make this relationship explicit by defining a one-one correspondence $\lambda_{\sum} : T(\sum) \to F(\sum)$, given by $\lambda_{\sum}(R)(t) = R(t)^+$, for all $t \in \tau(\sum)$. This correspondence is used frequently in the following discussion.

3.1 Operator Generalizations

It is easily seen that neutrosophic relations are a generalization of fuzzy relations, in that for each fuzzy relation there is a neutrosophic relation with the same information content, but not *vice versa*. It is thus natural to think of generalizing the operations on fuzzy relations such as union, join, and projection etc. to neutrosophic relations. However, any such generalization should be intuitive with respect to the belief system model of neutrosophic relations. We now construct a framework for operators on both kinds of relations and introduce two different notions of the generalization relationship among their operators.

An n-ary operator on fuzzy relations with signature $\langle \sum_1, ..., \sum_{n+1} \rangle$ is a function $\Theta : F(\sum_1) \times \cdots \times F(\sum_n) \to F(\sum_{n+1})$, where $\sum_1, ... \sum_{n+1}$ are any schemes. Similarly, an n- ary operator on neutrosophic relations with signature $\langle \sum_1, ..., \sum_{n+1} \rangle$ is a function $\Psi : V(\sum_1) \times \cdots V(\sum_n) \to V(\sum_{n+1})$.

Definition 12 An operator Ψ on neutrosophic relations with signature $\langle \sum_1, ..., \sum_{n+1} \rangle$ is *totality preserving* if for any total neutrosophic relations $R_1, ..., R_n$ on schemes $\sum_1, ..., \sum_n$, respectively, $\Psi(R_1, ..., R_n)$ is also total.

Definition 13 A totality preserving operator Ψ on neutrosophic relations with signature $\langle \sum_1, ..., \sum_{n+1} \rangle$ is a *weak generalization* of an operator Θ on fuzzy relations with the same

signature, if for any total neutrosophic relations $R_1,...,R_n$ on scheme $\sum_1,...,\sum_n$, respectively, we have

$$\lambda_{\sum_{n+1}}(\Psi(R_1,...,R_n)) = \Theta(\lambda_{\sum_1}(R_1),...,\lambda_{\sum_n}(R_n)).$$

The above definition essentially requires Ψ to coincide with Θ on total neutrosophic realtions (which are in one-one correspondence with the fuzzy relations). In general, there may be many operators on neutrosophic relations that are weak generalizations of a given operator Θ on fuzzy relations. The behavior of the weak generalizations of Θ on even just the consistent neutrosophic relations may in general vary. We require a stronger notion of operator generalization under which, at least when restricted to consistent neutrosophic relations, the behavior of all the generalized operators is the same. Before we can develop such a notion, we need that of 'representation' of a neutrosophic relation.

We associate with a consistent neutrosophic relation R the set of all (fuzzy relations corresponding to) total neutrosophic relations obtainable from R by filling the gaps between the belief and doubt factors for each tuple. Let the map $reps_{\sum} : C(\sum) \to 2^{F(\sum)}$ be given by

$$reps_{\sum}(R) = \{Q \in F(\sum) \mid \underset{t_i \in \tau(\sum)}{\wedge} (R(t_i)^+ \le Q(t_i) \le 1 - R(t_i)^-)\}.$$

The set $reps_{\sum}(R)$ contains all fuzzy relations that are 'completions' of the consistent neutrosophic relation R. Observe that $reps_{\sum}$ is defined only for consistent neutrosophic relations and produces sets of fuzzy relations. Then we have following observation.

Proposition 1 For any consistent neutrosophic relation R on scheme $\sum$, $reps_{\sum}(R)$ is the singleton $\{\lambda_{\sum}(R)\}$ iff R is total.

Proof It is clear from the definition of consistent and total neutrosophic relations and from the definition of $reps$ operation.

We now need to extend operators on fuzzy relations to sets of fuzzy relations. For any operator $\Theta : F(\sum_1)\cdots F(\sum_n) \to F(\sum_{n+1})$ on fuzzy relations, we let $S(\Theta) : 2^{F(\sum_1)} \times \cdots \times 2^{F(\sum_n)} \to 2^{F(\sum_{n+1})}$ be a map on sets of fuzzy relations defined as follows. For any sets $M_{1,...,}M_n$ of fuzzy relations on schemes $\sum_1,...,\sum_n$, respectively,

$$S(\Theta)(M_1,...,M_n) = \{\Theta(R_1,...,R_n) \mid R_i \in M_i, \text{ for all } i, 1 \le i \le n\}.$$

In other words, $S(\Theta)(M_1,...,M_n)$ is the set of Θ - images of all tuples in the Cartesian product $M_1 \times \cdots \times M_n$. We are now ready to lead up to a stronger notion of operator generalization.

Definition 14 An operator Ψ on neutrosophic relations with signature $\langle \sum_1,...,\sum_{n+1} \rangle$ is *consistency preserving* if for any consistent neutrosophic relations $R_1,...,R_n$ on schemes $\sum_1,...,\sum_n$, respectively, $\Psi(R_1,...,R_n)$ is also consistent.

Definition 15 A consistency preserving operator Ψ on neutrosophic relations with signature $\langle \sum_1,...,\sum_{n+1} \rangle$ is a *strong generalization* of an operator Θ on fuzzy relations with the same signature, if for any consistent neutrosophic relations $R_1,...,R_n$ on schemes $\sum_1,...,\sum_n$, respectively, we have

$$reps_{\sum_{n+1}}(\Psi(R_1,...,R_n)) = S(\Theta)(reps_{\sum_1}(R_1),...,reps_{\sum_n}(R_n)).$$

Given an operator Θ on fuzzy relations, the behavior of a weak generalization of Θ is 'controlled' only over the total neutrosophic relations. On the other hand, the behavior of a strong generalization is 'controlled' over all consistent neutrosophic relations. This itself suggests that strong generalization is a stronger notion than weak generalization. The following proposition makes this precise.

Proposition 2 If Ψ is a strong generalization of Θ, then Ψ is also a weak generalization of Θ.

Proof Let $\langle \sum_1,...,\sum_{n+1} \rangle$ be the signature of Ψ and Θ, and let $R_1,...,R_n$ be any total neutrosophic relations on schemes $\sum_1,...,\sum_n$, respectively. Since all total relations are consistent, and Ψ is a strong generalization of Θ, we have that

$reps_{\sum_{n+1}}(\Psi(R_1,...,R_n)) = S(\Theta)(reps_{\sum_1}(R_1),...,reps_{\sum_n}(R_n))$,

Proposition 1 gives us that for each $i, 1 \le i \le n$, $reps_{\sum_i}(R_i)$ is the singleton set $\{\lambda_{\sum_i}(R_i)\}$. Therefore, $S(\Theta)(reps_{\sum_1}(R_i),...,reps_{\sum_n}(R_n))$ is just the singleton set: $\{\Theta(\lambda_{\sum_1}(R_1),...,\lambda_{\sum_n}(R_n))\}$. Here, $\Psi(R_1,...,R_n)$ is total, and $\lambda_{\sum_{n+1}}(\Psi(R_1,...,R_n)) = \Theta(\lambda_{\sum_1}(R_1),...,\lambda_{\sum_n}(R_n))$, i.e. Ψ is a weak generalization of Θ.

Though there may be many strong generalizations of an operator on fuzzy relations, they all behave the same when restricted to consistent neutrosophic relations. In the next section, we propose strong generalizations for the usual operators on fuzzy relations. The proposed

generalized operators on neutrosophic relations correspond to the belief system intuition behind neutrosophic relations.

First we will introduce two special operators on neutrosophic relations called split and combine to transform inconsistent neutrosophic relations into pseudo-consistent neutrosophic relations and transform pseudo-consistent neutrosophic relations into inconsistent neutrosophic relations.

Definition 16 (**Split Operator** Δ) Let R be a neutrosophic relation on scheme $\sum$. Then,

$\Delta(R) = \{\langle t,b,d\rangle \mid \langle t,b,d\rangle \in R \quad and \quad b+d \le 1\} \cup$

$\{\langle t,b',d'\rangle \mid \langle t,b,d\rangle \in R \quad and \quad b+d>1 \quad and \quad b' = b \quad and \quad d' = 1-b\} \cup$

$\{\langle t,b',d'\rangle \mid \langle t,b,d\rangle \in R \quad and \quad b+d>1 \quad b' = 1-d \quad and \quad d' = d\}.$

It is obvious that $\Delta(R)$ is pseudo-consistent if R is inconsistent.

Definition 17 (**Combine Operator** ∇) Let R be a neutrosophic relation on scheme $\sum$. Then,

$\nabla(R) = \{\langle t,b',d'\rangle \mid (\exists b)(\exists d)((\langle t,b',d\rangle \in R \quad and \quad (\forall b_i,d_i)(\langle t,b_i,d_i\rangle \to b' \ge b_i) \quad and$

$\langle t,b,d'\rangle \in R \quad and \quad (\forall b_i)(\forall d_i)(\langle t,b_i,d_i\rangle \to d' \ge d_i))\}.$

It is obvious that $\nabla(R)$ is inconsistent if R is pseudo-consistent.

Note that strong generalization defined above only holds for consistent or pseudo-consistent neutrosophic relations. For any arbitrary neutrosophic relations, we should first use split operation to transform them into non-inconsistent neutrosophic relations and apply the set-theoretic and relation-theoretic operations on them and finally use combine operation to transform the result into arbitrary neutrosophic relation. For the simplification of notation, the following generalized algebra is defined under such assumption.

4. Generalized Algebra on Neutrosophic Relations

In this section, we present one strong generalization each for the fuzzy relation operators such as union, join, and projection. To reflect generalization, a hat is placed over a fuzzy relation operator to obtain the corresponding neutrosophic relation operator. For example, ∞ denotes the natural join mong fuzzy relations, and $\hat{\infty}$ denotes natural join on neutrosophic relations. These generalized operators maintain the belief system intuition behind neutrosophic relations.

4.1 Set-Theoretic Operators

We first generalize the two fundamental set-theoretic operators, union and complement.

Definition 18 Let R and S be neutrosophic relations on scheme $\sum$. Then,

(a) the *union* of R and S, denoted $R \hat{\cup} S$, is a neutrosophic relation on scheme $\sum$, given by

$$(R \hat{\cup} S)(t) = \left\langle \max\{R(t)^+, S(t)^+\}, \min\{R(t)^-, S(t)^-\} \right\rangle, \text{ for any } t \in \tau(\sum);$$

(b) the *complement* of R, denoted $\hat{-} R$, is a neutrosophic relation on scheme $\sum$, given by

$$(\hat{-} R)(t) = \left\langle R(t)^-, R(t)^+ \right\rangle, \text{ for any } t \in \tau(\sum).$$

An intuitive appreciation of the union operator can be obtained as follows: Given a tuple t, since we believed that it is present in the relation R with confidence $R(t)^+$ and that it is present in the relation S with confidence $S(t)^+$, we can now believe that the tuple t is present in the "either - R - or - S" relation with confidence which is equal to the larger of $R(t)^+$ and $S(t)^+$. Using the same logic, we can now believe in the absence of the tuple t from the "either - R - or - S" relation with confidence which is equal to the smaller (because t must be absent from both R and S for it to be absent from the union) of $R(t)^-$ and $S(t)^-$. The definition of *complement* and of all the other operators on neutrosophic relations defined later can (and should) be understood in the same way.

Proposition 3 The operators $\hat{\cup}$ and unary $\hat{-}$ on neutrosophic relations are strong generalizations of the operators $\cup$ and unary $-$ on fuzzy relations.

Proof Let R and S be consistent neutrosophic relations on scheme $\sum$. Then $reps_{\sum}(R \hat{\cup} S)$ is the set

$$\{Q \mid \underset{t_i \in \tau(\sum)}{\wedge} (\max\{R(t_i)^+, S(t_i)^+\} \le Q(t_i) \le 1 - \min\{R(t_i)^-, S(t_i)^-\})\}$$

This set is the same as the set

$$\{r \cup s \mid \underset{t_i \in \tau(\sum)}{\wedge} (R(t_i)^+ \le r(t_i) \le 1 - R(t_i)^-), \quad \underset{t_i \in \tau(\sum)}{\wedge} (S(t_i)^+ \le s(t_i) \le 1 - S(t_i)^-)\}$$

which is $S(\cup)(reps_{\sum}(R),\ \ reps_{\sum}(S))$. Such a result for unary $\hat{-}$ can also be shown similarly.

For sake of completeness, we define the following two related set-theoretic operators:

Definition 19 Let R and S be neutrosophic relations on scheme $\sum$. Then,

(a) the *intersection* of R and S, denoted $R \hat{\cap} S$, is a neutrosophic relation on scheme $\sum$, given by

$$(R \hat{\cap} S)(t) = \left\langle \min\{R(t)^+, S(t)^+\}, \max\{R(t)^-, S(t)^-\} \right\rangle, \text{ for any } t \in \tau(\sum).$$

(b) the *difference* of R and S, denoted $R \hat{-} S$, is a neutrosophic relation on scheme $\sum$, given by

$$(R \hat{-} S)(t) = \left\langle \min\{R(t)^+, S(t)^-\}, \max\{R(t)^-, S(t)^+\} \right\rangle, \text{ for any } t \in \tau(\sum).$$

The following proposition relates the intersection and difference operators in terms of the more fundamental set-theoretic operators union and complement.

Proposition 4 For any neutrosophic relations R and S on the same scheme, we have

$$R \hat{\cap} S = \hat{-}(\hat{-} R \hat{\cup} \hat{-} S), \ \ and$$

$$R \hat{-} S = \hat{-}(\hat{-} R \hat{\cup} S).$$

Proof By definition,

$$\hat{-} R(t) = \left\langle R(t)^-, R(t)^+ \right\rangle$$

$$\hat{-} S(t) = \left\langle S(t)^-, S(t)^+ \right\rangle$$

$$and \ \ (\hat{-} R \hat{\cup} \hat{-} S)(t) = \left\langle \max(R(t)^-, S(t)^-), \min(R(t)^+, S(t)^+) \right\rangle$$

$$so, \ \ (\hat{-}(\hat{-} R \hat{\cup} \hat{-} S))(t) = \left\langle \min(R(t)^+, S(t)^+), \max(R(t)^-, S(t)^-) \right\rangle = R \hat{\cap} S(t).$$

The second part of the result can be shown similarly.

4.2 Relation-Theoretic Operators

We now define some relation-theoretic algebraic operators on neutrosophic relations.

Definition 20 Let R and S be neutrosophic relations on schemes $\sum$ and Δ, respectively. Then, the *natural join* (further for short called *join*) of R and S, denoted $R \overset{\wedge}{\infty} S$, is a neutrosophic relation on scheme $\sum \cup \Delta$, given by

$$(R \overset{\wedge}{\infty} S)(t) = \left\langle \min\{R(\pi_{\sum}(t))^{+}, S(\pi_{\Delta}(t))^{+}\}, \max\{R(\pi_{\sum}(t))^{-}, S(\pi_{\Delta}(t))^{-}\} \right\rangle,$$

where π is the usual projection of a tuple.

It is instructive to observe that, similar to the intersection operator, the minimum of the belief factors and the maximum of the doubt factors are used in the definition of the join operation.

Proposition 5 $\overset{\wedge}{\infty}$ is a strong generalization of ∞.

Proof Let R and S be consistent neutrosophic relations on schemes $\sum$ and Δ, respectively. Then $reps_{\sum \cup \Delta}(R \overset{\wedge}{\infty} S)$ is the set $\{Q \in F(\sum \cup \Delta) \mid \wedge_{t_i \in \tau(\sum \cup \Delta)}(\min\{R_{\pi_{\sum}}(t_i)^{+}, S_{\pi_{\Delta}}(t_i)^{+}\} \le Q(t_i) \le$

$1 - \max\{R_{\pi_{\sum}}(t_i)^{-}, S_{\pi_{\Delta}}(t_i)^{-}\})\}$ and $S(\infty)(reps_{\sum}(R), reps_{\Delta}(S)) = \{r \infty S \mid r \in reps_{\sum}(R),$

$s \in reps_{\Delta}(S)\}$.

Let $Q \in reps_{\sum \cup \Delta}(R \overset{\wedge}{\infty} S)$. Then $\pi_{\sum}(Q) \in reps_{\sum}(R)$, where $\pi_{\sum}$ is the usual projection over $\sum$ of fuzzy relations. Similarly, $\pi_{\Delta}(Q) \in reps_{\Delta}(R)$, Therefore, $Q \in S(\infty)(reps_{\sum}(R), reps_{\Delta}(S))$.

Let $Q \in S(\infty)(reps_{\sum}(R), reps_{\Delta}(S))$. Then $Q(t_i) \ge \min\{R_{\pi_{\sum}}(t_i)^{+}, S_{\pi_{\Delta}}(t_i)^{+}\}$ and

$Q(t_i) \le \min\{1 - R_{\pi_{\sum}}(t_i)^{-}, 1 - S\pi_{\Delta}(t_i)^{-}\} = 1 - \max\{R_{\pi_{\sum}}(t_{i,})^{-}, S_{\pi_{\Delta}}(t_i)^{-}\}$, for any $t_i \in \tau(\sum \cup \Delta)$, because R and S are consistent. Therefore, $Q \in reps_{\sum \cup \Delta}(R \overset{\wedge}{\infty} S)$.

We now present the projection operator.

Definition 21 Let R be a neutrosophic relation on scheme $\sum$, and $\Delta \subseteq \sum$. Then, the *projection* of R onto Δ, denoted $\overset{\wedge}{\pi}_{\Delta}(R)$, is a neutrosophic relation on scheme Δ, given by

$$(\hat{\pi}_{\Delta}(R))(t) = \left\langle \max\{R(u)^{+} \mid u \in t^{\Sigma}\}, \min\{R(u)^{-} \mid u \in t^{\Sigma}\} \right\rangle.$$

The belief factor of a tuple in the projection is the maximum of the belief factors of all of the tuple's extensions onto the scheme of the input neutrosophic relation. Moreover, the doubt factor of a tuple in the projection is the minimum of the doubt factors of all of the tuple's extensions onto the scheme of the input neutrosophic relation.

We present the selection operator next.

Definition 22 Let R be a neutrosophic relation on scheme $\sum$, and let F be any logic formula involving attribute names in $\sum$, constant symbols (denoting values in the attribute domains), equality symbol $=$, negation symbol $\neg$, and connectives $\vee$ and $\wedge$. Then, the *selection* of R by F, denoted $\hat{\sigma}_F(R)$, is a neutrosophic relation on scheme $\sum$, given by

$$(\hat{\sigma}_F(R))(t) = \langle \alpha, \beta \rangle, \quad where$$

$$\alpha = \begin{cases} R(t)^{+} & if \quad t \in \sigma_F(\tau(\sum)) \\ 0 & otherwise \end{cases} \quad and \quad \beta = \begin{cases} R(t)^{-} & if \quad t \in \sigma_F(\tau(\sum)) \\ 1 & otherwise \end{cases}$$

where σ_F is the usual selection of tuples satisfying F from ordinary relations.

If a tuple satisfies the selection criterion, its belief and doubt factors are the same in the selection as in the input neutrosophic relation. In the case where the tuple does not satisfy the selection criterion, its belief factor is set to 0 and the doubt factor is set to 1 in the selection.

Proposition 6 The operators $\hat{\pi}$ and $\hat{\sigma}$ are strong generalizations of π and σ, respectively.

Proof Similar to that of Proposition 5.

Example 2 Relation schemes are sets of attribute names, but in this example we treat them as ordered sequence of attribute names (which can be obtained through permutation of attribute names), so tuples can be viewed as the usual lists of values. Let $\{a, b, c\}$ be a common domain for all attribute names, and let R and S be the following neutrosophic relations on schemes $\langle X, Y \rangle$ and $\langle Y, Z \rangle$ respectively.

t	$R(t)$
(a,a)	<0,1>
(a,b)	<0,1>
(a,c)	<0,1>
(b,b)	<1,0>
(b,c)	<1,0>
(c,b)	<1,1>

t	$S(t)$
(a,c)	<1,0>
(b,a)	<1,1>
(c,b)	<0,1>

For other tuples which are not in the neutrosophic relations $R(t)$ and $S(t)$, their $\langle \alpha, \beta \rangle = \langle 0,0 \rangle$ which means no any information available. Because R and S are inconsistent, we first use split operation to transform them into pseudo-consistent and apply the relation-theoretic operations on them and transform the result back to arbitrary neutrosophic set using combine operation. Then, $T_1 = \nabla(\Delta(R) \overset{\wedge}{\infty} \Delta(S))$ is a neutrosophic relation on scheme $\langle X, Y, Z \rangle$ and $T_2 = \nabla(\overset{\wedge}{\pi}_{\langle X,Z \rangle}(\Delta(T_1)))$ and $T_3 = \overset{\wedge}{\sigma}_{X \neg = Z}(T_2)$ are neutrosophic relations on scheme $\langle X, Z \rangle$. T_1, T_2, and T_3 are shown below:

t	$T_1(t)$
(a,a,a)	<0,1>
(a,a,b)	<0,1>
(a,a,c)	<0,1>

(a,b,a)	<0,1>
(a,b,b)	<0,1>
(a,b,c)	<0,1>
(a,c,a)	<0,1>
(a,c,b)	<0,1>
(a,c,c)	<0,1>
(b,b,a)	<1,1>
(b,c,b)	<0,1>
(c,b,a)	<1,1>
(c,b,b)	<0,1>
(c,b,c)	<0,1>
(c,c,b)	(<0,1>

t	$T_2(t)$
(a,a)	<0,1>
(a,b)	<0,1>
(a,c)	<0,1>
(b,a)	<1,0>
(c,a)	<1,0>

t	$T_3(t)$
(a,a)	<0,1>
(a,b)	<0,1>
(a,c)	<0,1>

(b,a)	<1,0>
(b,b)	<0,1>
(c,a)	<1,0>
(c,c)	<0,1>

5. An Application

Consider the target recognition example presented in [36]. Here, an autonomous vehicle needs to identify objects in a hostile environment such as a military battlefield. The autonomous vehicle is equipped with a number of sensors which are used to collect data, such as speed and size of the objects (tanks) in the battlefield. Associated with each sensor, we have a set of rules that describe the type of the object based on the properties detected by the sensor.

Let us assume that the autonomous vehicle is equipped with three sensors resulting in data collected about radar readings, of the tanks, their gun characteristics, and their speeds. What follows is a set of rules that associate the type of object with various observations.

Radar Readings:

- Reading r_1 indicates that the object is a T-72 tank with belief factor 0.80 and doubt factor 0.15.
- Reading r_2 indicates that the object is a T-60 tank with belief factor 0.70 and doubt factor 0.20.
- Reading r_3 indicates that the object is not a T-72 tank with belief factor 0.95 and doubt factor 0.05.
- Reading r_4 indicates that the object is a T-80 tank with belief factor 0.85 and doubt factor 0.10.

Gun Characteristics:

- Characteristic c_1 indicates that the object is a T-60 tank with belief factor 0.80 and doubt factor 0.20.

- Characteristic c_2 indicates that the object is not a T-80 tank with belief factor 0.90 and doubt factor 0.05.
- Characteristic c_3 indicates that the object is a T-72 tank with belief factor 0.85 and doubt factor 0.10.

Speed Characteristics:

- Low speed indicates that the object is a T-60 tank with belief factor 0.80 and doubt factor 0.15.
- High speed indicates that the object is not a T-72 tank with belief factor 0.85 and doubt factor 0.15.
- High speed indicates that the object is not a T-80 tank with belief factor 0.95 and doubt factor 0.05.
- Medium speed indicates that the object is not a T-80 tank with belief factor 0.80 and doubt factor 0.10.

These rules can be captured in the following three neutrosophic relations:

Radar Rules

Reading	Object	Confidence Factors
r_1	T-72	<0.80,0.15>
r_2	T-60	<0.70,0.20>
r_3	T-72	<0.05,0.95>
r_4	T-80	<0.85,0.10>

Gun Rules

Reading	Object	Confidence Factors
c_1	T-60	<0.80,0.20>

c_2	T-80	<0.05,0.90>
c_3	T-72	<0.85,0.10>

Speed Rules

Reading	Object	Confidence Factors
low	T-60	<0.80,0.15>
high	T-72	<0.15,0.85>
high	T-80	<0.05,0.95>
medium	T-80	<0.10,0.80>

The autonomous vehicle uses the sensors to make observations about the different objects and then uses the rules to determine the type of each object in the battlefield. It is quite possible that two different sensors may identify the same object as of different types, thereby introducing inconsistencies.

Let us now consider three objects o_1, o_2 and o_3 which need to be identified by the autonomous vehicle. Let us assume the following observations made by the three sensors about the three objects. Once again, we assume certainty factors (maybe derived from the accuracy of the sensors) are associated with each observation.

Radar Data

Object-id	Reading	Confidence Factors
o_1	r_3	<1.00,0.00>
o_2	r_1	<1.00,0.00>
o_3	r_4	<1.00,0.00>

Gun Data

Object-id	Reading	Confidence Factors
o_1	c_3	<0.80,0.10>
o_2	c_1	<0.90,0.10>
o_3	c_2	<0.90,0.10>

Speed Data

Object-id	Reading	Confidence Factors
o_1	high	<0.90,0.10>
o_2	low	<0.95,0.05>
o_3	medium	<0.80,0.20>

Given these observations and the rules, we can use the following algebraic expression to identify the three objects:

$$\hat{\pi}_{Object-id,Ojbect}(Radar\ \ Data\ \ \hat{\infty}\ \ Radar\ \ Rules)\hat{\cap}$$
$$\hat{\pi}_{Object-id,Object}(Gun\ \ Data\ \ \hat{\infty}\ \ Gun\ \ Rules)\hat{\cap}$$
$$\hat{\pi}_{Object-id,Object}(Speed\ \ Data\ \ \hat{\infty}\ \ Speed\ \ Rules)$$

The intuition behind the intersection is that we would like to capture the common (intersecting) information among the three sensor data. Evaluating this expression, we get the following neutrosophic relation:

Object-id	Reading	Confidence Factors
o_1	T-72	<0.05,0.00>
o_2	T-80	<0.00,0.05>

o_3	T-80	<0.05,0.00>

It is clear from the result that by the given information, we could not infer any useful information that is we could not decide the status of objects o_1, o_2 and o_3.

6. Conclusions and Future Work

We have presented a generalization of fuzzy relations, intuitionistic fuzzy relations (interval-valued fuzzy relations), and paraconsistent relations, called neutrosophic relations, in which we allow the representation of confidence (belief and doubt) factors with each tuple. The algebra on fuzzy relations is appropriately generalized to manipulate neutrosophic relations.

Various possibilities exist for further study in this area. Recently, there has been some work in extending logic programs to involve quantitative paraconsistency. Paraconsistent logic programs were introduced in [37] and probabilistic logic programs in [38]. Paraconsistent logic programs allow negative atoms to appear in the head of clauses (thereby resulting in the possibility of dealing with inconsistency), and probabilistic logic programs associate confidence measures with literals and with entire clauses. The semantics of these extensions of logic programs have already been presented, but implementation strategies to answer queries have not been discussed. We propose to use the model introduced in this paper in computing the semantics of these extensions of logic programs. Exploring application areas is another important thrust of our research.

We developed two notions of generalizing operators on fuzzy relations for neutrosophic relations. Of these, the stronger notion guarantees that any generalized operator is “well-behaved” for neutrosophic relation operands that contain consistent information.

For some well-known operators on fuzzy relations, such as union, join, and projection, we introduced generalized operators on neutrosophic relations. These generalized operators maintain the belief system intuition behind neutrosophic relations, and are shown to be “well-behaved” in the sense mentioned above.

Our data model can be used to represent relational information that may be incomplete and inconsistent. As usual, the algebraic operators can be used to construct queries to any database systems for retrieving vague information.

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Neutrosophic Logic Based Semantic Web Services Agent

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Abstract

In this paper, we propose a framework called Semantic Web Services (SWS) agent for providing high QoS Semantic Web services. The SWS agent is based on the neutrosophic logic. The neutrosophic logic was recently proposed by Smarandache to model and reason with fuzzy, incomplete and inconsistent information. The SWS agent can solve two challenges facing practicability of current Web services technology. One is how to locate the services Registries having requested Web services efficiently; another is how to retrieve the requested services from these Registries with high QoS. We use neutrosophic neural networks with Genetic Algorithms (GA) to do the simulation. Simulation results show that the SWS agent is extensible and scalable to handle uncertain QoS metrics effectively.

Key words: Quality of service, Semantic Web, Web services, intelligent agents, neutrosophic logic, neural networks, genetic algorithms

1 Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [1]. Since then fuzzy sets and fuzzy logic have been applied to many real applications to handle uncertainty. The traditional fuzzy set uses one real value $\mu_A(x) \in [0,1]$ to represent the grade of membersip of fuzzy set A defined on universe X. The corresponding fuzzy logic associates each proposition p with a real value $\mu(p) \in [0,1]$ which represents the degree of truth. Sometimes $\mu_A(x)$ itself is uncertain and hard to be defined by a crisp value. So the concept of interval valued fuzzy sets was proposed [2] to capture the uncertainty of grade of membership. The traditional fuzzy logic can be easily extended to the interval valued fuzzy logic. There are related works such as type-2 fuzzy sets and type-2 fuzzy logic [3–5]. The family of fuzzy sets and fuzzy logic can only handle "complete" information that is if a grade of truth-membership is μ_A then a grade of falsity-membership is $1-\mu_A(x)$ by default. In some applications such as expert systems, decision making systems and information fusion systems, the information is both uncertain and incomplete. Traditional fuzzy sets and fuzzy logic cannot handle such situation. In 1986, Atanassov introduced the intuitionistic fuzzy set [6] which is a generalization of a fuzzy set. The intuitionistic fuzzy sets consider both truth-membership and falsity-membership. The corresponding intuitionistic fuzzy logic [7] associates each proposition p with two real values $\mu(p)$-truth degree and $\nu(p)$-falsity degree, respectively, where $\mu(p), \nu(p) \in [0,1], \mu(p) + \nu(p) \leq 1$. So intuitionistic fuzzy sets and intuitionistic fuzzy logic can handle uncertain and incomplete information.

However, inconsistent information exists in a lot of real situations such as those mentioned above. It is obvious that the intuitionistic fuzzy logic cannot reason with inconsistency because it requires $\mu(p) + \nu(p) \leq 1$. In 1995, Smarandache introduced the concept of neutrosophic sets and neutrosophic logic which can model and reason with fuzzy, incomplete and inconsistent information at the same time. A special case of the neutrosophic sets and neutrosophic logic is studied in [8,9].

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In this paper, we propose a framework called Semantic Web services (SWS) agent based on neutrosophic logic to provide high QoS Semantic Web services based on specific domain ontology such as gnome. The SWS agent can solve two challenges existing for automatic discovery and invocation of Web services. One is how to locate the services Registries advertising requested Web services efficiently; another is how to retrieve the requested services from these Registries with the highest quality of service (QoS). The Semantic Web services technologies can be exploited to solve the first challenge. For the second challenge, we believe the QoS of Semantic Web services should cover both functional and non-functional properties. Here we must be aware that on the one hand, the degree of capability matching [10–13] and non-functional properties are all fuzzy, incomplete and even inconsistent; and on the other hand, different application domains have different requirements on non-functional properties. It is not flexible to use classical mathematical modelling methods to evaluate the whole QoS of Semantic Web services.

The paper is organized as follows. In section 2, we present the necessary background knowledge of neutrosophic logic and QoS model. Section 3 provides details of design of architecture of the SWS agent. Section 4 gives the design of neutrosophic neural network with GA and simulation result. In section 5, we present the related work. And finally, in section 6, we conclude this paper and give the future research direction.

2 Background

This section details the background material related to this research. We give a brief review of neutrosophic logic and QoS model.

2.1 Neutrosophic Propositional Logic

In this section, we introduce the elements of the neutrosophic propositional logic based on the definition of neutrosophic sets [14] by using the notation from the theory of classical propositional logic.

2.1.1 Syntax of Neutrosophic Propositional Logic

Here we give the formalization of syntax of the neutrosophic propositional logic.

Definition 1 *An* alphabet *of the neutrosophic propositional logic consists of three classes of symbols:*

(1) A set of neutrosophic propositional variables*, denoted by lower-case letters, sometimes indexed;*
(2) Five connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ *which are called conjunction, disjunction, negation, implication, and biimplication symbols respectively;*
(3) The parentheses (and).

□

The alphabet of the neutrosophic propositional logic has combinations obtained by assembling connectives and neutrosophic propositional variables in strings. The purpose of the construction rules is to have the specification of distinguished combinations, called formulas.

Definition 2 *The set of formulas (well-formed formulas) of the neutrosophic propositional logic is defined as follows.*

(1) Every interval neutrosophic propositional variable *is a formula;*
(2) If p is a formula, then so is $(\neg p)$;
(3) If p and q are formulas, then so are
  (a) $(p \wedge q)$,
  (b) $(p \vee q)$,
  (c) $(p \rightarrow q)$, and
  (d) $(p \leftrightarrow q)$.
(4) No sequence of symbols is a formula which is not required to be by 1, 2, and 3.

□

To avoid having formulas cluttered with parentheses, we adopt the following precedence hierarchy, with the highest precedence at the top:

$$\begin{gathered} \neg, \\ \wedge, \vee, \\ \rightarrow, \leftrightarrow. \end{gathered}$$

Here is an example of the neutrosophic propositional logic formula:

$$\neg p_1 \wedge p_2 \vee (p_1 \rightarrow p_3) \rightarrow p_2 \wedge \neg p_3$$

Definition 3 *The* language of interval neutrosophic propositional calculus *given by an alphabet consists of the set of all formulas constructed from the symbols of the alphabet.* □

2.1.2 Semantics of Neutrosophic Propositional Logic

The study of neutrosophic propositional logic comprises, among others, a *syntax*, which has the distinction of well-formed formulas, and a *semantics*, the purpose of which is the assignment of a meaning to well-formed formulas.

To each neutrosophic proposition p, we associate it with an ordered triple components $\langle t(p), i(p), f(p)\rangle$, where $t(p), i(p), f(p) \in [0,1]$. $t(p), i(p), f(p)$ is called truth-degree, indeterminacy-degree and falsity-degree of p, respectively. Let this assignment be provided by an *interpretation function* or *interpretation* NL defined over a set of propositions P in such a way that

$$NL(p) = \langle t(p), i(p), f(p)\rangle.$$

Hence, the function NL gives the truth, indeterminacy and falsity degrees of all propositions in P. We assume that the interpretation function NL assigns to the logical truth $T : NL(T) = \langle 1, 1, 0\rangle$, and to $F : NL(F) = \langle 0, 0, 1\rangle$.

An interpretation which makes a formula true is a *model* of the formula.

The semantics of four neutrosophic propositional connectives is given in Table 1. Note that $p \leftrightarrow q$ if and only if $p \rightarrow q$ and $q \rightarrow p$.

Table 1
Semantics of Four Connectives in Neutrosophic Propositional Logic

Connectives	Semantics
$NL(\neg p)$	$\langle f(p), 1 - i(p), t(p)\rangle$
$NL(p \wedge q)$	$\langle \max(t(p), t(q)), \max(i(p), i(q)), \min(f(p), f(q))\rangle$
$NL(p \vee q)$	$\langle \max(t(p), t(q)), \max(i(p), i(q)), \min(f(p), f(q))\rangle$
$NL(p \rightarrow q)$	$\langle \min(1, 1 - t(p) + t(q)), \min(1, 1 - i(p) + i(q)), \max(0, f(q) - f(p))\rangle$

2.2 Neutrosophic Predicate Logic

In this section, we will extend our consideration to the full language of first order neutrosophic predicate logic. First we give the formalization of syntax of first order neutrosophic predicate logic as in classical first-order predicate logic.

2.2.1 Syntax of Neutrosophic Predicate Logic

Definition 4 *An* alphabet *of the first order neutrosophic predicate logic consists of seven classes of symbols:*

(1) variables, *denoted by lower-case letters, sometimes indexed;*
(2) constants, *denoted by lower-case letters;*
(3) function symbols, *denoted by lower-case letters, sometimes indexed;*
(4) predicate symbols, *denoted by lower-case letters, sometimes indexed;*
(5) Five connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ *which are called the conjunction, disjunction, negation, implication, and biimplication symbols respectively;*
(6) Two quantifiers, *the* universal quantifier $\forall$ *(for all) and the* existential quantifier $\exists$ *(there exists);*
(7) The parentheses (and).

□

To avoid having formulas cluttered with brackets, we adopt the following precede nce hierarchy, with the highest precedence at the top:

$\neg, \forall, \exists$

$\wedge, \vee$

$\rightarrow, \leftrightarrow$

Next we turn to the definition of the first order neutrosophic language given by an alphabet.

Definition 5 *A* term *is defined as follows:*

(1) A variable is a term.
(2) A constant is a term.
(3) If f is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, f_n)$ is a term.

□

Definition 6 *A* (well-formed)formula *is defined inductively as follows:*

(1) If p is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $p(t_1, \ldots, t_n)$ is a formula (called an atomic formula *or, more simply, an* atom*).*
(2) If F and G are formulas, then so are $(\neg F), (F \wedge G), (F \vee G), (F \rightarrow G)$ and $(F \leftrightarrow G)$.
(3) If F is a formula and x is a variable, then $(\forall x F)$ and $(\exists x F)$ are formulas.

□

Definition 7 *The* first order neutrosophic language *given by an alphabet consists of the set of all formulas constructed from the symbols of the alphabet .* □

Definition 8 *The* scope *of* $\forall x$ *(resp.* $\exists x$*) in* $\forall xF$ *(resp.* $\exists xF$*) is* F*. A* bound occurrence *of a variable in a formula is an occurrence immediately following a quantifier or an occurrence within the scope of a quantifier, which has the same variable immediately after the quantifier. Any other occurrence of a variable is* free. □

2.2.2 Semantics of Neutrosophic Predicate Logic

In this section, we study the semantics of neutrosophic predicate logic, the purpose of which is the assignment of a meaning to well-formed formulas. In the neutrosophic propositional logic, an interpretation is an assignment of truth values (ordered triple component) to propositions. In the first order neutrosophic predicate logic, since there are variables involved, we have to do more than that. To define an interpretation for a well-formed formula in this logic, we have to specify two things, the domain and an assignment to constants and predicate symbols occurring in the formula. The following is the formal definition of an interpretation of a formula in the first order neutrosophic predicate logic.

Definition 9 *An* interpretation function (or interpretation) *of a formula* F *in the first order neutrosophic predicate logic consists of a nonempty domain* D*, and an assignment of "values" to each constant and predicate symbol occurring in* F *as follows:*

(1) To each constant, we assign an element in D*.*
(2) To each n*-ary function symbol, we assign a mapping from* D^n *to* D*. (Note that* $D^n = \{(x_1, \ldots, x_n) | x_1 \in D, \ldots, x_n \in D\}$*).*
(3) Predicate symbols get their meaning through evaluation functions E *which assign to each variable* x *an element* $E(x) \in D$*. To each* n*-ary predicate symbol* p*, there is a function* $NP(p) : D^n \to N$*. So* $NP(p(x_1, \ldots, x_n)) = NP(p)(E(x_1), \ldots, E(x_n))$*.*

□

That is, $NP(p)(a_1, \ldots, a_n) = \langle t(p(a_1, \ldots, a_n)), i(p(a_1, ldots, a_n)), f(p(a_1, \ldots, a_n)),$ where $t(p(a_1, \ldots, a_n)), i(p(a_1, \ldots, a_n)), f(p(a_1, \ldots, a_n)) \in [0, 1]$. They are called truth-degree, indeterminacy-degree and falsity-d egree of $p(a_1, \ldots, a_n)$ respectively. We assume that the interpretation f unction NP assigns to the logical truth $T : NP(T) = \langle 1, 1, 0 \rangle$, and to $F : NP(F) = \langle 0, 0, 1 \rangle$.

The semantics of four neutrosophic predicate connectives and two quantifiers is given in Table 2. For simplification of notation, we use p to denote $p(a_1, \ldots, a_i)$. Note th at $p \leftrightarrow q$ if and only if $p \rightarrow q$ and $q \rightarrow p$.

Table 2
Semantics of Four Connectives and Two Quantifiers in Neutrosopphic Predicate Logic

Connectives	Semantics
$NP(\neg p)$	$\langle f(p), 1 - i(p), t(p)\rangle$
$NP(p \wedge q)$	$\langle \min(t(p), t(q)), \min(i(p), i(q)), \max(f(p), f(q))\rangle$
$NP(p \vee q)$	$\langle \max(t(p), t(q)), \max(i(p), i(q)), \min(f(p), f(q))\rangle$
$NP(p \rightarrow q)$	$\langle \min(1, 1 - t(p) + t(q)), \min(1, 1 - i(p) + i(q)), \max(0, f(q) - f(p))\rangle$
$NP(\forall x F)$	$\langle \min t(F(E(x))), \min i(F(E(x))), \max f(F(E(x)))\rangle,\ E(x) \in D$
$NP(\exists x F)$	$\langle \max t(F(E(x))), \max i(F(E(x))), \min f(F(E(x)))\rangle,\ E(x) \in D$

2.3 QoS Model

Different applications generally have different requirements of QoS dimensions. [15,16] investigated the features with which successful companies assert themselves in the competitive world markets. Their result showed that success is based on three essential dimensions: time, cost and quality. [17] associates eight dimensions with quality, including performance and reliability.

In order to be more precise, we give our definitions of the three dimensions. (1) For a Semantic Web services, the capability can be defined as the degree that functional properties a Semantic Web service providde match with the functional properties a Semantic Web service requestor requires; (2) The response time of a Semantic Web service represents the time that elapses between service requests arrival and the completion of that service request. Response time is the sum of waiting time and actual processing time; (3) The trustworthiness of bioinformatics Semantic Web services should capture the consistency, reliability, competence and honesty of the service.

3 Architecture of neutrosophic logic based SWS agent

The neutrosophic logic based SWS agent can provide high QoS SWS based on specifi c ontology. The extensible SWS agent uses centalized client/server architecture internally. But itself can also be and should be implemented as a Semantic Web service based on specific service ontology. The neutrosophic

logic based SWS agent comprises of six components: (a)Registries Crawler; (b)SWS Repository; (c)Inquiry Server; (d)Publish Server; (e)Agent Communication Server; (f)Intelligent Inference Engine. The high level architecture of the neutrosophic logic based SWS agent is shown in Figure 1.

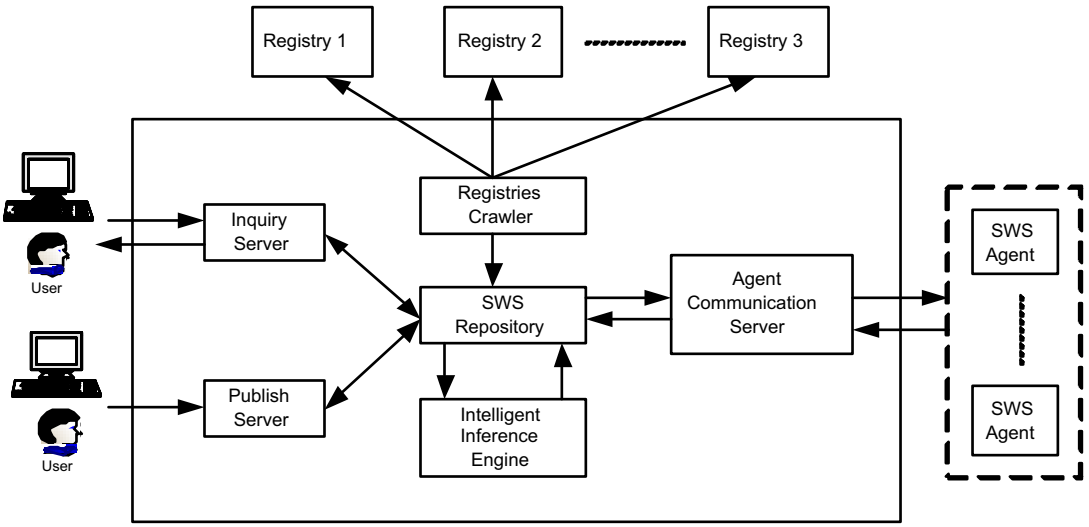

Fig. 1. Architecture of the neutrosophic logic based SWS agent

The Intelligent Inference Engine (IIE) is the core of the neutrosophic logic bas ed SWS agent. The neutrosophic logic based SWS agent is extensible because IIE uses neutrosoph ic logic inference system to calculate the QoS of the Semantic Web services with multidimensional QoS metrics . IIE gets the degree of capability matching and non-functional properties' values from OWL-S Matching Engine and return back the whole QoS to OWL-S Matchin g Engine. In the next section, we show the design of the IIE using neutrosophic logic, neural networks and genetic algorithm.

3.1 Design of Intelligent Inference Engine

This section shows one implementation of IIE based on neutrosophic logic, neural network and genetic algorithm. A schematic diagram of the four-layered neutrosophic neural network is shown in Figure 3. Nodes in layer one are input nodes representing input linguistic variables. Nodes in layer two are membership nodes. Membership nodes are truth-membership node, indeterminacy-membership node and falsity-membership node, which are responsible for mapping an input linguistic variable into three possibility distributions for that variable. The rule nodes reside in layer three. The last layer contains the output variable nodes [18].

As we mentioned before, the metrics of QoS of Semantic Web services are multidimensional. For illustration of specific ontology based Semantic Web

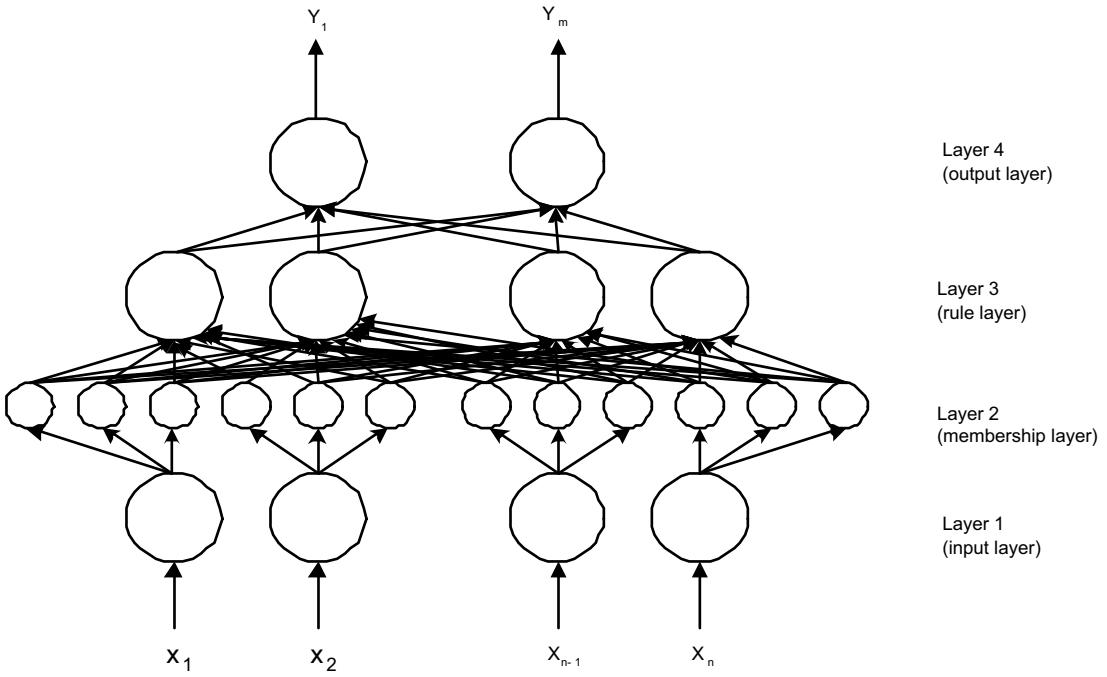

Fig. 2. Schematic diagram of Neutrosophic Neural Network

services for bioinformatics, we decide to use capability, response time and trustworthiness as our inputs and whole QoS as output. The neutrosophic logic system is based on TSK model.

3.2 Input neutrosophic sets

Let x represent capability, y represent response time and z represent trustworthiness. We scale the capability, response time and trustworthiness to [0,10] respectively. The graphical representation of membership functions of x, y, and z are shown in Figure 4.

3.3 Neutrosophic rule bases

Here, we design the neutrosophic rule base based on the TSK model. A neutrosophic rule is shown below:

IF x is I_1 and y is I_2 and z is I_3 THEN O is $a_{i,1} * x + a_{i,2} * y + a_{i,3} * Z + a_{i,4}$.

where, I_1, I_2 and I_3 are in low, middle, and high respectively and i in [1,27]. There are totally 27 neutrosophic rules. The $a_{i,j}$ are consequent parameters which will be obtained by training phase of neutrosophic neural network usng genetic algorithm.

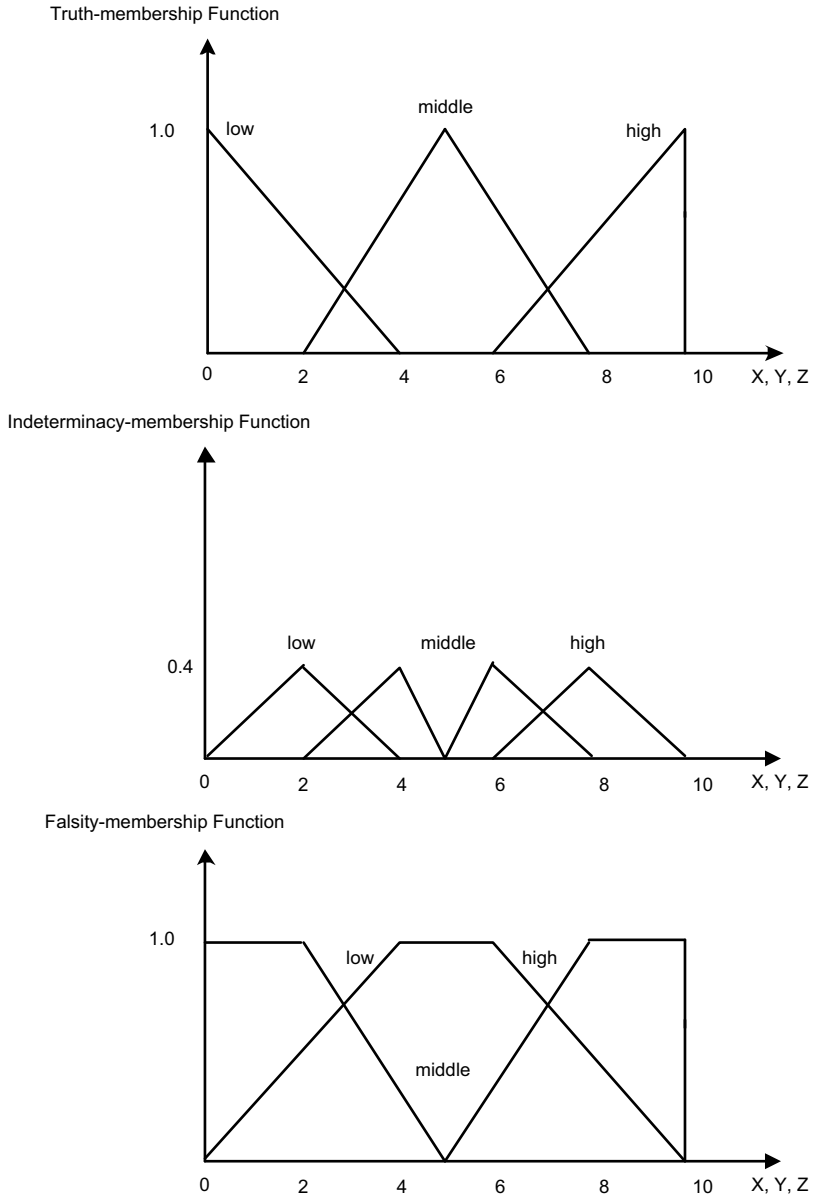

Fig. 3. Membership functions of inputs

3.4 Design of deneutrosophication

Suppose, for certain inputs x, y and z, there are m fired neutrosophic rules. To calculate the firing strength of jth rule, we use the formula:

$$W^j = W_x^j * W_y^j * W_z^j, \tag{1}$$

where
$W_x^j = (0.5 * t_x(x) + 0.35 * (1 - f_x(x)) + 0.025 * i_x(x) + 0.05),$
$W_y^j = (0.5 * t_y(y) + 0.35 * (1 - f_y(y)) + 0.025 * i_y(y) + 0.05),$

$W_z^j = (0.5 * t_z(z) + 0.35 * (1 - f_z(z)) + 0.025 * i_z(z) + 0.05)$,
where $t_x, f_x, i_x, t_y, f_y, i_y, t_z, f_z, i_z$, are the truth-membership, falsity-membership, indeterminacy-membership of neutrosophic inputs x, y, z, respectively.

So the crisp output is:

$$O = \sum_{j=1}^{m} W^j * (a_{j,1} * x + a_{j,2} * y + a_{j,3} * z + a_{j,4}) / (\sum_{j=1}^{m} W^j) \tag{2}$$

3.5 Genetic algorithms

GA is a model of machine learning which derives its behavior form a metaphor of the processes of evolution in nature. This is done by creation within a machine of a population of individuals represented by chromosomes. Here we use real-coded scheme. Given the range of parameters (coefficients of linear equations in TSK model), the system uses the derivate-free random search-GA to learn to find the near optimal solution by the fitness function through the training data.

(1) Chromosome: The genes of each chromosome are 108 real numbers (there are 108 parameters in the fuzzy fule base) which are initially generated randomly in the given range. So each chromosome is a vector of 108 real numbers.
(2) Fitness function: The fitness function is defined as

$$E = 1/2 \sum_{j=1}^{m} (d_i - o_i)^2 \tag{3}$$

(3) Elitism: The tournament selection is used in the elitism process.
(4) Crossover: The system will randomly select two parents among the population, then randomly select the number of cross points, and simply exchange the corresponding genes among these two parents to generate a new generation.
(5) Mutation: For each individual in the population, the system will randomly select genes in the chromosome and replace them with randomly generated real numbers in the given range.

3.6 Simulations

There are two phases for applying a fuzzy neural network: training and predicting. In the training phase, we use 150 data entries as training data set. Each entry consists of three inputs and one expected output. We tune the

Table 3
Prediction Result of Neutrosophic Neural Network

Input x	Input y	Input z	Desired output	Real output o
1	0	1	0	0.51
1	2	5	1	1.71
1	4	7	2	2.59
3	2	9	3	3.52
3	6	7	4	3.81
3	10	7	5	4.92
5	8	9	6	5.43
7	10	7	7	5.90
7	10	9	8	6.45
9	10	9	9	7.36

performance of the system by adjusting the size of population, the number of generation and probability of crossover and mutation. Table 1 gives the part of prediction results with serveral parameters for output o.

In Table 1, No. of generation = 10000, No. of population = 100, probability of crossover = 0.7, probability of mutation = 0.3. The maximum error of prediction result is 1.64. The total prediction error for 150 entries is 19functions and choosing reasonable training data set which is based on specific application domain can reduce the prediction error a lot. Here the example is just for illustration.

4 Conclusion and future work

In this paper, we discuss the design of Intelligent Inference Engine of extensible neutrosophic logic based SWS agent. The neutrosophic logic based SWS agent supports both the keyword based discovery and capability based discovery of the Semantic Web services. The primary motivation of our work is to solve two challenges facing current Web services advertising and discoverying techniques. One is how to locate the Registry hosting required Web service description; another is how to find the required Web service with highest QoS in the located Registry. The neutrosophic logic based SWS agent solves these two problems efficiently and effectively. The neutrosophic logic based SWS agent is built upon the Semantic Web, Web services and neutrosophic logic. The neutrosophic logic based SWS agent could be used in WWW, P2P and Grid infrastructure. The neutrosophic logic based SWS agent is flexible and extensi-

ble. In the future, we plan to extend the architecture of the neutrosophic logic based SWS agent to compute the whole QoS workflow of Semantic Web services to facilitate the composition and monitoring of complex Semantic Web services and apply it to Semantic Web-based bioinformatics applications.

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НЕЙТРОСОФИЯ (neutrosophy, от фр. и латин. neuter, что значит «нейтральное» и греч. sophia, что значит «мудрость», «учение») – знание о нейтральный объектах. Н. является теорией, созданной как обобщение *диалектики*. Эта теория рассматривает каждое понятие или идею «A» вместе с соответствующей противоположностью или отрицанием «antiA» и спектром нейтральностей «neutA» (т.е. вместе с понятиями или идеями, расположенными между двумя крайностями и при этом не подпадающими ни под «A», ни под «antiA»). Как «neutA», так и «antiA» отсылают к «nonA». В этой теории каждая идея «A» имеет тенденцию нейтрализоваться и уравновешиваться посредством «antiA» и «nonA». Следовательно, Н. базируется не только на анализе противоположных суждений, как это имеет место в диалектике, но также и на анализе нейтральностей, промежуточных между ними. В классическом смысле «A», «neutA», «antiA» между собой не пересекаются. Но с учетом того, что границы между понятиями часто являются нечеткими, случается, что «A», «neutA», «antiA» имеют общие части. Н. может быть расширена до *нейтрософичной логики*, *нейтрософичного множества*, *нейтрософичной вероятности* и *нейтрософичной статистики*, которые используются в технических приложениях (особенно в программном обеспечении), медицине, кибернетике, физике. Н. как научное направление была оформлена в работах Флорентина Смарандаке в 1995.

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Ф. Смарандаке (перевод А.Н. Шумана)

НЕЙТРОСОФИЧНАЯ ЛОГИКА – направление в логике, позволяющее унифицировать множество существующих логических систем, скажем такие как *нечеткая логика* (особенно интуиционистская нечеткая логика), *паранепротиворечивая логика*, *интуиционистская логика* и т.д. Основная идея Н.Л. состоит в оценке каждого логического суждения в трехмерном нейтрософичном пространстве, где каждая размерность соответственно представлена «истиной» (T), «ложью» (F) и «неопределенностью» (I) рассматриваемого суждения, где T, I, F являются стандартными или нестандартными подмножествами интервала $]^{-}0, 1^{+}[$. Для программного обеспечения может использоваться классический интервал [0, 1]. T, I, F являются независимыми компонентами, предполагающими неполную информацию (когда их наибольшая сумма < 1), паранепротиворечивую и противоречащую информацию (когда их наибольшая сумма > 1) или полную информацию (сумма компонентов = 1). Например, суждение может быть истинным в интервале (0.4, 0.6), неопределенным в интервале (0.15,0.25) и ложным в точке либо 0.4, либо 0.6. См. *нейтрософия*.

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Ф. Смарандаке (перевод А.Н. Шумана)

НЕЙТРОСОФИЧНОЕ МНОЖЕСТВО – обобщение понятия множества в рамках *нейтрософии*. Пусть U – универсум дискурса, M – множество, включенное в U. Элемент x из U обозначается по отношению к множеству M как x(T, I, F) и принадлежит M следующим образом: элемент истинно принадлежит M (t%), элемент неопределенно принадлежит M (i%), элемент ложно принадлежит M (f%), где t принимает свои значения во множестве T, i – во множестве I, f – во множестве F. Статически, T, I, F являются подмножествами, но динамически T, I, F – функции, операторы, зависящие от разных параметров. Н.М. обобщает нечеткое множество (в особенности интуиционистское нечеткое множество), паранепротиворечивое множество, интуиционистское множество и т.д. См. *нейтрософичная логика.*

Ф. Смарандаке (перевод А.Н. Шумана)

НЕЙТРОСОФИЧНАЯ ВЕРОЯТНОСТЬ – обобщение классической вероятности и нечеткой вероятности, в котором шанс того, что событие A наступит, истинно на t%, где t пробегает множество T, неопределенно на i%, где i пробегает множество I, и ложно на f%, где f пробегает множество F. В классической вероятности n_sup <= 1, в то время как в Н.В. n_sup <= 3^+. В нечеткой вероятности вероятность произвольного события есть подмножество T в [0, 1], а не число p в [0, 1], его невероятность – подмножество F (также из интервала [0, 1]); неопределенного подмножества I здесь не существует. См. *нейтрософичная статистика*.

Ф. Смарандаке (перевод А.Н. Шумана)

НЕЙТРОСОФИЧНАЯ СТАТИСТИКА – анализ событий в терминах *нейтрософичной вероятности*. Функция, которая моделирует нейтрософичную вероятность переменной x называется нейтрософичным распределением: NP(x) = (T(x), I(x), F(x)), где T(x) представляет вероятность того, что x наступит, F(x) представляет вероятность того, что x не наступит, и I(x) вероятность неопределенности или неизвестность вероятности переменной x. См. *нейтрософия*.

Ф. Смарандаке (перевод А.Н. Шумана)

ПАРАДОКСИЗМ (от слова «парадокс») – культурное направление, использующее парадоксы для создания произведений литературы, искусства, философии, социологии, науки и т.д. П. – авангардное течение как форма проявления протеста, предложен писателем Флорентином Смарандаке в 1980-ых, основывается на частом употреблении антитез, антиномий, противоречий, парадоксов. См. *нейтрософия*.

Ф. Смарандаке (перевод А.Н. Шумана)

Neutrosophic Transdisciplinarity

(Multi-Space & Multi-Structure)

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A) Definition:

Neutrosophic Transdisciplinarity means to find common features to uncommon entities,
i.e., for vague, imprecise, not-clear-boundary entity <A> one has:
<A> ∩ <nonA> ≠ Ø (empty set),
or even more <A> ∩ <antiA> ≠ Ø,
similarly <A> ∩ <neutA> ≠ Ø and <antiA> ∩ <neutA> ≠ Ø,
up to <A> ∩ <neutA> ∩ <antiA> ≠ Ø;
where <nonA> means what is not A, and <antiA> means the opposite of <A>.

There exists a Principle of Attraction not only between the opposites <A> and <antiA>
(as in dialectics),
but also between them and their neutralities <neutA> related to them,
since <neutA> contributes to the Completeness of Knowledge.
<neutA> means neither <A> nor <antiA>, but in between;
<neutA> is included in <nonA>.

As part of Neutrosophic Transdisciplinarity we have:

B) Multi-Structure and Multi-Space:

B1) Multi-Concentric-Structure:
Let S_1 and S_2 be two distinct structures, induced by the ensemble of laws L, which verify the ensembles of axioms A_1 and A_2 respectively, such that A_1 is strictly included in A_2.
One says that the set M, endowed with the properties:
a) M has an S_1-structure;
b) there is a proper subset P (different from the empty set Ø, from the unitary element, from the idempotent element if any with respect to S_2, and from the whole set M) of the initial set M, which has an S_2-structure;
c) M doesn't have an S_2-structure; is called a 2-concentric-structure.
We can generalize it to an n-concentric-structure, for $n \geq 2$ (even infinite-concentric-structure).
(By default, *1-concentric structure* on a set M means only one structure on M and on its proper subsets.)

An **n-concentric-structure** on a set S means a weak structure {w(0)} on S

such that there exists a chain of proper subsets
$P(n-1) < P(n-2) < \ldots < P(2) < P(1) < S$,
where '<' means 'included in',
whose corresponding structures verify the inverse chain
$\{w(n-1)\} > \{w(n-2)\} > \ldots > \{w(2)\} > \{w(1)\} > \{w(0)\}$,
where '>' signifies 'strictly stronger' (i.e., structure satisfying more axioms).

For example:
Say a groupoid D, which contains a proper subset S which is a semigroup, which in its turn contains a proper subset M which is a monoid, which contains a proper subset NG which is a non-commutative group, which contains a proper subset CG which is a commutative group, where D includes S, which includes M, which includes NG, which includes CG.
[This is a 5-concentric-structure.]

B2) **Multi-Space**:

Let $S_1, S_2, ..., S_n$ be n structures on respectively the sets $M_1, M_2, ..., M_n$, where $n \geq 2$ (n may even be infinite).
The structures S_i, i = 1, 2, …, n, may not necessarily be distinct two by two; each structure S_i may be or not n_i-concentric, for $n_i \geq 1$.
And the sets M_i, i = 1, 2, …, n, may not necessarily be disjoint, also some sets M_i may be equal to or included in other sets M_j, j = 1, 2, …, n.
We define the Multi-Space M as a union of the previous sets:
$M = M_1 \cup M_2 \cup \ldots \cup M_n$, hence we have n (different or not, overlapping or not) structures on M.
A multi-space is a space with many structures that may overlap, or some structures may include others or may be equal, or the structures may interact and influence each other as in our everyday life.

Therefore, a region (in particular a point) which belong to the intersection of $1 \leq k \leq n$ sets M_i may have k different structures in the same time. And here it is the difficulty and beauty of the a multi-space and its overlapping multi-structures.

{We thus may have $\mathbf{<R> \neq <R>}$, i.e. a region R different from itself, since R could be endowed with different structures simultaneously.}

For example we can construct a geometric multi-space formed by the union of three distinct subspaces: an Euclidean subspace, a Hyperbolic subspace, and an Elliptic subspace.

As particular cases when all M_i sets have the same type of structure, we can define the Multi-Group (or n-group; for example; bigroup, tri-group, etc., when all sets M_i are groups), Multi-Ring (or n-ring, for example biring, tri-ring, etc. when all sets M_i are rings), Multi-Field (n-field), Multi-Lattice (n-lattice), Multi-Algebra (n-algebra), Multi-Module (n-module), and so on -

which may be generalized to Infinite-Structure-Space (when all sets have the same type of structure), etc.

Conclusion.
The multi-space comes from reality, it is not artificial, because our reality is not homogeneous, but has many spaces with different structures.

A multi-space means a combination of any spaces (may be all of the same dimensions, or of different dimensions – it doesn't matter).

For example, a Smarandache geometry (SG) is a combination of geometrical (manifold or pseudo-manifold, etc.) spaces, while the multi-space is a combination of any (algebraic, geometric, analytical, physics, chemistry, etc.) space.
So, the multi-space can be interdisciplinary, i.e. math and physics spaces, or math and biology and chemistry spaces, etc.
Therefore, an SG is a particular case of a multi-space.
Similarly, a Smarandache algebraic structure is also a particular case of a multi-space.

This multi-space is a combination of spaces on the horizontal way, but also on the vertical way (if needed for certain applications).
On the horizontal way means a simple union of spaces (that may overlap or not, may have the same dimension or not, may have metrics or not, the metrics if any may be the same or different, etc.).
On the vertical way means more spaces overlapping in the same time, every one different or not.
The multi-space is really very general because it tries to model our reality. The parallel universes are particular cases of the multi-space too.
So, they are multi-dimensional (they can have some dimensions on the horizontal way, and other dimensions on the vertical way, etc.).

The multi-space with its multi-structure is a *Theory of Everything*. It can be used, for example, in the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak, and strong interactions (in physics).

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Neutrosophic Logic as a Theory of Everything in Logics

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Abstract.
Neutrosophic Logic (*NL*) is a Theory of Everything in logics, since it is the most general so far. In the Neutrosophic Propositional Calculus a neutrosophic proposition has the truth value *(T, I, F),* where *T* is the degree of truth, *I* is the degree of indeterminacy (or neutral, i.e. neither truth nor falsehood), and *F* is the degree of falsehood, where *T, I, F* standard or non-standard subsets of the non-standard unit interval $]^{-}0, 1^{+}[$. In addition, these values may vary over time, space, hidden parameters, etc.
Therefore, *NL* is a triple-infinite logic but, by splitting the Indeterminacy, we prove in this article that *NL* is a *n-infinite logic*, with *n = 1, 2, 3, 4, 5, 6, ...* .
Also, we present a total order on Neutrosophic Logic.

1. Introduction.

The neutrosophic component of Indeterminacy can be split into more subcategories, for example Belnap split Indeterminacy into: the paradox (<*A*> *and* <*antiA*>) and uncertainty (<*A*> *or* <*antiA*>), while truth would be <*A*>, and falsehood <*antiA*>. This way Belnap got his four-valued logic.

In neutrosophy we can combine <*A*> and <*nonA*>, getting a degree of <*A*> a degree of <*neutA*> and a degree of <*antiA*>.
<*A*> actually gives birth to <*antiA*> and <*neutA*> (not only to <*antiA*> as in dialectics).

But Indeterminacy can be split, depending on each application, in let's say: vagueness, ambiguity, unknown, unpredicted, error, etc. given rise to <u>n-infinite logic</u>, for $n \geq 1$.

2. History of Infinite Logics.

A *1-infinite logic* is the **fuzzy logic**, since in fuzzy logic $t + f = 1$, where t = truth value and f = false value.
Intuitionistic fuzzy logic is a *2-infinite logic*, since t and f vary in the interval *[0, 1],* while $i = 1 - t - f$, where i = indeterminacy.
Neutrosophic logic is a *3-infinite logic*, since *t, i, f* are independent, and their sum is not necessarily equal to *1,* but with *3,* since *NL* also generalizes the intuitionistic logic which supports incomplete theories (the sum of the components is less than 1), and paraconsistent logic (when the sum of components is greater than 1). In *NL* all three components *t, i, f* vary in the non-standards interval $]^{-}0, 1^{+}[$.
Belnap Logic can be consider as a *3-infinite logic,* by taking *(t, p, u, f)* truth value of a proposition, where $t + p + u + f = 1$.
It can be generalized to a **Neutrosophic Belnap Logic**, which will be a *4-infinite logic*, by letting $t + p + u + f \leq 4$ in order to include the Paraconsistent Neutrosophic Belnap

Logic (sum of all four components is greater than *1,* but less than or equal to *4*) and the Intuitionistic Neutrosophic Belnap Logic (sum of components is less than *1*).
(*2+k*)-infinite neutrosophic logic. In we split the Indeterminacy in *k*-parts {like paradox (true and false simultaneously), ignorance (true or false), unknown, vagueness, error, etc.} then we get a *(2+k)-infinite logic*, for $k \geq 1$.

NL is, so far, the most general logic, that's why we can call it a Theory of Everything in Logics.

Etymologically, *neutro-sophic* means a logic based on a 'neutral' component (indeterminacy, unknown, i.e. neither true nor false, hidden parameters, and tight result).

3. **A Total Order in Crisp Neutrosophic Logic**.

Umberto Rivieccio recommended in his article [13] that "it would be very useful to define suitable order relations on the set of neutrosophic truth values".
Yes, but I think for each application we might have a different order relation;
I am not sure if one can get one such order relation workable for all problems;

About the total order on *NL* with crisp components, here it is a small extension of Charles Ashbacher's order defined in the book:
http://www.gallup.unm.edu/~smarandache/IntrodNeutLogic.pdf, page 119, i. e.
for crisp values t, i, f we can define a total order:
$(t_1, f_1, i_1) < (t_2, f_2, i_2)$ if:
a) either $t_1 < t_2$;
b) or $t_1 = t_2$ but $f_1 > f_2$;
c) or $t_1 = t_2, f_1 = f_2,$ but $i_1 > i_2$.
Ashbacher has only the first two conditions: a) and b).
Condition c) is needed in the case when the sum of components is not *1* {I mean when $t+f+i < 1$ for intuitionistic (incomplete) logic; or $t+f+i > 1$ for paraconsistent logic; if $t+f+i = 1$ the third condition is not needed - it is implicit}.
We can re-write the components as:
(t, f, i) since *f* is more important than *i*.

Ashbacher also does a splitting of Indeterminacy into more components, as I wrote to Umberto Rivieccio in some e-mails, giving rise to different neutrosophic logics.

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Blogs on Applications of Neutrosophics and Multispace in Sciences

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The following blogs on applications of neutrosophics and multispace in sciences include meditations / reflections on science, comments, hypotheses, proposals, comparisons of ideas, possible projects, extensions or deviations or alternatives to classical knowledge, etc. selected from e-mails, letters, drafts, conversations, impressions, my diary, etc.
We introduce the *non-standard quaternion space* and *non-standard biquaternion space* [and even a generalization of them to a general non-standard vector space of any dimension] as possible working spaces for connecting the micro- and macro-levels in physics. Neutrosophy is a MetaPhilosophy.

Also, we consider that our multispace (with its multistructure of course) unifies many science fields. We write about parallel quantum computing and mu-bit, about multi-entangled states or particles and up to multi-entangles objects, about multispace and multivalued logics, about possible connection between unmatter with dark matter (what about investigating a possible existence of dark antimatter and dark unmatter?), about parallel time lines and multi-curve time, projects about writing SF at the quantum level as for example the adventure of a particle-man or "Star Shrek" – a satire to *Star Trek* (just for fun), about parallel universes as particular case of the multispace, and advance the hypothesis that more models of the atom are correct not only the standard model of the atom, etc.

I coined the name *unmatter* as a combination of matter and antimatter – and a possible third form of matter - since 2004, in a paper uploaded in the CERN website, and I published papers about "unmatter" that is now the predecessor of unparticles, which are a type of unmatter (mixtures of particles and antiparticles).

These fragments of ideas and believes have to be further investigated, developed and check experimentally if possible. {Actually, no knowledge is definitive in no domain!}

1. Blog on the Applications of Neutrosophics in Sciences.

1.1. I supposed that the readers will find it helpful trying to calculate for example Schrödinger's equation from quantum mechanics taking into consideration the indeterminacy.

1.2. As a physics example, in Schrödinger's Cat the neutrosophic logic offer the possibility of considering the cat neither dead nor alive, but undecided, while the fuzzy logic does not do that.

We can, let's say, instead of introducing in the problem a pair of twin cats - one cat in the box and another outside, we might also think at introducing a pair of entangled particles *A* (in the box) and *B* (outside the box). Then everything what happens to particle *A* inside will happen the opposite to particle *B* outside, hence measuring *B* will know the state of *A* (since *A* and *B* are complementary).
Readers, what are your opinions?
It would be nice to have entangled macro-objects (for example Entangled... Cats!). Let's name the two cats C_1 *and* C_2, but in such a way that cat C_2 is made from all entangled particles which form cat C_1. Then cat C_2 will be the entangled cat of C_1.
Similarly to the possible teleportation of objects: decompose an object in particle, then teleportate each particle, and then reassemble the teleportated particles - in this way it might be possible to teleportate objects.
Hence, decompose C_1 in particle, get an entangle particle for each particle of C_1, then reassemble the entangled particles and get a cat C_2 that is the entangled of C_1.

Herein the neutrosophic logic, based on three components, truth component, falsehood component, indeterminacy component *(T, I, F),* works very well.
We agree that Multivalued interpretation offers better explanation, but it seems to me that physicists in particular those who work with experiments prefer some quantitative effect: what can be measured?
For instance, some people began to test this paradox using Bose condensate... so we should translate this issue in a math language, like Bell inequality, which is testable;
see the below references in arXiv.org:

cond-mat/0508659:
Title: Creation, detection and decoherence of Schrödinger cat states in Bose-Einstein condensates
Authors: Y. P. Huang, M. G. Moore

cond-mat/0310573:
Title: Generating Schr\"{o}dinger-cat states in momentum and internal-state space from Bose-Einstein condensates with repulsive interactions
Authors: J. Higbie, D. M. Stamper-Kurn

cond-mat/0006349:
Title: Decoherence and long-lived Schrödinger cats in BEC
Authors: Diego A. R. Dalvit, Jacek Dziarmaga, Wojciech H. Zurek

I think that perhaps we should find first Schrödinger's original paper on this issue.
This problem has deep linkage with decoherence, in my opinion.
In the meantime, my viewpoint is that Schrodinger wanted to emphasize the inadequacy of statistical interpretation of his wavefunction. Therefore, multivaluedness offer significant advantage… .
I find this *n-valued logic* interpretation of Schrödinger equation goes unnoticed in almost all physics literature... .

The cat can be dead, or can be alive, or it is undecided (i.e. we don't know if she is dead or alive).

Frédéric Dambreville, a French expert in logics [I was amazed by his “referee function” in sensors’ fusion, being able to deduce many fusion rules], wrote to me that the notion of “true” is very subtle in the deterministic (non classical) logics.
Certain modal logics can include three states: *certain*, *possible*, and *impossible*. These are not habitual multi-valued logics, but they ‘simulate’ a kind of three-valued logic.
I replied back that there also are other logics, such as: the dynamic logic, tense logic, dialethism, etc. and also that:

> *Dans l’intuitionistique logique je suis d'accord qu'il n'est pas toujours vrai que d'une contradiction on peut trouver n'importe quoi [tu m'as ecrit ça dans un e-mail antérieur];*
> *par example: "il pleut" et "il ne pleut pas" n'implique pas que "ma voiture est blanche";*
> *car dans la logique classique tu as:*
> *$A \wedge nonA \rightarrow B$ est vrai pour n'importe quelles propositions logiques A, B.*
> *Aussi pour la logique trivalente, introduite par Lukasiewicz, avec les valeures 1, 1/2, 0.*
> *Mais, dans les applications de la logique floue, ça marche mieux que dans la logique modale qui donne l'etat de possibilité (il peut etre 1% possible ou bien 99% possible, mais entre ceux-ci est une grande difference).*
> *Aussi, la logique néutrosophique marche mieux que la logique trivalente, qui dit que 1 = vrai, 0 = faux, et 1/2 = inconnu, car la logique néutrosophique donne des pourcents (est plus rafinée, plus éxacte).*

Splitting indeterminacy may help to a better accuracy, depending on the type of indeterminacy. Normally indeterminacy *(I)* is split into uncertainty *(U)* and paradox / conflicting *(P)*.
The sum of the components is *1* in complete information, but we may have less than *1* for incomplete information, and greater than 1 for paraconsistent information.
In the cases when it is not *1*, we can normalize the components if needed (dividing each component by the sum of all components) and the sum becomes *1*.

A known approach explaining the *n*-value logic is via hydrodynamics (Fokker) process.

1.3. But it seems to me, that an unexplored part of information theory in physics, is what we know as **the physics of information**. For instance, physicists used to exchange scientific articles and cite each other. Can we consider it as unit of 'information bit'?

I've got a draft article on this issue, albeit not rigorous yet. The idea is quite simple: each time a scientist cites another article by his/her peer, we can count it as 1 quantum bit of information. At the end, citation analysis would end up to become another 'lab' where we could study quantum of decision making... Perhaps we could extend this approach.
Maybe we'd like to write a story/novel where philosophy ideas become the main theme, though in 'playful' tone or perhaps humorous, like Borges or Umberto Eco who blends

ancient tales with philosophical maze. Borges used mysticism somehow, while Eco science and religion.
I'd like to write this in popular style: the connection between the universe, cosmology, and number theory. But, of course, it should go beyond simple numerology or gematria, but perhaps a touch on information theory, Shannon entropy, Riemann zeta hypothesis, and perhaps also neutrosophy (and world peace?)...
Or, as alternative, what if a kind of weird *UFO*-like culture want to wreck havoc to this Earth, and there is no way to stop it except we can come into their computer-base, crack their own computer code (which requires some cryptography exposition), and reprogramming all their scenario from the ground?

In information fusion, the neutrosophic bba (basic believe assignment) is a generalization of the classical bba, because if for example the frame of discernment is $\Theta = \{A, B\}$
and $m_1(A)=0.2$, $m_1(B)=0.5$, $m_1(A \cup B)=0.3$, then you can write it as a neutrosophic bba:
$nm_1(A)=(0.2,0,0)$, $nm_1(B)=(0.5,0,0)$, $nm_1(A \cup B)=(0.3,0,0)$,
where each triplet for example $nm(A) = (a_1, a_2, a_3)$ means the believe in A is a_1, the disbelieve in A is a_3, and the unknown/indeterminate believe in A is a_2.
Better example is when we have the complement:
$m_2(A)=0.4$, $m_2(nonA)=0.1$, $m_2(nonB)=0.3$, $m_2(A \cup B)=0.2$,
then the neutrosophic bba associated *to* $m_2(.)$ is:
$nm_2(A)=(0.4, 0, 0.1)$, $nm_2(B)=(0, 0.3, 0)$, $nm_2(A \cup B)=(0.2,0,0)$.
And then you combine $nm_1(., ., .)$ with $nm_2(., ., .)$ using an *N-norm* and/or *N-conorm*.

1.4. Other questions: Do **advanced cultures beyond Earth** have advanced computation system using neutrosophy logic, instead of binary logic? If yes, does it mean that it is possible to write a programming language based on neutrosophy logic?

We say yes, there are programs based on fuzzy logic and on neutrosophic logic.
The binary logic does not work in all our everyday life events. We deal with approximations every time.
We cannot say exactly "Team X will win in the game", hence we cannot use binary logic.
In binary logic there is only black or white, but you know that in between there are more colors.
When we predict, we cannot predict 100% (we are not Gods!), we predict a certain percent, say 60% (but this is fuzzy or neutrosophic logic).

1.5. In an Ancient book of Chinese, The Book of Change, the TAIJI diagram in fact presents positive with that of negative, and they harmoniously exists.

In neutrosophy not only positive and negative harmoniously exist (as in the Ancient Chinese philosophy and as in dialectics) but also the positive, negative, and neutralities in between them harmoniously exist.

You might get an idea, according to some application for example, to split Indeterminacy differently, and thus to form a new neutrosophic logic and study it.

Do we mean that Lukasiewicz argument meant that multiple-value logic could be real?

Yes. See in the voting process or in a soccer game.
You have three possibilities: to gain, to loose, or to get a tight game. Hence the binary logic does not work any longer in such cases.

1.6. Observation does not always reflect the reality; there might be distorted observation due to various objective factors (various media of transmission/propagation of light, optical illusions, disturbed sounds, etc.).

So the experiment is best [not many done to prove quantum or astrophysical theories unfortunately], then observation, then ideas.
But all three have to prove/implicate each other for a perfect/correct/real theory.

1.7. About the deSitter space: I was neither against it nor for it, but as in neutrosophic logic, as you already remarked, in between [thus, we realize very well how important is multi-valued logic in physics]. So I am not under scientific pressure for this subject. But I confess I had better/smoother relationships with Dmitri Rabounski, Larissa Borissova, and Stephen Crothers.

I think, there are cases when deSitter is degenerated, and others where it is not [= neutrosophic logic!].
So, this has to be pointed out to all three fighting people, therefore everybody is right and wrong in some degree, and in another degree ambiguous/unclear.

1.8. Most physicists are busy with other things, except with revising of 'logic'. Only a few take a look of modifying physics from the scratch (defining logic, see for instance Rauscher or Bearden).

You know I'm more inclined to applications to physics, and for physicists things start to become interesting if we present it as geometry. Therefore I think more physicists appreciate my *NL* than mathematicians, perhaps we shall define the Neutrosophy as geometry of logic. This is why I think this modal logic seems interesting.
Possible paper may be something like 'Geometric Logic representation of Neutrosophy'...
Therefore, physicists can use their knowledge in geometry to study implications of NL, actually this is what I choose to do in a paper, using simple coin tossing problem as example.

We could also analyze something concerning the 'quantization of decision making' in quantitative way. I don't find yet any similar approach in the physics literature.

1.9. John A. Wheeler concludes, following Niels Bohr, that the universe is **self-organizing**. I subscribe to this self-organization in any macro- or micro-field. This is connected with topology and with Peirce's process philosophy.

I thought at defining and developing a quantum topology – for example.

1.10. About ‘our’ Lunar Base (by V. Christianto & F. Smarandache):

What do you think about the life effect under various gravities?
For example, after living for decades on a lunar base, where gravity is 6 times smaller, how a human being will look like? What physical and psychical changes would occur?

Or, on a big planet, say Jupiter, with a big mass and big gravity, might be difficult to walk...
Or will it be possible to increase or decrease gravity and atmosphere pressure in such of way to have the same as on the earth?

1.11. If neither Hilbert space, nor axiomatization (I hate this in quantum theory - which is more chaos), then **multiple valued logic** should do better.

But I thought at "multiple spaces overlapping" [*multi-space* or shortly *mu-space*], so we can explain how a particle can be in two places in the same time, or how a particle can be and cannot be in one place simultaneously, and maybe other odd phenomena in *QM*.

R. Garden's three-valued logic is referring to: true, false, undecided;
while neutrosophic logic is referring to: percentage of truth, percentage of falsehood, and percentage of undecidability/indeterminacy, so *NL* is more refined, more general.

More applicable in quantum physics would be neutrosophic probability *(NP)* {than neutrosophic logic *(NL)*;
a neutrosophic probability space and its axioms are defined in my fourth edition book:
www.gallup.unm.edu/~smarandache/eBook-neutrosophics4.pdf
at the end of the book.
For example, the neutrosophic probability that a particle *A* is in a place P_1 and particle *A* is not in P_1 in the same time could be for example: *NP(A in P1) = (0.4, 0.2, 0.4),* i.e. *40% A is in place* P_1*, 40% A is not in* P_1*, and 20%* unknown.
And so on.

1.12. It is possible using graphs in Tifft redshift quantization.

The book of Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps, by W. B. Vasantha Kandasamy and F. Smarandache is the most used/read (online) book about neutrosophic cognitive maps: www.gallup.unm.edu/~smarandache/NCMs.pdf.

1.13. About modern logics: many of them differ from the way the logical operators (*and, or, implication, negation*) are defined. But also they depend on the introduction of new operators and their study: for example "it is possible that" or "it is sufficient" for modal logic. In neutrosophic logic I introduced the **neuterization** and **antonymization** operators, in addition of the classical ones adjusted to the neutrosophic way.

We can define ourselves more operators – if needed in practice - and make them work on a logical space, for example on the neutrosophic logical space since it is the most extended.

For example: one physics logical operator might be "it is a potential of/for" (or something similar), another physics logical operator could be another needed idea from physics, maybe "it is a condition of existence of", then another similar operator: "it is a hypothesis for", etc.

There is a book where a friend (Dr. Andrew Schumann) and I used some non-linear logics (are called non-linear since the logical operators are some non-linear functions): http://www.gallup.unm.edu/~smarandache/Neutrality.pdf .

How to relate the modal operators and other logical operators to neutrosophic logic? Just extending the operators from the Boolean (or other logical space) to the neutrosophic logic, i.e. instead of using 1 or 2 truthiness components for the logical propositions, we have to use 3. Then we need to adjust the logical operator's function.

I read the article *Matrix method to solve multivalued logic differential equations*, by Svetlana Yanushkevich.
We can extend the Boolean Differential Calculus (BDC) to Neutrosophic Differential Calculus (NDC).
The problem might be the usefulness of NDC in physics, hence we need first to know how is BDC used in physics?

1.14. 'Paraconsistency logic' is used in theoretical physics, which perhaps the reader may find interesting.

There also are good articles by W. Smilga (2005) discussing 'information theory' viewpoint of particle physics, based on binary logic *(1/0).* See below some references from arXiv.org:

**physics/0505040:
Title: Informational Structures in Particle Physics
Authors: Walter Smilga

**physics/0502040:
Title: Elementary Informational Structures of Particle
Physics and their Relation to Quantum Mechanics and Space-Time
Authors: Walter Smilga
Comments: 16 pages, in German, presented at the spring meeting 2005 of the German Physical Society, Berlin, 4-9 March 2005

I guess this is a good starting point; could it be particles 'interfere' with each other if we use multivalued-logic, perhaps resembles 'bootstrap theory' of Geopffrey Chew from UCLA, Berkeley? Furthermore, information (bit) could be related to Shannon entropy, then to thermodynamics... See this from arXiv.org:

physics/0401002:
Title: Information Flow and Computation in the Maxwell Demon Problem
Authors: Roger D. Jones, Sven G. Redsun, Roger E. Frye
Submitted to J. Stat. Phys.

1.15. Another interesting question, if it's real that information → entropy → theormodynamics, then does it offer any clue on reversibility problem? I find only very few articles discussing this issue → Time flows backward are allowed or not?

I think this is right:
Shannon entropy → thermodynamics
since entropy means disorder, the bigger entropy the higher temperature, pressure.
See:
physics/0406137:
Title: The (absence of a) relationship between thermodynamic and logical reversibility
Authors: O. J. E. Maroney
Based on talk at ESF Conference on Philosophical and Foundational Issues in Statistical Physics, Utrecht, November 2003.

Disorder also mean timearrow (Gibbs theorem). So one can say if we could arrange such that entropy becomes less, it would mean time flows backward... right?
Have you heard of Srinivasan's work on NAFL (non-Aristotelian finitary logic), which discusses among other things a new interpretation of wavefunction. Perhaps you'd like to see it:

PhilSci Archive - Quantum superposition justified in "A new non-Aristotelian finitary logic (NAFL)", proposed by Srinivasan Radhakrishnan.
Platonism in classical logic versus formalism;
philsci-archive.pitt.edu/archive/00000635/

[math/0506475] Foundations of real analysis and computability ...
From: Radhakrishnan Srinivasan
... theory in the recently proposed non-Aristotelian finitary logic (NAFL)...
arxiv.org/abs/math.LO/0506475

1.16. There are many ways of defining neutrosophic operators and the neutrosophic orders, so we might adjust them or define new ones according to each problem/application to solve.

In technical applications *NL* and *NS* components *(T, I, F)* are subsets of the interval *[0,1],* but in philosophy they are subsets of the non-standard unit interval $]^{-}0, 1^{+}[$, since we need to be able to distinguish between absolute truth, which is 1^{+}, and (relative) truth, which is *1*. Similarly for absolute falsehood and (relative) falsehood, absolute indeterminacy and (relative) indeterminacy.
So, you might catch this in any of your future papers.
Then the negation of 1^{+} is $^{-}0$.

$1^{+} = 1+\varepsilon$ (where ε is epsilon), and ε is a very tiny (close to zero) positive number (infinitesimal), and $^{-}0 = 0 - \varepsilon$.

I coined the name "neutrosophy-neutrosophic" in English language, since I

needed something related to the middle part: i.e. neutral component (neither true nor false, unknown, or indeterminacy).
So, the name "neutrosophic" is much better and natural name, than Atanassov's "intuitionistic".

Even more, "*I*" can be split depending on the application, into *E* = error, *V* = vague, *K* = unknown, etc. and we get a logic on more than four components (actually on *n* components).

The theory of interval neutrosophic logic/set [14] can be extended in an easy way to subset neutrosophic logic/set, by taking *inf(S)* and *sup(S),* where *S* is the subset included in *[0, 1]*.

We can define a neutrosophic lattice, i.e. a *pseudo-lattice* on *3* components or more, that do not necessarily verify all axioms of a classical lattice, since nor *NL* verifies all properties of classical logic.

There is an attempt to define the *Neutrosophic Modal Logic* in my first book on neutrosophics (1998) but, of course, it should be developed.

I also think that (as in fuzzy logic/set) the connectives form classes of connectives (*N-norm, N-conorm* for example for neutrosophic conjunction and respectively neutrosophic disjunction), and in each case/application/problem we have to choose the best connectives that work for the respective problem;
neither in fuzzy logic there is used only one conjunction and one disjunction for example.
A paper of mine on *N-norm* and *N-conorm* partially solves this question.

Rivieccio also suggested to define suitable syntactical consequence relations and to prove completeness for each semantic neutrosophic system.
I fully agree; all people who worked in the neutrosophics were focusing either on using it in philosophy (I employed the "law of included middle" by designing the third component "*I*" = indeterminacy), or in technical applications (so that's why they/we mostly focused on $\wedge$, $\vee$, and $\neg$ *(negation)* neutrosophic connectives.

Rivieccio said: let *T, I, F* be subsets of some partially or linearly ordered lattice *L* instead of the real unit interval *[0, 1],* or even to consider different lattices L_1, L_2, L_3 such that $T \subseteq L_1$, $I \subseteq L_2$ and $F \subseteq L_3$.
Yes, this can be done, or this HAS to be done especially when *T, I, F* are subsets of *[0,1]* and it would be much harder to define a order relation between subsets like for example *[0.1. 0.3]* and *[0.2,0.4];* but in a lattice we can better define the two binary operations "*meet*" and "*join*" and then a (partial) order relationship.

N*eutrosophic modal logic* and *neutrosophic temporal logic* could be developed for physics application, especially at the quantum level.

1.17. The use of neutrosophy in **nucleon model** is unexplored yet. I find it at least quite similar to Barut's binucleon.

The basic point is: physicists normally think of +/- like in electromagnetic field. I find Prof. Kaivarainen's work interesting in this regard (bi-vacuum model). But to include another 'neutral' aspect would require significant revision... I guess.

For instance, if we accept Don Borghi experiment supporting Rutherford's initial model, then it seems that *neutron = electron + proton*. The experimental fact that neutron radii ~ electron radii seems to verify this conjecture. But you know, this hypothesis has been almost forgotten in standard literature…

For information theory, I could only mention *Planck radiation*. Can we derive *Bose-Einstein condensate* from information theory (Shannon entropy), for instance?

This would be interesting, if possible.

In the meantime, I could only mention one reference book, albeit rather outdated: *Entropy, Complexity and the Physics of Information*, by Santa-Fe Studies, edited by W. Zurek (1990). This is one of the most mind-boggling books I've read.

1.18. While I'm sure that we can describe Schrödinger's paradox in terms of Neutrosophy, I guess most other people will not be quite happy to abandon their simple common-logic system, unless we can prove the advantages of using the multiple-value logic.

In this regard, I guess explaining the paradox is the prerequisite, but another problem is what we called **ultraviolet spectral lines**:

NASA scientists observed extreme ultraviolet spectral lines from space for hydrogen and helium.

This is what I don't find clue how to prove it in terms of *QM*.

Alternatively, there are few guys claiming to be able to explain the spectral lines using *'classical electromagnetic radiation'*.

The best *QM* approach so far is to use nonlinear field by introducing double-wavefunction into *NSE*. But it can't explain yet the anomalous spectral lines. I also remember herein a paper by Oleinik discussing this idea.

Another question: do you think there is linkage between *Schrödinger's paradox* and *Heisenberg's uncertainty* and also *Einstein-Podolsky-Rosen paradox*?

I'm not sure yet with Heisenberg's uncertainty, but it seems to me that *EPR* could be explained using superluminal lightspeed (I wrote on this issue in one of my controversial articles about entangled particles[1]). Other references which may be useful to check.

But of course it will require more work to explore the 'parallelism' between theories. To quote Banach:

"Good mathematician finds analogy between theories,
Great mathematician finds analogy between analogies."

1.19. Non-standard vector space of any dimension to be used in physics.

[1] This was called the *Smarandache Hypothesis* that there is no speed barrier in the universe and one can construct any speed.

I thought that we could work with vectors v not only in the real vector space of dimension four, but to extend them to a complex vector space as follows:

$$v = (a_1+b_1i)x + (a_2+b_2i)y + (a_3+b_3i)z + (a_4+b_4i)t$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are real numbers,
$i = \sqrt{-1}$, $(x, y, z) = 3D$-space, t = time.
Don't we get Minkovski's $4D$-space when $b_1 = b_2 = b_3 = b_4 = 0$?
A further extension would like to introduce now in physics is the non-standard analysis and therefore the non-standard vector space:

$$V = (A_1+B_1i)x + (A_2+B_2i)y + (A_3+B_3i)z + (A_4+B_4i)t$$

where $A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4$ are non-standard numbers, i.e.
$A_k = a_k \pm \varepsilon$ and similarly $B_k = b_k \pm \varepsilon$, where ε is a very tiny positive number close to zero; for particular cases we may consider some ε *'s* equal to zero, therefore the corresponding $A_k = a_k$ or $B_k = b_k$ become just ordinary real numbers.
Having infinitesimals vector spaces may insure the connection between quantum level and macro-level in science, not explored before upon our knowledge.
We can generalize even more, considering a real vector space of dimension n, and then its corresponding complex space, and further their non-standard vector space:

$$V = (A_1+B_1i)x_1 + (A_2+B_2i)x_2 + ... + (A_n+B_ni)x_n + (A_{n+1}+B_{n+1}i)t$$

$A_k = a_k \pm \varepsilon$ and similarly $B_k = b_k \pm \varepsilon$, where $\varepsilon \geq 0$ (when $\varepsilon = 0$ we can get into the macrolevel for that coordinate x_k, but if $\varepsilon > 0$ we could be in a quantum level for that coordinate x_k),
and even more coordinates for the time t as well: $t_1, t_2, \ldots$.
Therefore, as particular cases, the quaternion and biquaternion spaces can be well extended to non-standard quaternion space and non-standard biquaternion space respectively.

1.20. Neutrosophy as a MetaPhilosophy.

Student:

Then the question: Which is the real happiness and what contribute to our real happiness. What would we feel if we suddenly died? Happy or miserable? What would we have if all those we possess became the past?

Neutrosophic Philosopher:

It depends on what everybody understands by "happiness"? Myself I am interested in discovering new ideas in science, arts. Someone else would like to become a leader, another to have a nice family, etc.

Student:

Is fate destined by some imaginary power or accumulated by ourselves in the long life cycles?

Neutrosophic Philosopher:

It is created by us, by chance, by friends, by society.

Student:

Then what should we do to create real happiness?

Neutrosophic Philosopher:

Learn to attain happiness defined in your terms. Know how to measure it.

Student:

I am really limited to give any further insight, and you can follow plenty of Buddhist URLs from our previous contact (or other great teachings I am not familiar with). There is only sincerity in this practice.
The slightest bargain can definitely result in the opposite destiny. May you be really genius and understand it.

Neutrosophic Philosopher:

Right, the theory of paradoxism: when I want something, it almost surely happen the contradictory!

Feng Liu mentioned in his presentation to the International Congress of Mathematicians (Beijing, P.R. China, August 2002) which I attended, that *T, I, F* in neutrosophy can signify intentions or confidence, not necessarily figures.
He also mentioned in his Dialogues and Sushi's poem: we fail to see the true face of Lushan mountain just because we are in it. Therefore a preliminary means is to abandon logic.
He said in the Chinese Translation of Neutrosophics [1] that the evolution and retrogression of human society is more or less a misleading concept, and people have followed a wrong teacher like Darwin, since those who diligently follow the Way would often be sniffed at. The "wise" adhere fast to their accomplishments and would actually understand nothing different from their own referential world, and fail to reach the light.

Meanwhile the Chinese are adapting more and more wastes than treasures from the west – added Feng Liu - the whole globe is merely a neglectable tiny spot in the universe. Man is sin by nature, and also from his sin, he possesses by nature the limitless power of self-enlightenment. When one is deprived of all his treasures - or replaces them with means, he begins his beggar's life. What an exchange of culture! (Clearly such Neutrosophic Dialogues are needed.)

Years ago, Master Chin Kung suggested that one should reach *arhat* stage before he can translate Buddhist sutras. As we know all written words are symbols to illustrate the underlying truth. One needs to actually see the truth instead of imaging the truth.

Arhat stage is the final goal of *Buddhist practice*; the highest rank in early Indian Buddhism, when there is nothing left to learn.
Literally it means “foe destroyer” or “worthy of respect”.

Buddhist sutras are referring to Buddha’s teaching the *Bodhisattva’s path* to the fifty-five stages of the enlightenment, the specific working of individual and collective karma, the existence, the *Instructions to Purity*, how to get a *Bodhimanda* and the *Shurangama Mantra*, and understanding the fifty kinds of deviant *Samadhi-concentrations* that delude us.

2. Blog on Applications of Multispace (and Multistructure) in Science

2.1. Think about parallel quantum computing [somehow similar to parallel computer programming that we are very familiar].

Since we work in a multispace $S = S_1 \cup S_2 \cup ... \cup S_n$, (a union of spaces that may overlap) which consequently has a multistructure, we may consider a quantum computing in the same time in each space $S_1, S_2, ..., S_n$ – connecting this to the mu-bit.

This is an interesting idea. I could imagine we may propose a novel nanocomputer using this concept...

http://www.foresight.org/updates/Update12/Update12.2.html.

We should call it multi-nanocomputer or *munanocomputer*.

Can we connect the mu-bit and parallel quantum computing to new energy creation (what everybody is looking for today)?

Not sure, but it is useful to check it. Just find an interesting Stanford thesis discussing quantum computation and quaternion number... May be they could be related...

http://www.stanford.edu/~eboyden3/mengthesis/thesis1.htm.

Now perhaps we could combine the mu-bit with Cantor's infinity

http://www.earlham.edu/~peters/writing/infinity.htm

and get an *infinite-bit*?

2.2. Imagine some multi-entangled states or particles (if possible?) + multispace + multi-parallel quantum computing.

One particle (or one particle state) in each space of the multispace.

My best thought so far is to compare this multispace-bit with Young double slit experiment - like the paradox that light could resemble particle (photon) yet it could behave like wave (Fresnel).

We can try a similar experiment with more slits at different distances from each other, or slits of various shapes, and the source light coming from many angles either separately i.e. each time from an angle, or in the same time from many angles.

It seems to me that this paradox could be reconciled once we introduce multispace-bit, that creation of 'bit' will multiplicate itself into multi-space, which triggers wave pattern in Young slit experiment.

But how to put in into a quantum-mechanics language?

In each space S_i the bit may be different, as for the light: particle in S_1, wave in another space S_2. But there might be possible another form of the light (neither particle, nor wave - maybe both of them simultaneously, or none of them as in neutrosophics - i.e. a form that we are not aware of in today's science) occurring in space S_3, and so on.

I tried to develop the (geometrical or algebraic or analytical, etc.) multi-space, and then to consider a similar/adopted correspondent for bit, qu-bit, up to mu-bit (= **multi-bit** in a multi-space).

I read a paper about: "Multivalued logic and gates for quantum computation" by Ashok Muthukhrishnnan *et al*.

The authors define the 'qudit' [as a generalization of the qubit] as a *d*-dimensional quantum system with the basis states $|0\rangle$, $|1\rangle$, ..., $|d-1\rangle$, and also they talk about 'quantum computing in multilevel systems'.

I think the mu-bit and multi-parallel quantum computing are different from 'qudit' and 'quantum computing in multilevel systems' respectively.

In our (mine and Vic Christianto's) 'multispace bit' we consider that each space may have a different structure, while I feel that in the 'qudit' the structures are the same for each level.
Hence I think the mu-bit is a generalization of the qudit.
We are able to explain the particle/wave duality of the light, while the qudit is not.

Similarly our multi-parallel quantum computing is a generalization of 'quantum computing in multilevel systems' since we may have different types of quantum computing at each level/space since we accept different structures.

2.3. A question perhaps: does it mean that multispace corresponds to multivalued logic, and if yes could we apply it to make 'multispace-multivalued' interpretation of Schrödinger equation? And how to find differential equation in multispaces?...

We can apply the multi-valued logic in the following way:
a logic L_1 in a field F_1, another logic L_2 in another field F_2, etc. (as many as necessary).
About a common *differential equation* there might be a problem, but I think it would be possible to get a differential equation E_1 in a field F_1, another differential equation E_2 in another field F_2, and so on. Then try to connect all of them under a general differential equation.

In a multi-space, a point (or an entire region which is common to more spaces) may have in the same time many metrics and pseudo-metrics (metrics that do not verify all metrics' axioms).
So, the multi-space becomes more complex. I wrote this to the physicist D. Rabounski, since this might give him a new impulse of reinterpretation of singular points in various theories.

2.4. I read Dr. Kaivarainen's idea about overunity machine and the possibility of using virtual particles and real particles and antiparticles.

I wonder if another category of particles, called unparticles which are a kind of unmatter (neither particles, nor antiparticles, but something in between: combinations of quarks and antiquarks bounding together even for a very short period of time, such as: pions, pentaquarks, etc.) would be of any usefulness for the interacting with those virtual particles, real particles, and antiparticles? Thus, having four categories of particles interacting each other.

Curiously, in the journal "Telegraph", from 23rd January 2008, there is an article by Roger Highfield, called *Is our cosmos teeming with alien 'unmatter'*, where the name

"unmatter" is just in the article's title, but my name is not even cited: http://www.telegraph.co.uk/science/large-hadron-collider/3322840/Is-our-cosmos-teeming-with-alien-unmatter.html ;
only Prof. Howard Georgi from Harvard University is cited for "unparticle" that is a kind of unmatter particle,
and everybody knows I coined the name **unmatter** as a combination of matter and antimatter – and a possible third form of matter - since 2004, in CERN website, and I published papers about "unmatter":

1. "A New Form of Matter – Unmatter, Formed by Particles and Anti-Particles" (EXT-2004-182), in CERN's web site: http://cdsweb.cern.ch/record/798551 and http://cdsweb.cern.ch/record/798551/files/ext-2004-142.pdf?version=1 (2004);
   - and in <Progress in Physics>, UNM-G, Vol. 1, 9-11, 2005;
   - and under the title of "Matter, Antimatter, and Unmatter" in <Infinite Energy>, Concord, NH, USA, Vol. 11, Issue 62, 50-51, 2005;
   - and in <Bulletin of Pure and Applied Sciences>, Vol. 23D, No. 2, 173-177, 2004.
2. "Verifying Unmatter by Experiments, More Types of Unmatter, and A Quantum Chromodynamics Formula", in Progress in Physics, Vol. 2, 113-116, 2005;
   - an improved version in "Infinite Energy", Concord, NH, USA, 36-39, Vol. 12, Issue 67, 2006.
3. "Unmatter Entities inside Nuclei, Predicted by the Brightsen Nucleon Cluster Model", Progress in Physics, Vol. 1, 14-18, 2006 (with D. Rabounski).
4. On Emergent Physics, "Unparticles" and Exotic "Unmatter" States, by Ervin Goldfain and Florentin Smarandache, Progress in Physics, Vol. 4, 10-15, 2008;
   - and in <Hadronic Journal>, Vol. 31, No. 6, 591-604, December 2008, http://www.hadronicpress.com/hadronic_journal.htm.

Or my abstract on unmatter in American Physical Society's meetings:
http://meetings.aps.org/Meeting/GEC09/Event/107355 ;
Bulletin of the American Physical Society**,** 62nd Annual Gaseous Electronics Conference Volume 54, Number 12 **,** Tuesday–Friday, October 20–23, 2009; Saratoga Springs, New York, USA.

In the below site there is a blog and a link to my CERN article:
http://novaspivack.typepad.com/nova_spivacks_weblog/2005/03/unmatter_a_poss.html
"**Unmatter: A Possible Third Form of Matter.**
This interesting abstract from CERN proposes a third form of matter that is between matter and antimatter: "unmatter." Interesting idea to track. It could have huge implications if confirmed."

Doing a search on Google for "unmatter" there are 2,680 sites (end of March 2010), and my unmatter articles are among the first, therefore anybody could read them and my definition of "unmatter".

From our above paper *On Emergent Physics, "Unparticles" and Exotic "Unmatter" States*, by E. Goldfain and F. Smarandache, unparticles represent exotic quantum states

that can occur at energies greater than the Standard Model's energy (bigger than *1 TeV*), with abnormal properties:

a) they are mixtures of particles and antiparticles [= unmatter];
b) their spin is not integer but fractional (i.e. their spin is different from 1 or 0 for Bosons, and different from1/2, 3/2, 5/2, etc. for Fermions); they are neither particles of matter (leptons and quarks), nor particles that transmit forces {gauge Bosons, as for example photons, gravitons, particles that transmit the weak interaction (*W, Z*), or gluons that connect the quarks in nuclei and transmit the strong interaction}; therefore, unparticles represent arbitrary mixtures of particles of matter with particles that tie the matter;
c) they are mixtures between Left and Right states (i.e. an arbitrary combination of quantum polarizations; they do not satisfy today's theory of the quantum field);
d) they are very unstable and decompose almost instantaneously.

I hope the scientific justice will eventually prevail.

2.5. I've studied that Ginzburg-Landau model can be related to 'compact sphere model', therefore this approach may be connected (somehow) to CGLE method of Ervin Goldfain. Others may find linkage with Schrödinger equation, Brownian motion etc.

2.6. I'd be happy to cooperate with Dr. Ervin Goldfain into connecting Unmatter with Dark Matter {and an idea just coming into my mind is about… Dark Antimatter and Dark Unmatter if these might exist (?)} – but this has to be proved in Geneva's Large Hadron Collider.

Ervin e-mailed me that the properties of "dark matter" seem to be close to unmatter and unparticles, but the experiments in this domain are barely starting.

Matter, antimatter, and unmatter; or particle, antiparticle, and unparticle – as in neutrosophy <*M*>, <*antiM*>, and <*unM*>; or <*P*>, <*antiP*>, and <*unP*>.
It would be interesting in finding out if Dark Matter is composed from aggregates of unmatter and unparticles (without electrical charge, and formed from mixtures of particles with fractional spin).

2.7. Apparently why normal scientists could not accept CF (cold fusion) easily (Taleyarkhan, Putterman etc.) is because there is an obvious problem herein: how to overcome Coulomb barrier? This is why hot-fusioners should create large Tokamak chamber like in ITER project (now to be built in France)...

This is partly why we introduced the 'Hulthen potential' term in Ginzburg-Landau equation, because Hulthen potential permits Coulomb barrier reversal, therefore it could be attractive instead of repulsive at some conditions...
We should envisage all fusions to be done simultaneously in each space $S_1, \ldots, S_m$, at various (lower, medium, higher) temperatures, pressures, etc..

2.8. Nanotechnology is stressing here in academia as a hot research subject. I bought a handbook from Amazon.com, but it has much organic chemistry.

We can unite nanoscale space with our world scale and with cosmic scale, or inorganic nanoscale with organic nanoscale.

2.9. Multi-space (or multispace) unifies science fields; actually the whole universe is a multi-space.

It is curious that we accepted all quantum theories, not yet proven, many based on imagination (maybe science fiction), while others are reluctant to multi-space which constitutes so obvious our reality. Sure, I acknowledge (and it is my fault and incompleteness) that there is not much theory behind multi-space (some theory is about Sm. Geometries, that are a particular type of multi-space formed as unions of geometrical spaces only).

Multispace is a qualitative notion, not a metrics notion, since multispace is too large and such including metric and non-metric spaces.

Weyl and Kahler geometries are used in quantization somehow, but my geometries look to be more general and including these geometries; how should we use them?

www.gallup.unm.edu/~smarandache/geometries.htm

Laurent Schadeck has asked if the Smarandache geometries relate to non-associative structures.

Non-associative structures are structures whose laws are not associative.

These geometries are regarded from another point of view, considering a union of hybrid geometrical spaces (that form together a multi-space).

In general, this multi-space embedded with a law is a non-associative structure.

2.10. Parallel universes are also a particular case of multi-space anyway.

You can have parallel universes on each dimension.

But "combinatorial manifolds" do not include the possibility to use pseudomanifolds as used in the Theory of Relativity.

While combinatorial pseudomanifolds will be part of Sm. multi-space.

2.11. I was still in a trip in Egypt, when an idea came to me that - following the neutrosophic logic where $<A>$ and $<antiA>$ can be simultaneously true, or using the multispace - maybe more models of the atom might be correct, not only the standard model of the atom.

But the type of model might depend of the substance atom, and on other parameters (temperature, pressure, etc.).

2.12. I read papers on parallel timelines, but those explanations look somehow fantastic to me.

Time is considered a line; shouldn't it be a curve since in the text it is taken for a spiral?

Multi-space could work for time in the sense of multi-curve time, I mean curve C_1 (like space C_1) representing time t_1, curve C_2 representing time t_2, etc. Hence, do we need infinitely many curve-times?

2.13. Did you hear that Pluto is not considered a planet any longer, so we now have only eight planets in the solar system (so far… who knows in the future?).

2.14. What about writing **sci-fi** using nanotechnology? [say a nanorobot guided inside the body to fight viruses, or nanomachines that auto-reproduce, etc.]. Like Asimov's stories, but adopted for today's science.

Microchips as additional human memories implemented in man's brain, etc.
Would it be exciting writing such stories? Maybe first we can start with short stories and make a collection.
Another choice I just think is writing a 'satire' version of Star Trek with the purpose of humor, for instance: "**Star Shrek**". Therefore, we can write another fiction. The character 'Shrek', from Disney, now is going to space-travel...
http://www.magicdragon.com/UltimateSF/SF-Index.html.
Just Google to see that there is no 'Star Shrek' novel so far...

2.15. But I'd imagine something in microuniverse, since in macrouniverse many people wrote before us, as "the adventure of a particle-man", etc... or a man who can move from our normal space to a nucleus space...

using a lot of fantasist quantum physics...
with things which are in two places in the same times, and with superluminal speeds inside...
Also, to be innovatory, we can include differential equations in the novel [why not first starting with short SF stories, and then go to a SF novel?];
for example, the equation may describe the trajectory of the particle-man inside a hydrogen atom...
or even parts of a scientific article [from ours, in order not having copyright problems], explaining the *SC potential* of the particle-man inside the nucleus, or how the particle-man moves from a space into another space with its multispace, for example from the quantum space to the macrospace and becomes a real man...
something funny...

In conclusion:
Should we keep these ideas for ourselves only so nobody would claim later he/she developed them before us?
Well, there is paradox here: if we don't discuss with anybody else, our ideas cannot grow, but if we discuss it others could steal them (with people from Seattle I mean is: 'Most Intelligent Customers Realize Our Software Only Fools Teenagers' - you know this joke? (www.ngkhai.net/blog/category/general-interest/jokesquotes).
Therefore I think the best way is to secure them first (US/ Europe is the best place, because if we apply to developing countries somehow a patent registered at Europe could supersede it), then begin writing articles for journals, etc...

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NON-EUCLIDEAN GEOMETRY

DEGREE OF NEGATION OF AN AXIOM

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In this article we present the two *classical negations* of Euclid's Fifth Postulate (done by Lobachevski-Bolyai-Gauss, and respectively by Riemann), and in addition of these we propose a *partial negation* (or a degree of negation) of an axiom in geometry.

The most important contribution of this article is the introduction of the degree of negation (or partial negation) of an axiom and, more general, of a scientific or humanistic proposition (theorem, lemma, etc.) in any field - which works somehow like the negation in fuzzy logic (with a degree of truth, and a degree of falsehood) or like the negation in neutrosophic logic [with a degree of truth, a degree of falsehood, and a degree of neutrality (i.e. neither truth nor falsehood, but unknown, ambiguous, indeterminate)].

The Euclid's Fifth postulate is formulated as follows: if a straight line, which intersects two straight lines, form interior angles on the same side, smaller than two right angles, then these straight lines, extended to infinite, will intersect on the side where the interior angles are less than two right angles.

This postulate is better known under the following formulation: through an exterior point of a straight line one can construct one and only one parallel to the given straight line.

The Euclid's V postulate (323 BC - 283 BC) is worldwide known, logically consistent in itself, but also along with other four postulates with which to form a consistent axiomatic system.

The question, which has been posted since antiquity, is if the fifth postulate is dependent of the first four?

An axiomatic system, in a classical vision, must be:

1) **Consistent** (the axioms should not contradict each other: that is some of them to affirm something, and others the opposite);
2) **Independent** (an axiom must not be a consequence of the others by applying certain rules, theorems, lemmas, methods valid in that system; if an axiom is proved to be dependent (results) of the others, it is eliminated from that system; the system must be minimal);
3) **Complete** (the axioms must develop the complete theory, not only parts of it).

The geometers thought that the V postulate (= axiom) is a consequence of the Euclid's first four postulates. Euclid himself invited others in this research. Therefore, the system proposed by Euclid, which created the foundation of classical geometry, seemed not be independent.

In this case, the V postulate could be eliminated, without disturbing at all the geometry's development.

There were numerous tentative to "proof" this "dependency", obviously unsuccessful. Therefore, the V postulate has a historic significance because many mathematicians studied it.

Then, ideas revolved around negating the V postulate, and the construction of an axiomatic system from the first four unchanged Euclidean postulates plus the negation of the fifth postulate. It has been observed that there could be obtained different geometries which are bizarre, strange, and apparently not connected with the reality.

a) Lobachevski (1793-1856), Russian mathematician, was first to negate as follows: "Through an exterior point to a straight line we can construct an infinite number of parallels to that straight line", and it has been named **Lobachevski's geometry** or hyperbolic geometry. This negation is 100%.

After him, independently, the same thing was done by Bolyai (1802-1860), Hungarian from Transylvania, and Gauss (1777-1855), German. But Lobachevski was first to publish his article.

Beltrami (1835-1900), Italian, found a model (= geometric construction and conventions in defining the notions of space, straight line, parallelism) of the hyperbolic geometry, that constituted a progress and assigning an important role to it. Analogously, the French mathematician Poincaré (1854-1912).

b) Riemann (1826-1866), German, formulated another negation: "Through an exterior point of a straight line one cannot construct any parallel to the given straight line", which has been named **Riemannian geometry** or elliptic geometry. This negation is also 100%.

c) Smarandache (b. 1954) partially negated the V postulate: "There exist straight lines and exterior points to them such that from those exterior points one can construct to the given straight lines:

1. only one parallel – in a certain zone of the geometric space [therefore, here functions the Euclidean geometry];
2. more parallels, but in a finite number – in another space zone;
3. an infinite number of parallels, but numerable – in another zone of the space;
4. an infinite number of parallels, but non-numerable – in another zone of the space [therefore, here functions Lobachevski's geometry];
5. no parallel – in another zone of the space [therefore, here functions the Riemannian geometry].

Therefore, the whole space is divided in five regions (zones), and each zone functions differently. This negation is not 100% as the previous ones.

I was a student at that time; the idea came to me in 1969. Why? Because I observed that in practice the spaces are not pure, homogeneous, but a mixture of different structures. In this way I united the three (Euclidean, hyperbolic, and elliptic) geometries connected by the V postulate, and I even extended them (with other two adjacent zones).

The problem was: how to connect a point from one zone, with a point from another different zone (how crossing the "frontiers")?

In "Bulletin of Pure and Applied Science" (Delhi, India), then in the prestigious German magazine which reviews articles of mathematics "Zentralblatt für Mathematik" (Berlin) there exist four variants of Smarandache Non-Euclidean Geometries [following the tradition: Euclid's (classical, traditional) geometry, Lobachevski's geometry, Riemannian geometry, Smarandache geometries].

The most important contribution of **Smarandache geometries** was the introduction of the **degree of negation of an axiom** (and more general the degree of negation of a theorem, lemma, scientific or humanistic proposition) which works somehow like the negation in fuzzy logic (with a degree of truth, and a degree of falsehood) or more general like the negation in neutrosophic logic (with a degree of truth, a degree of falsehood, and a degree of neutrality (neither true nor false, but unknown, ambiguous, indeterminate) [not only Enclid's geometrical axioms, but any scientific or humanistic proposition in any field] or partial negation of an axiom (and, in general, partial negation of a scientific or humanistic proposition in any field).

These geometries connect many geometrical spaces with different structures into a heterogeneous multi-space.

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[Translated from Romanian by the Author.]

NUMBER THEORY

Generalization and Alternatives of Kaprekar's Routine

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Abstract.
We extend Kaprekar's Routine for a large class of applications. We also give particular examples of this generalization as alternatives to Kaprekar's Routine and Number. Some open questions about the length of the iterations until reaching either zero or a constant or a cycle, and about the length of the cycles are asked at the end.

1. Generalization of Kaprekar's Routine.

Let f be an operator that maps a finite set $A = \{a_1, a_2, ..., a_p\}$, with $p \geq 1$, into itself:

$$f: A \rightarrow A.$$

Then, for any value $a \in A$ we have $f(a) \in A$ too. If we iterate this operator multiple times we get a chain like this:

$a \in$ A involves $f(a) \in A$, which involves $f(f(a)) \in A$ (f iterated with itself twice), which in its turn involves $f(f(f(a))) \in A$, and so on, $f(...f(a)...) \in A$ { where, in the last case, f has been iterated with itself $r \geq 1$ times; let's denote it by $f_r(a)$ }. Let's also denote, for the symmetry of notation,

$$a = f_0(a).$$

Since cardinal of A is a finite positive integer, $card(A) = p < +\infty$, after at most $r = p$ iterations we get two equal iterations

$$f_i(a) = f_j(a)$$

for $i \neq j$ with $0 \leq i < j \leq r$, in the above chain $a=f_0(a), f_1(a), f_2(a), ..., f_r(a)$.

Hence, this chain of values can form a cycle:

$$f_0(a), f_1(a), ..., f_i(a), f_{i+1}(a), ..., f_{j-1}(a), f_i(a), f_{i+1}(a), ..., f_{j-1}(a),$$

with the cycle $f_i(a), f_{i+1}(a), ..., f_{j-1}(a)$ of length $j-i$.

If $j=i+1$ then the chain reaches a constant, since the cycle has only one element $f_i(a)$.

2. Kaprekar's Routine.

Kaprekar's Routine {see [1], [2], [3]}, extended to *k*-digit numbers, is a particular case of the above algorithm.

For k = 1 and 2 the Kaprekar's Routine reaches zero, for k = 3 or 4 the algorithm reaches zero or 495, respectively zero or 6174. For $k \geq 6$ it also reaches cycles.

The *set A = {0, and all k-digit positive integers}*, so its cardinal is finite, *card(A) = 90…01*, where in this number we have *k-2* zeroes.

The operator *f* does the following: arranges the digits of number *a* in descending order (*a'*) and in ascending order (*a''*) and then subtracts them: *a`-a``*. Since *a`-a``* is also a *k*-digit number or zero (in the degenerate case when all the digits are equal), then f(*a`-a``*) is a *k*-digit number as well or zero, therefore f(*a`-a``*) $\in A$. And the iteration continues in the same way. After a finite number of iterations the algorithm reaches a constant (which can be zero in the degenerate case), or a constant, or gets into a cycle.

3. Alternatives to Kaprekar's Routine.

3.1. Let's consider the group of permutations *P* of the digits of the *k*-digit *(k ≥ 1)* number *a*. We define the operator

$$f_P(a) = |P_1(a)-P_2(a)|$$

where P_1 *and* P_2 are some permutations of *k* elements *{1, 2, …, k}*, and |.| means absolute value.

And the *set A = {0; and all m-digit positive integers, m ≤ k }*.

Then the sequence of iterations reaches zero, a constant, or a cycle.

Let's see an **example**:

$P_1(\{1,2,3\}) = \{2,3,1\}$ and $P_2(\{1,2,3\}) = \{1,3,2\}$; *a* = 125, then we have |251-152| = 099;

|990-099|=891; |918-819|=099; |990-099|=891; |918-819|=099; … .

So, we reached a cycle: 125, *099, 891*, 099, 891, … .

3.2. Let's have the same group of permutations and same set *A* as in Example 3.1, but taking as operator:

$$fa_P(a) = |a-P(a)|.$$

See another example:

P({1,2,3}) = {3,1,2}; *a* = 125, then |125-512| = 387, |387-738| = 351, |351-135| = 216,

|216-621| = 405, |405-540| = 135, |135-513| = 378, |378-837| = 459, |459-945| = 486,

|486-648| = 162, |162-216| = 054, |054-405| = 351, … .

So, we got: 125, 387, 351, 216, 405, 135, 378, 459, 486, 162, 054, 351, … .

3.3. Similarly if we take the operator: the absolute value of a number minus its reverse:

$$|a\text{-}reverse(a)|.$$

For example: 125, |125-521| = 396, |396-693| = 297, |297-792| = 495, |495-594| = 099,

|099-990| = 891, |891-198| = 693, |693-396| = 297, |297-792| = 495, … .

3.4. Let's consider the *Smarandache Form* of a number. A *Smarandache Palindrome* (SP) {see [4]-[13]} is a number of the form $a_1a_2...a_{n-1}a_na_na_{n-1}..a_2a_1$ or $a_1a_2...\ a_{n-1}a_na_{n-1}..a_2a_1$ where each a_i is a positive integer of any number of digits. For example, 143431 is a SP since it can be written as 1(43)(43)1. A Smarandache Form is any number $a_1a_2...a_{n-1}a_n$, where each a_i is a positive integer of any number of digits.
Then we can take the operator mapping SFs into SFs in the following way.
Example: Consider the following SF: 3-digit numbers under the SF of 1-digit and 2-digit groups: for example 5(76). Then we switch the groups and add them. Take modulo 1000 of the result.
Start with 5(76), 5(76) + (76)5 = 1342 whole module 1000 is 342 = 3(42);
3(42) + (42)5 = 767 = 7(67); 7(67) + (67)7 = 1444 whose modulo 1000 is 444 = 4(44);
4(44) + (44)4 = 888 = 8(88); 8(88) + (88)8 = 1776 whose module 1000 is 776 = 7(76);
7(76) + (76)7 = 1543 whose modulo 1000 is 543 = 5(43);
5(43) + (43)5 = 9(78); 9(78) + (78)9 = 1767 whose modulo 1000 is 767 = 7(67);
7(67) + (67)7 = 1444 whose modulo 1000 is 4(44).
We got: 5(76), 3(42), 4(44), 7(76), 5(43), 7(67), 4(44), … .

3.5. Or one consider another operator that subtracts two *k*-digit numbers in the following way: adding 1 to each digit less than 9, then subtracting 1 from each non-zero digit, then subtracting the numbers.
Example: 495, 596-384 = 212, 323 – 101 = 222, 333-111 = 222, … . We reached a constant.
Or add 2 to each digit strictly less than 8, and subtract 3 from each digit strictly greater than 2. Etc.

3.6. Consider any function *f* defined on the set of *k*-digit numbers whose range is the set of positive integers, and then calculate *modulo* 10^k of the result.

There are infinitely many such operators in order to choose from nice examples.

4. Open Questions:

1. What is the longest number of iterations until one reaches either zero or a constant or a cycle that one can have for each case of the above generalizations?
2. What is the longest cycle that one can have for each particular case of the above generalizations?
3. In what conditions one reaches a constant, not a cycle? By cycle we understand a sequence of two or more numbers that repeat indefinitely.
4. Study the cases when $f(a) = a$ for interesting particular cases of this generalization.
5. Study the case when $f(a) = 0$ for interesting particular cases of this generalization.
6. If the operator defined in the above Generalization of the Kaprekar's Routine is a random operator (i.e. for a given k-digit number a one randomly generate another k-digit number b), is it still possible to reach a constant or a cycle?
   It is possible for sure to generate two equal k-digit numbers, c, after at most 10^k random operations, but then the next k-digit number following the first c would not necessarily be the same as the previous k-digit number following the previous c.

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Three Conjectures and Two Open Generalized Problems in Number Theory

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1. On a Problem with Primes.

In [1] it was asked if $\prod_{n=1}^{m}\frac{P_n+1}{P_n-1}$, where P_n is the n^{th} prime, is an integer for others $m \notin \{1,2,3,4,8\}$?

1) We conjecture no.
2) We also conjecture that

$$R_m = \prod_{n=1}^{m}\frac{P_n+k}{P_n-k},$$

with $k \in \mathrm{N}^* = \{1, 2, \ldots\}$, is an integer for a finite number of values of m.

3) Another conjecture: there is an infinite number of k's for which no R_m is an integer.

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2. On a Problem with Infinite Sequences.

Let $1 \le a_1 < a_2 < \ldots$ be an infinite sequence of integers such that any three members do not constitute an arithmetical progression. Is it true that always $\sum\frac{1}{a_i} \le 2$?

Is the function $S\left(\{a_n\}_{n\ge1}\right) = \sum_{n\ge1}\frac{1}{a_n}$ bijective (biunivocal)?

For example, $a_n = p^{n-1}$, $n \ge 1$, p is an integer >1, has the property of the assumption, and $\sum_{n\ge1}\frac{1}{a_i} = 1 + \frac{1}{p-1} \le 2$,

Analogously for geometrical progressions.

4) More generally: let f be a function $f : \mathrm{R}_+^m \to \mathrm{R}_+^*$. We construct a sequence $0 < a_1 < a_2 < \ldots$ such that there is no $\left(a_{i_1},\ldots,a_{i_m},a_{i_{m+1}}\right)$ such that $a_{i_{m+1}} = f(a_{i_1},\ldots,a_{i_m})$.

Find $\max_{\{a_n\}_{n\geq 1}} \sum^{m}_{n\geq 1} \frac{1}{a_n}$

(It's a generalization of a question from the Problem E28, p. 127, in *Unsolved Problems in Number Theory*, by R. K. Guy, Springer-Verlag, 1981.)

5) Is the function

$$S\left(\{a_n\}_{n\geq 1}\right) = \sum_{n\geq 1} \frac{1}{a_n}$$

bijective?

[This manuscript was confiscated by the Romanian Secret Police [Securitate] in 1980's. A copy of it was recovered by the Author after the 1989 Revolution through the CNSAS = National Council of Studying the Archives of the Secret Police.

Unfortunately, other confiscated manuscripts (a few hundreds of pages of mathematical proposed problems and conjectures, rebus, literary works) were never recovered, despite the fact that the Author required the CNSAS for returning his manuscripts.]

Addendum:
On the next page see a copy of my holograph manuscript stamped by the C.N.S.A.S. on 23 July 2002.

ON A PROBLEM WITH PRIMES

by FLORENTIN SMARANDACHE

David Silverman asked if $\prod_{n=1}^{m} \frac{p_n+1}{p_n-1}$, where p_n is the nth prime, is an integer for others $m \notin \{1,2,3,4,8\}$. We conjecture NO.

We conjecture that

$$R_m = \prod_{n=1}^{m} \frac{p_n+k}{p_n-k}, \text{ with } k \in \mathbb{N}^*,$$

is an integer for a finite number of values of m.

There is an infinite number of k for which no R_m is an integer.

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R.K. Guy, Unsolved Problems in Number Theory, Springer-Verlag, 1981, problem B48, p. 57.

ON A PROBLEM WITH INFINITE SEQUENCES

by FLORENTIN SMARANDACHE

Let $1 \leq a_1 < a_2 < \ldots$ be an infinite sequence of integers such that any three members do not constitute an arithmetical progression. Is it true that always $\sum 1/a_i \leq 2$?

Is the function

$$S(\{a_n\}_{n \geq 1}) = \sum_{n \geq 1} 1/a_n$$

bijective? (biunivocal)

For example, $a_n = p^{n-1}$, $n \geq 1$, p is an integer > 1, has the property of the assumption, and $\sum_{n \geq 1} 1/a_i = 1 + \frac{1}{p-1} \leq 2$.

Analogously for geometrical progressions.

More generaly: let f be a function $f: \mathbb{R}_+^m \to \mathbb{R}_+^*$. We construct a sequence $0 < a_1 < a_2 < \ldots$ such that there be no $(a_{i_1}, \ldots, a_{i_m}, a_{i_{m+1}})$ with $f(a_{i_1}, \ldots, a_{i_m}) = a_{i_{m+1}}$.

Find $\max_{\{a_n\}_{n \geq 1}} \sum_{n \geq 1} 1/a_n$.

(It's a generalisation of a question from the problem E28, R.K. Guy, Unsolved Problems in Number Theory, Springer-Verlag, 1981, p. 127.)

Is the function

$$S(\{a_n\}_{n \geq 1}) = \sum_{n \geq 1} 1/a_n$$

bijective?

C.N.S.A.S.
SERVICIUL
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23.-07- 2002

Open Questions about Concatenated Primes and Metasequences

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Abstract.
We define a metasequence as a sequence constructed with the terms of other given sequence(s). In this short note we present some open questions on concatenated primes involved in metasequences.

First Class of Concatenated Sequences.

1) Let $a_1, a_2, ..., a_{k-1}, a_k$ be given $k \geq 1$ digits in the numeration base b.
a) There exists a prime number P of the concatenated form:

$$P = \overline{*_{...}*a_1*_{...}*a_2*_{...}*\ldots*_{...}*a_{k-1}*_{...}*a_k*_{...}*}$$

where the stars "$*_{...}*$" represent various (from none to any finite positive integer) numbers of digits in base b.

Of course, if a_k is the last digit then a_k should belong to the set {1, 3, 7, 9} in base *10*. Similar restriction for the last number's digit a_k in other base b.

b) Are there infinitely many such primes?
c) What about considering fixed positions for the given digits: i.e. each given a_i on a given position n_i ?
d) As a consequence, for any group of given digits $a_1, a_2, ..., a_{k-1}, a_k$ do we have finitely or infinitely many primes starting with this group of digits (i.e. in the following concatenated form):

$$\overline{a_1a_2...a_{k-1}a_k*_{...}*}$$

?
e) As a consequence, for any group of given digits $a_1, a_2, ..., a_{k-1}, a_k$ do we have finitely or infinitely many primes ending with this group of digits (i.e. in the following concatenated form):

$$\overline{*_{...}*a_1a_2...a_{k-1}a_k}$$

(of course considering the primality restriction on the last digit a_k) ?

f) As a consequence, for any group of given digits $a_1, a_2, ...,a_{k-1}, a_k$ and any given digits $b_1, b_2, ...,b_{j-1}, b_j$ do we have finitely or infinitely many primes beginning with the group of digits $a_1, a_2, ...,a_{k-1}, a_k$ and ending with the group of digits $b_1, b_2, ...,b_{j-1}, b_j$ (i.e. in the following concatenated form):

$$\overline{a_1a_2...a_{k-1}a_k*...*b_1b_2...b_{j-1}b_j}$$

(of course considering the primality restriction on the last digit bj) ?

g) As a consequence, for any group of given digits $a_1, a_2, ...,a_{k-1}, a_k$ do we have finitely or infinitely many primes having inside of their concatenated form this group of digits (i.e. in the following concatenated form):

$$\overline{*...*a_1a_2...a_{k-1}a_k*...*}$$

?

h) As a consequence, for any groups of given digits $a_1, a_2, ...,a_{k-1}, a_k$ and $b_1, b_2, ...,b_{j-1}, b_j$ and $c_1, c_2, ...,c_{i-1}, c_i$ do we have finitely or infinitely many primes beginning with the group of digits $a_1, a_2, ...,a_{k-1}, a_k$, ending with the group of digits $b_1, b_2, ...,b_{j-1}, b_j$, and having inside the group of digits $c_1, c_2, ...,c_{i-1}, c_i$ (i.e. in the following concatenated form):

$$\overline{a_1a_2...a_{k-1}a_k*...*\,c_1c_2...c_{i-1}c_i*...*b_1b_2...b_{j-1}b_j}$$

(of course considering the primality restriction on the last digit bj) ?

i) What general condition has a sequence $s_1, s_2, ..., s_n, ...$ to satisfy in order for the concatenated metasequence

$$\overline{s_1s_2...s_n}$$

for $n = 1, 2, ...$ to contain infinitely many primes?

Second Class of Metasequences.

2) Let's note the sequence of prime numbers by $p_1 = 2, p_2 = 3, p_3 = 5, ..., p_n$ the n^{st} prime number, for any natural number n.

a) Does the metasequence

$$p_1 \cdot p_2 \cdot ... \cdot p_n + 1$$

for $n = 1, 2, \ldots$ contains finitely or infinitely many primes?

b) What about the metasequence:

$$p_1 \cdot p_2 \cdot \ldots \cdot p_n - 1$$

?

c) What general condition has a sequence $s_1, s_2, \ldots, s_n, \ldots$ to satisfy in order for the metasequence

$$s_1 \cdot s_2 \cdot \ldots \cdot s_n \pm 1$$

for $n = 1, 2, \ldots$ to contain infinitely many primes?

Reference:

F. Smarandache, Sequences of Numbers Involved in Unsolved Problems, 139 p., HeXis, 2006.

k-Factorial

Florentin Smarandache
Arizona State Univ., Special Collections
1972

Let n and k be positive integers, with 1 =< k =< n-1.

As a generalization of the factorial and double factorial one defines the k-factorial of n as the below product of all possible strictly positive factors:

SKF(n) = n(n-k)(n-2k)… .

Particular Cases:

S1F(n) is just the well-known factorial of n, i.e. n! = n(n-1)(n-2)…1.

S2F(n) is just the well-known double factorial of n, i.e. n!! = n(n-2)(n-4)… .

S3F(n) is the triple factorial of n, i.e. n!!! = n(n-3)(n-6)… .

S4F(n) is the fourth factorial of n, i.e. S4F(n) = n(n-4)(n-8)… .

Examples:

S3F(7) = 7(7-3)(7-6) = 28.

S4F(8) = 8(8-4)=32.

S10F(27) = 27(27-10)(27-20) = 27(17)7 = 3213.

Remark:

Many Smarandache type functions, such as the Smarandache (classical) function, double factorial function, ceil functions, etc. can be extended/transformed to this k-factorial definition.

Back and Forth Factorials

Florentin Smarandache
Arizona State Univ., Special Collections
1972

Let n>k≥1 be two integers. Then the Smarandacheial is defined as:

$$!n!_k = \prod_{\substack{0<|n-k\cdot i|\le n \\ i\in N}} (n-k\cdot i)$$

For examples:

1) In the case k=1:

$$!n!_1 \overset{conv}{\equiv} !n! = \prod_{\substack{0<|n-i|\le n \\ i=0,1,2,\dots}} (n-i) = n(n-1)(n-2)\dots(2)(1)(-1)(-2)\dots(-n+2)(-n+1)(-n) = (-1)^n (n!)^2.$$

Thus !5! = 5(5-1)(5-2)(5-3)(5-4)(5-6)(5-7)(5-8)(5-9)(5-10)=5·4·3·2·1·(-1)·(-2)·(-3)·(-4)·(-5) = -14400.

The sequence is: 4, -36, 576, -14400, 518400, -25401600, 1625702400, -131681894400, 13168189440000, -1593350922240000, 229442532802560000, -38775788043632640000, 7600054456551997440000, -1710012252724199424000000, … .

2) In the case k=2:
a) If n is odd, then

$$!n!_2 = \prod_{\substack{0<|n-2i|\le n \\ i=0,1,2,\dots}} (n-2i) = n(n-2)(n-4)\dots(3)(1)(-1)(-3)\dots(-n+4)(-n+2)(-n) = (-1)^{(n+1)/2}(n!!)^2.$$

a) If n is even, then

$$!n!_2 = \prod_{\substack{0<|n-2i|\le n \\ i=0,1,2,\dots}} (n-2i) = n(n-2)(n-4)\dots(4)(2)(-2)(-4)\dots(-n+4)(-n+2)(-n) = (-1)^{n/2}(n!!)^2.$$

Thus: $!3!_2 = 3(3-2)(3-4)(3-6) = 9$ and $!4!_2 = 4(4-2)(4-6)(4-8) = 64$.

The sequence is: 9, 64, -225, -2304, 11025, 147456, -893025, -14745600, 108056025, 2123366400, … .

3) In the case k=3:

$$!n!_3 = \prod_{\substack{0<|n-3i|\le n \\ i=0,1,2,\dots}} (n-3i) = n(n-3)(n-6)\dots .$$

Thus $!7!_3 = 7(7-3)(7-6)(7-9)(7-12) = 7(4)(1)(-2)(-5) = 280$.

The sequence is: -8, 40, 324, 280, -2240, -26244, -22400, 246400, 3779136, 3203200, -44844800, … .

4) In the case k=4:

$$!n!_4 = \prod_{\substack{0<|n-4i|\le n \\ i=0,1,2,\ldots}} (n-4i) = n(n-4)(n-8)\ldots .$$

Thus $!9!_4 = 9(9-4)(9-8)(9-12)(9-16) = 9(5)(1)(-3)(-7) = 945.$

The sequence is: -15, 144, 105, 1024, 945, -14400, -10395, -147456, -135135, 2822400, 2027025, … .

5) In the case k=5:

$$!n!_5 = \prod_{\substack{0<|n-5i|\le n \\ i=0,1,2,\ldots}} (n-5i) = n(n-5)(n-10)\ldots .$$

Thus $!11!_5 = 11(11-5)(11-10)(11-15)(11-20) = 11(6)(1)(-4)(-9) = 2376.$

The sequence is: -24, -42, 336, 216, 2500, 2376, 4032, -52416, -33264, -562500, -532224, -891072, 16039296, … .

More general:
Let n>k≥1 be two integers and m≥1 another integer. Then the generalized Smarandacheial is defined as:

$$!n!m_k = \prod_{\substack{0<|n-k\cdot i|[m \\ i\ cN}} (n-k\cdot i)$$

For examples:

$$!7!3_2 = 7(7-2)(7-4)(7-6)(7-8)(7-10) = 7(5)(3)(1)(-1)(-3) = 315.$$

$$!7!9_2 = 7(7-2)(7-4)(7-6)(7-8)(7-10)(7-12)(7-14)(7-16) = 7(5)(3)(1)(-1)(-3)(-5)(-7)(-9)$$
$$= -99225.$$

References:

J. Dezert, editor, "Smarandacheials", Mathematics Magazine, Aurora, Canada, No. 4/2004; http://www.mathematicsmagazine.com/corresp/J_Dezert/JDezert.htm, and www.gallup.unm.edu/~smarandache/Smarandacheials.htm.
F. Smarandache, "Back and Forth Factorials", Arizona State Univ., Special Collections, 1972.

[These Back and Forth Factorials have been called Smarandacheials.]

Back and Forth Summands

Florentin Smarandache
Arizona State Univ., Special Collections
1972

Let n>k≥1 be two integers. Then a Back and Forth Summand is defined as:

$$S(n, {}_k) = \sum_{\substack{0<|n-k\cdot i|\le n \\ i=0,1,2,\dots}} (n-k\cdot i) \quad \text{[for signed numbers]}$$

$$S|n, {}_k| = \sum_{\substack{0<|n-k\cdot i|\le n \\ i=0,1,2,\dots}} |n-k\cdot i| \quad \text{[for absolute value numbers]}$$

which are duals and semi-duals respectively of Smarandacheials.

$S(n, {}_1)$ and $S(n, {}_2)$ with corresponding $S|n, {}_1|$ and $S|n, {}_2|$ are trivial.

a) In the case k=3:

$$S(n, {}_3) = \sum_{\substack{0<|n-3i|\le n \\ i=0,1,2,\dots}} (n-3i) = n+(n-3)+(n-6)+\dots \text{ ; [for signed numbers].}$$

$$S|n, {}_3| = \sum_{\substack{0<|n-3i|\le n \\ i=0,1,2,\dots}} |n-3i| = n+|n-3|+|n-6|+\dots \text{ ; [for absolute value numbers].}$$

Thus $S(7, {}_3) = 7+(7-3)+(7-6)+(7-9)+(7-12) = 7+(4)+(1)+(-2)+(-5) = 5$; [for signed numbers].
Thus $S|7, {}_3| = 7+|7-3|+|7-6|+|7-9|+|7-12| = 7+4+1+2+5 = 19$; [for absolute value numbers].

The sequence is $S(n, {}_3)$: 3, 2, 0, 5, 3, 0, 7, 4, 0, 9, 5, 0, … ; [for signed numbers].
The sequence is $S|n, {}_3|$: 7, 12, 18, 19, 27, 36, 37, 48, … ; [for absolute value numbers].

4) In the case k=4:

$$S(n, {}_4) = \sum_{\substack{0<|n-4i|\le n \\ i=0,1,2,\dots}} (n-4i) = n+(n-4)+(n-8)\dots \text{ ; [for signed numbers].}$$

$$S|n, {}_4| = \sum_{\substack{0<|n-4i|\le n \\ i=0,1,2,\dots}} |n-4i| = n+|n-4|+|n-8|\dots \text{ ; [for absolute value numbers].}$$

Thus $S(9, {}_4) = 9+(9-4)+(9-8)+(9-12)+(9-16) = 9+(5)+(1)+(-3)+(-7) = 5$; for signed numbers.
Thus $S|9, {}_4| = 9+|9-4|+|9-8|+|9-12|+|9-16| = 9+5+1+3+7 = 25$; [for absolute value numbers].

The sequence is $S(n, _4) = 3, 0, 4, 0, 5, 0, 6, 0, 7, 0, 8, 0, 9, 0, 10, 0, 11, \dots .$
The sequence is $S|n, _4| = 9, 16, 16, 24, 25, 36, 36, 48, 49, 64, 64, 80, 81, 100, 100, \dots .$

5) In the case k=5:

$$S(n, _5) = \sum_{\substack{0<|n-5i|\le n \\ i=0,1,2,\dots .}} (n-5i) = n+(n-5)+(n-10)\dots .$$

$$S|n, _5| = \sum_{\substack{0<|n-5i|\le n \\ i=0,1,2,\dots .}} |n-5i| = n+|n-5|+|n-10|\dots .$$

Thus $S(11, _5) = 11+(11-5)+(11-10)+(11-15)+(11-20) = 11+6+1+(-4)+(-9) = 5.$
Thus $S|11, _5| = 11+|11-5|+|11-10|+|11-15|+|11-20| = 11+6+1+4+9 = 31.$

The sequence is $S(n, _5)$: 3, 6, 2, 6, 0, 5, 10, 3, 9, 0, 7, 14, 4, 12, 0, … .
The sequence is $S|n, _5|$: 11, 12, 20, 20, 30, 31, 32, 33, 45, 60, 61, 62, 80, 80, 100, … .

More general:
Let $n>k\ge 1$ be two integers and $m\ge 0$ another integer.
Then the Generalized Back and Forth Summand is defined as:

$$S(n, m, _k) = \sum_{i=0,1,2,\dots,\text{floor}[(n+m)/k]} (n-k\cdot i) \quad \text{[for signed numbers].}$$

$$S|n, m, _k| = \sum_{i=0,1,2,\dots,\text{floor}[(n+m)/k]} |n-k\cdot i| \quad \text{[for absolute value numbers].}$$

For examples:
$S(7, 9, _2) = 7+(7-2)+(7-4)+(7-6)+(7-8)+(7-10)+(7-12)+(7-14)+(7-16)$
$= 7+(5)+(3)+(1)+(-1)+(-3)+(-5)+(-7)+(-9) = -2.$
$S|7, 3, _2| = 7+|7-2|+|7-4|+|7-6|+|7-8|+|7-10| = 7+5+3+1+1+3 = 20.$

References:
J. Dezert, editor, "Smarandacheials", Mathematics Magazine, Aurora, Canada, No. 4/2004;
www.gallup.unm.edu/~smarandache/Smarandacheials.htm.
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M. Bencze, Some Properties of the Smarandache Summands, mss.

A Numerical Experiment on Fermat's Theorem
(not intended as formal proof or disproof)

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Fermat's "Last Theorem" asserts that if n > 2, the equation
x^n + y^n = z^n cannot be solved in integers x, y, z, with xyz <> 0:
http://www.fortunecity.com/emachines/e11/86/mathex5.html.

Theorem:
For any triplets of numbers (a,b,c) obeying Pythagorean theorem
we have a^2+b^2=c^2.

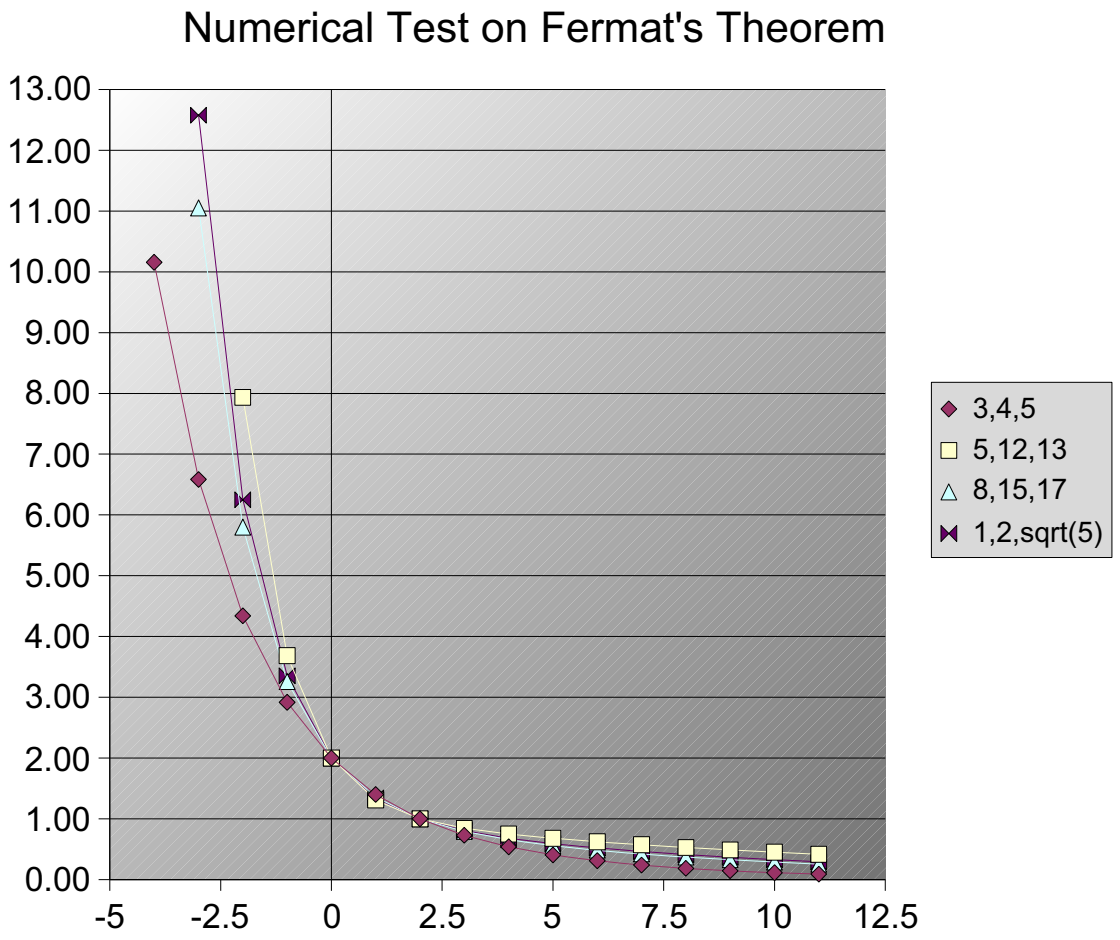
It perhaps could be shown (numerically) that :
a^n+b^n=c^n,
or:
(a^n+b^n)/c^n=k =1 (Fermat's Surface)
holds true if and only if n=2. (Generalized Fermat's Last Theorem)

First try: 3, 4, 5 (3^2 + 4^2 = 5^2)
Second try: 5,12,13 (5^2+12^2=13^2)
Third try: 6, 8,10 (6^2 + 8^2 = 10^2)
Fourth try: 1, 2, sqrt (1^2 + 2^2 = 2.236^2)

N	K First try	K Second try	K Third try	K Fourth try
	3,4,5	5,12,13	8,15,17	1,2,sqrt(5)
-5				
-4	10.16			
-3	6.58		11.05	12.58
-2	4.34	7.93	5.80	6.25
-1	2.92	3.68	3.26	3.35
0	**2.00**	**2.00**	2.00	**2.00**
1	1.40	1.31	1.35	1.34
2	**1.00**	**1.00**	**1.00**	**1.00**
3	0.73	0.84	0.79	0.81
4	0.54	0.75	0.66	0.68
5	0.41	0.68	0.56	0.59
6	0.31	0.62	0.48	0.52
7	0.24	0.57	0.42	0.46
8	0.18	0.53	0.37	0.41
9	0.14	0.49	0.33	0.37
10	0.11	0.45	0.29	0.33
11	0.09	0.41	0.25	0.29

Conclusions:

(i) It is clear from the diagram that for the triplets (3,4,5) and (5,12,13) k=1 only at n=2.
(ii) For other triplets of numbers it perhaps does not obey the same formula.
(iii) But generally speaking, from the Chart given below it appears that:
=> **For n < 2 --> k tends > 1;**
=> **For n > 2 --> k tends < 1.**
(iv) For triplets of numbers (a,b,c), which do not follow the Pythagorean Triangle (> 180 degrees or < 180 degrees), i.e. when the triangle is on curved-surface, then Fermat theorem could be broken.
(v) We can make an 'associated condition': for the same triplets of (a,b,c) following Pythagorean theorem a^2+b^2=c^2, it follows that for n=0 then (a^n+b^n)/c^n=k will yield k=2 (of course).



References (for similar simplified proof of Fermat's Theorem):

[1] http://www.fortunecity.com/emachines/e11/86/mathex5.html
[2] http://www.economics.ox.ac.uk/Members/giuseppe.mazzarino/Fermat_March_2003.pdf
[3] http://www.skidmore.edu/academics/theater/productions/arcadia/math.html
[4] http://www.fermatproof.com/
[5] http://www.itsoc.org/review/05pl1.pdf
[6] http://yacas.sourceforge.net/Algochapter3.html
[7] http://www.blackdouglas.com.au/webpapr/workmath/workmath.htm

(Jan. 18, 2006)

About Factorial Sums

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Abstract. In this paper, we present some new inequalities for factorial sum.

Application 1.We have the following inequality

$$\sum_{k=1}^{n} k! \leq \frac{2((n+1)!-1)}{n+1}$$

Proof. If $x_k,\ y_k > 0\ \ (k = 1,2,...,n)$, have the same monotonity, then

$$\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) \leq \frac{1}{n}\sum_{k=1}^{n} x_k y_k \qquad (1)$$

the Chebishev's inequality.

If $x_k,\ y_k$ have different monotonity, then holds true the reverse inequality, we take $x_k = k,\ y_k = k!\ \ (k = 1,2,...,n)$ and use that $\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$.

Application 2. We have the following inequality

$$\sum_{k=1}^{n} k! \leq \frac{3(n+1)(n+1)!}{n^2+3n+5}$$

Proof. In (1) we take

$$x_k = k^2 + k + 1;$$
$$y_k = k!\ \ (k = 1,2,...,n)$$

and the identity

$$\sum_{k=1}^{n} (k^2+k+1)k! = (n+1)(n+1)!$$

Application 3. We have the following inequality

$$\sum_{k=1}^{n} \frac{1}{k!} \geq \frac{n^2(n+1)}{2((n+1)!-1)}$$

Proof. Using the Application 1, we take

$$\sum_{k=1}^{n}\frac{1}{k!}\geq\frac{n^2}{\sum_{k=1}^{n}k!}\geq\frac{n^2(n+1)}{2((n+1)!-1)}$$

Application 4. We have the following inequality

$$\sum_{k=1}^{n}\frac{1}{k!}\geq\frac{n^2(n^2+3n+5)}{3(n+1)(n+1)!}$$

Proof. Using the Application 2, we take

$$\sum_{k=1}^{n}\frac{1}{k!}\geq\frac{n^2}{\sum_{k=1}^{n}k!}\geq\frac{n^2(n^2+3n+5)}{3(n+1)(n+1)!}$$

Application 5. We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{k!}\geq 1+\frac{2}{n}\left(1-\frac{1}{n!}\right)$$

Proof. In (1) we take $x_k=k,\quad y_k=\frac{1}{(k+1)!},\ (k=1,2,...,n)$ and we obtain

$$\frac{1}{n}\left(\sum_{k=1}^{n}k\right)\left(\sum_{k=1}^{n}\frac{1}{(k+1)!}\right)\geq\sum_{k=1}^{n}\frac{k}{(k+1)!}=1-\frac{1}{(n+1)!}$$

therefore

$$\left(\sum_{k=1}^{n}\frac{1}{(k+1)!}\right)\geq\frac{2}{n+1}\left(1-\frac{1}{(n+1)!}\right)$$

or

$$\sum_{k=2}^{n}\frac{1}{k!}\geq\frac{2}{n}\left(1-\frac{1}{n!}\right)$$

therefore

$$\left(\sum_{k=1}^{n}\frac{1}{k!}\right)\geq 1+\frac{2}{n}\left(1-\frac{1}{n!}\right)$$

Application 6. We have the following inequality:

$$\left(\sum_{k=1}^{n}\frac{1}{(k+2)^2k!}\right)\geq\frac{2}{n+5}\left(1-\frac{1}{(n+2)!}\right)$$

Proof. In (1) we take $x_k=k+2,\ y_k=\frac{1}{(k+2)^2k!},\ (k=1,2,...,n)$

therefore

$$\frac{1}{n}\left(\sum_{k=1}^{n}(k+2)\right)\sum_{k=1}^{n}\frac{1}{(k+2)^2k!}\geq\sum_{k=1}^{n}\frac{1}{(k+2)^2k!}=1-\frac{1}{(n+2)!}$$

therefore

$$\sum_{k=1}^{n}\frac{1}{(k+2)^2 k!}\geq\frac{2}{n+5}\left(1-\frac{1}{(n+2)!}\right)$$

Application 7.We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)!}\geq\frac{6}{2n^2+9n+1}\left(\frac{1}{2}-\frac{1}{(n+1)(n+2)!}\right)$$

Proof. In (1) we take

$$x_k=k^2+2k+2,\ y_k=\frac{1}{k(k+1)(k+2)!},\ (k=1,2,...,n)$$

then

$$\frac{1}{n}\sum_{k=1}^{n}(k^2+2k+2)\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)!}\geq\sum_{k=1}^{n}\frac{k^2+2k+2}{k(k+1)(k+2)!}=$$

$$=\sum_{k=1}^{n}\frac{1}{k(k+1)!}-\frac{1}{(k+1)(k+2)!}=\frac{1}{2}-\frac{1}{(n+1)(n+2)!}$$

Application 8. We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{4k^4+1}\geq\frac{n}{2n^2+2n+1}$$

Proof. In (1) we take $x_k=4k,\ y_k=\frac{1}{4k^4+1},\ (k=1,2,...,n)$,

therefore

$$\frac{1}{n}\left(\sum_{k=1}^{n}4k\right)\left(\sum_{k=1}^{n}\frac{1}{4k^4+1}\right)\geq\sum_{k=1}^{n}\frac{4k}{4k^4+1}=\sum_{k=1}^{n}\left(\frac{1}{2k^2-2k+1}-\frac{1}{2k^2+2k+1}\right)=\frac{2n(n+1)}{2n^2+2n+1}$$

Application 9. We have the following inequality:

$$\sum_{k=1}^{n}\frac{1}{4k^4-1}\geq\frac{3n}{(2n+1)^2}$$

Proof. In (1) we take $x_k=k^2,\ y_k=\frac{1}{4k^2-1},\ (k=1,2,...,n)$ then

$$\frac{1}{n}\left(\sum_{k=1}^{n}k^2\right)\left(\sum_{k=1}^{n}\frac{1}{4k^2-1}\right)\geq\sum_{k=1}^{n}\frac{k^2}{4k^2-1}=\frac{n(n+1)}{2(2n+1)},\ \text{etc.}$$

Reference:

[1] Octogon Mathematical Magazine (1993-2007)

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Inequalities for Integer and Fractional Parts

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Abstract: In this paper we present some new inequalities relative to integer and functional parts.

Theorem 1. If $x>0$, then $\frac{[x]}{3x+\{x\}}+\frac{\{x\}}{3x+[x]}\geq\frac{4}{15}$, where $[\cdot]$ and $\{\cdot\}$ denote the integer part, and respectively the fractional part.

Proof. In inequality $\frac{a}{a+2b+2c}+\frac{b}{2a+b+2c}+\frac{c}{2a+2b+c}\geq\frac{3}{5}$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 2. If $a,\ b,\ c,\ x>0$, then

$$\frac{a}{[x]b+\{x\}c}+\frac{b}{[x]c+\{x\}a}+\frac{c}{[x]a+\{x\}b}\geq\frac{3}{x}.$$

Proof. In inequality $\frac{a}{ub+vc}+\frac{b}{uc+va}+\frac{c}{ua+vb}\geq\frac{3}{u+v}$, we take $u=[x]$ and $v=\{x\}$.

Theorem 3. If $x>0$ and $a\geq 1$, then

$$\frac{[x]}{(a+1)[x]+2\{x\}}+\frac{[x]}{(a+1)\{x\}+2[x]}\leq\frac{2a+1}{(a+1)(a+2)}.$$

Proof. In inequality $\frac{x}{ax+y+z}+\frac{y}{x+ay+z}+\frac{z}{x+y+az}\leq\frac{3}{a+2}$, we take $y=[x]$ and $z=\{x\}$.

Theorem 4. If $x>0$, then

$$[x]\left(\frac{1}{x[x]+x+1}+\frac{1}{[x]\{x\}+[x]+1}\right)+\{x\}\left(\frac{1}{x[x]+x+1}+\frac{1}{x\{x\}+\{x\}+1}\right)\le 1.$$

Proof. In inequality $\frac{x}{xy+x+1}+\frac{y}{yz+y+1}+\frac{z}{zx+z+1}\le 1$, we take $y=[x]$ and $z=\{x\}$.

Theorem 5. If $x>0$, then

$$\frac{x^3}{[x]\left(3[x]^2+3[x]\{x\}+\{x\}^2\right)}+\frac{x[x]^2}{\{x\}\left([x]^2+[x]\{x\}+\{x\}^2\right)}+\frac{x\{x\}^2}{[x]^2+3[x]\{x\}+3\{x\}^2}\ge\frac{3}{2}$$

Proof. In inequality $\sum\frac{x^2}{y\left(x^2+xy+y^2\right)}\ge\frac{3}{x+y+z}$, we take $y=[x]$ and $z=\{x\}$.

Theorem 6. If $x>0$, $\frac{1}{[x]+2\{x\}}+\frac{1}{2[x]+\{x\}}\ge\frac{1}{x}$.

Proof. In inequality $\sum\frac{a^2+bc}{b+c}\ge a+b+c$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 7. If $x>0$,

$$\frac{[x]^3}{[x]^2+[x]\{x\}+\{x\}^2}+\frac{\{x\}^3}{3\{x\}^2+3[x]\{x\}+[x]^2}\ge\frac{x\left(3[x]^2-\{x\}^2\right)}{3\left(3[x]^2+3[x]\{x\}+\{x\}^2\right)}$$

Proof. In inequality $\sum\frac{a^3}{a^2+ab+b^2}\ge\frac{a+b+c}{3}$, we take $a=x,\ b=[x]$, $c=\{x\}$.

Theorem 8. If $x>0$, then

$$\frac{1}{2[x]^3+4[x]^2\{x\}+4[x]\{x\}^2+\{x\}^3}+\frac{1}{[x]^3+[x]^2\{x\}+[x]\{x\}^2+\{x\}^3}+$$
$$+\frac{1}{[x]^3+4[x]^2\{x\}+4[x]\{x\}^2+2\{x\}^3}\le\frac{1}{x[x]\{x\}}.$$

Proof. In inequality $\sum\frac{1}{a^3+b^3+abc}\le\frac{1}{abc}$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 9. If $x>1$, then $4\left(\frac{[x]^3}{\{x\}}+\frac{\{x\}^3}{[x]}\right)\ge[x]^2+[x]\{x\}+\{x\}^2$.

Proof. In inequality $\sum\frac{1}{a}\left(-a+b+c\right)^3\ge a^2+b^2+c^2$, we take $a=x,\ b=[x]$, $c=\{x\}$.

Theorem 10. If $x>0$, then

$$\frac{x^4}{[x]^2-[x]\{x\}+\{x\}^2}+\frac{x\left([x]^3+\{x\}^3\right)}{[x]^2+[x]\{x\}+\{x\}^2}\geq\frac{3}{2}\left(x^2+[x]\{x\}\right)$$

Proof. In inequality $\sum\frac{a^3}{b^2-bc+c^2}\geq\frac{3\sum ab}{\sum a}$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 11. If $x>0$, then $\left|\frac{[x]\{x\}([x]-\{x\})}{x(x+[x])(x+\{x\})}\right|<1$.

Proof. In inequality $\left|\sum\frac{a-b}{a+b}\right|<1$ we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 12. If $x>0$, then $\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}}>1$.

Proof. In inequality $\sum\sqrt{\frac{x}{y+z}}>2$, we take $y=[x],\ z=\{x\}$.

Theorem 13. If $x>1$, then $3+\frac{\{x\}}{[x]}+\frac{[x]}{\{x\}}\geq 3\sqrt[3]{\frac{(x+[x])(x+\{x\})}{[x]\{x\}}}$.

Proof. In inequality $\left(\sum a\right)\left(\sum\frac{1}{a}\right)\geq 3\left(1+\sqrt[3]{\frac{\prod(a+b)}{abc}}\right)$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 14. If $x>0$, then $\left(\sqrt{[x]}+\sqrt{\frac{[x]\{x\}}{x}}+\sqrt{\{x\}}\right)^4\geq 32[x]\{x\}$.

Proof. In inequality $\sum\sqrt{xy}\geq 2\sqrt[4]{xyz\sum x}$, we take $y=[x],\ z=\{x\}$.

Theorem 15. If $x>0$, then $\left(x^2+[x]\{x\}\right)^2\geq 6x^2[x]\{x\}$.

Proof. In inequality $\left(\sum xy\right)^2\geq 3xyz\sum x$, we take $y=[x],\ z=\{x\}$.

Theorem 16. If $x>0$, then

$$x^2-x\sqrt{[x]\{x\}}+[x]\{x\}\geq\left([x]\sqrt{\{x\}}+\{x\}\sqrt{[x]}\right)\sqrt{x}$$

Proof. In inequality $\sum xy\geq\sum x\sqrt{yz}$, we take $y=[x],\ z=\{x\}$.

Theorem 17. If $x>0$, then $\sqrt{[x](x+\{x\})}+\sqrt{\{x\}(x+[x])}\leq\left(2\sqrt{2}-1\right)x$.

Proof. In inequality $\sum\sqrt{x(y+z)}\le\sqrt{2}\sum x$, we take $y=[x],\ z=\{x\}$.

Theorem 18. If $x>0$, then $\dfrac{[x]}{x+\{x\}}+\dfrac{\{x\}}{x+[x]}\ge\dfrac{1}{2}$.

Proof. In inequality $\sum\dfrac{a}{b+c}\ge\dfrac{3}{2}$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 19. If $x>0$, then $(x+[x])^3+(x+\{x\})^3\ge 21x[x]\{x\}+[x]^3+\{x\}^3$.

Proof. In inequality $\sum(x+y)^3\ge 21xyz+\sum x^3$, we take $y=[x],\ z=\{x\}$.

Theorem 20. If $x>1$, then $\sqrt{\dfrac{x+[x]}{x+\{x\}}}+\sqrt{\dfrac{x+\{x\}}{x+[x]}}\le\sqrt{\dfrac{[x]}{\{x\}}}+\sqrt{\dfrac{\{x\}}{[x]}}$.

Proof. In inequality $\sqrt{\dfrac{x+y}{x+z}}+\sqrt{\dfrac{x+z}{x+y}}\le\dfrac{y+z}{\sqrt{yz}}$, we take $y=[x],\ z=\{x\}$.

Theorem 21. If $x>0$, then $\dfrac{x}{x+[x]}+\dfrac{x}{x+\{x\}}\ge\dfrac{5}{2}$.

Proof. In inequality $2\sum\dfrac{1}{x+y}\ge\dfrac{9}{\sum x}$, we take $y=[x],\ z=\{x\}$.

Theorem 22. If $x>1$, then $\dfrac{\{x\}}{[x]}+\left(\dfrac{\{x\}}{[x]}\right)^2+\left(\dfrac{[x]}{\{x\}}\right)^2+\dfrac{\{x\}^2}{x^2}\ge\left(\dfrac{1}{x}+\dfrac{1}{\{x\}}\right)[x]$.

Proof. In inequality $\sum\dfrac{x^2}{y^2}\ge\sum\dfrac{x}{z}$, we take $y=[x],\ z=\{x\}$.

Theorem 23. If $x>0$, then

$$[x]^2-[x]\{x\}+\{x\}^2\ge\frac{3}{4}\max\left\{[x]^2;([x]-\{x\})^2;\{x\}^2\right\}.$$

Proof. In inequality $\sum x^2-\sum xy\ge\dfrac{3}{4}\max\left\{(x-y)^2;(y-z)^2;(z-x)^2\right\}$, we take $y=[x],\ z=\{x\}$.

Theorem 24. If $x>0$, then $e^{\{x\}}+e^{[x]}\ge 2+x$.

Proof. In inequality $e^y+e^z\ge 2+y+z$, we take $y=[x],\ z=\{x\}$.

Theorem 25. If $x\in\mathrm{R}$, then $|\sin[x]|+|\sin\{x\}|+|\cos x|\ge 1$.

Proof. In inequality $|\sin a|+|\sin b|+|\cos(a+b)|\ge 1$ we take $a=x,\ b=\{x\}$.

Theorem 26. If $x>0$, then $\left(3[x]^2+3[x]\{x\}+\{x\}^2\right)\cdot\left([x]^2+[x]\{x\}+\{x\}^2\right)\cdot$

$$\cdot\left(3\{x\}^2+3\{x\}[x]+[x]^2\right)\geq\left(x^2+[x]\{x\}\right)^3.$$

Proof. In inequality

$\left(a^2+ab+b^2\right)\cdot\left(b^2+bc+c^2\right)\cdot\left(c^2+ca+a^2\right)\geq\left(ab+bc+ca\right)^3$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 27. If $x>0$, then

$$\frac{[x]}{\left(3[x]+2\{x\}\right)\left([x]+2\{x\}\right)}+\frac{\{x\}}{\left(3\{x\}+2[x]\right)\left(\{x\}+2[x]\right)}\geq\frac{11}{48x}.$$

Proof. In inequality $\left(\sum x\right)\sum\frac{x}{\left(2x+y+z\right)\left(y+z\right)}\geq\frac{9}{8}$, we take $y=[x],\ z=\{x\}$.

Theorem 28. If $x>0$, then $\frac{[x]^2}{x+[x]}+\frac{\{x\}^2}{x+\{x\}}\geq\frac{x\left(2x^2+3[x]\{x\}\right)}{\left(x+[x]\right)\left(x+\{x\}\right)}$.

Proof. In inequality $\sum\frac{x^2}{\left(x+y\right)\left(x+z\right)}\geq\frac{3}{4}$, we take $y=[x],\ z=\{x\}$.

Theorem 29. If $x>1$, then

$$\sqrt{1+\frac{2\{x\}}{[x]}}+\sqrt{1+\frac{2[x]}{\{x\}}}\geq1+2\left(\sqrt{\frac{[x]}{x+\{x\}}}+\sqrt{\frac{\{x\}}{x+[x]}}\right).$$

Proof. In inequality $\sum\sqrt{\frac{y+z}{x}}\leq\sqrt{2}\sum\sqrt{\frac{x}{y+z}}$ we take $y=[x],\ z=\{x\}$.

Theorem 30. If $x>1$, then $\frac{\{x\}}{[x]}+\frac{[x]}{\{x\}}\geq1+2\left(\frac{[x]}{x+\{x\}}+\frac{\{x\}}{x+[x]}\right)$.

Proof. In inequality $\sum\frac{y+z}{x}\geq4\sum\frac{x}{y+z}$, we take $y=[x],\ z=\{x\}$.

Theorem 31. If $x>0$, then

1). $\min\left\{\left(\sqrt{2}+1\right)\sqrt{x}+\sqrt{\{x\}};\left(\sqrt{2}+1\right)\sqrt{x}+\sqrt{x+[x]}\right\}\geq\sqrt{5\left([x]+2\{x\}\right)}$

2). $\left(\sqrt{2}+1\right)\sqrt{x}+\sqrt{x+\{x\}}\geq\sqrt{5\left(\{x\}+2[x]\right)}$.

Proof. In $\sqrt{a+b+c}+\sqrt{b+c}+\sqrt{c}\geq\sqrt{a+4b+9c}$, we take $a=x,\ b=[x],\ c=\{x\}$, etc.

Theorem 32. If $x \in \mathrm{R}$, then

1). $|\sin x| \le |\sin[x]| + |\sin\{x\}|$

2). $|\cos x| \le |\cos[x]| + |\cos\{x\}|$

Proof. In inequalities $|\sin(a+b)| \le |\sin a| + |\sin b|$ and $|\cos(a+b)| \le |\cos a| + |\cos b|$, we take $a = x,\ b = [x]$.

Theorem 33. If $x > 1$, then $6 + \dfrac{\{x\}}{[x]} + \dfrac{[x]}{\{x\}} \ge \left(\sqrt[3]{\dfrac{x}{[x]}} + \sqrt[3]{\dfrac{[x]}{\{x\}}} + \sqrt[3]{\dfrac{\{x\}}{x}} \right)^3$.

Proof. In inequality $3\left(\sum a\right)\left(\sum \dfrac{1}{a}\right) \ge \left(\sum \sqrt[3]{\dfrac{a}{b}}\right)$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 34. If $x > 0$, then $\dfrac{[x]}{(x+\{x\})^2} + \dfrac{\{x\}}{(x+[x])^2} \ge \dfrac{1}{8x}$.

Proof. In inequality $\sum \dfrac{a}{(b+c)^2} \ge \dfrac{9}{4\sum a}$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 35. If $x > 0$, then

$$\frac{x[x]}{(x+\{x\})(2x+[x])} + \frac{\{x\}(x+\{x\})}{(x+[x])(2x+\{x\})} \ge \frac{[x]+5\{x\}}{12x}.$$

Proof. In inequality $\sum \dfrac{a(a+b)}{(b+c)(2a+b+c)} \ge \dfrac{3}{4}$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 36. If $x > 0$, then $\dfrac{[x]}{2x+\{x\}} + \dfrac{\{x\}}{2x+[x]} \le \dfrac{3[x]^2 + 4[x]\{x\} + 3\{x\}^2}{6x}$.

Proof. In inequality $\sum \dfrac{ab}{a+b+2c} \le \dfrac{1}{4}\sum a$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 37. If $x > 0$, then $\left([x]^5 - [x]^2 + 3\right)\left(\{x\}^5 - \{x\}^2 + 3\right) \ge \dfrac{8x^3}{x^5 - x^2 + 3}$.

Proof. In inequality $\prod\left(a^5 - a^2 + 3\right) \ge \left(\sum a\right)^3$, we take $a = x,\ b = [x],\ c = \{x\}$.

Theorem 38. If $x > 0$, then $\dfrac{(2x+[x])^2}{2[x]^2 + (x+\{x\})^2} + \dfrac{(2x+\{x\})^2}{2\{x\}^2 + (x+[x])^2} \le 5$.

Proof. In inequality $\sum \frac{(2a+b+c)^2}{2a^2+(b+c)^2} \le 8$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 39. If $x>0$, then

1). $\left(x+\sqrt{x[x]}+\sqrt[3]{x[x]\{x\}}\right)^3 \le 9x^2(x+[x])$

2). $\left([x]+\sqrt{[x]\{x\}}+\sqrt[3]{x[x]\{x\}}\right)^3 \le 9x^2[x]$

3). $\left(\{x\}+\sqrt{x\{x\}}+\sqrt[3]{x[x]\{x\}}\right)^3 \le 9x^2(x+\{x\})$.

Proof. In inequality $\frac{a+\sqrt{ab}+\sqrt[3]{abx}}{2} \le \sqrt[3]{a\left(\frac{a+b}{2}\right)\left(\frac{a+b+c}{b}\right)}$, we take $a=x,\ b=[x],\ c=\{x\}$, etc.

Theorem 40. If $x>0$, then $7(x+[x])^4+7(x+\{x\})^4 \ge 3x^4+4([x]^4+\{x\}^4)$.

Proof. In inequality $\sum(a+b)^4 \ge \frac{4}{7}\sum a^4$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 41. If $x>0$, then $\frac{\{x\}^2}{(x+\{x\})^2+[x]^2}+\frac{[x]^2}{(x+[x])^2+\{x\}^2} \ge \frac{3}{20}$.

Proof. In inequality $\sum \frac{(b+c-a)^2}{(b+c)^2+a^2} \ge \frac{3}{5}$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 42. If $x>0$, then $\sqrt{\frac{2x}{x+[x]}}+\sqrt{\frac{2[x]}{x}}+\sqrt{\frac{2\{x\}}{x+\{x\}}} \le 3$.

Proof. In inequality $\sum \frac{2a}{a+b} \le 3$, we take $a=x,\ b=[x],\ c=\{x\}$.

Theorem 43. If $x>0$, then $\frac{1}{(x+[x])^2}+\frac{1}{(x+\{x\})^2+\{x\}^2} \ge \frac{5x^2-4[x]\{x\}}{4x^2(x^2+[x]\{x\})}$.

Proof. In inequality $\left(\sum xy\right)\left(\sum \frac{1}{(x+y)^2}\right) \ge \frac{9}{4}$, we take $y=[x],\ z=\{x\}$.

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Souvenirs from the Empire of Numbers

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1. Forward.

Browsing through my fifth to twelfth grade years of preoccupation for creation I discovered a notebook of Number Theory.
I liked to play with numbers as Tudor Arghezi (1880-1967) – our second national Romanian poet {after the genial poet Mihai Eminescu (1850-1889)} – played with words.
I was so curious and amazed by the numbers' properties.
Interesting theorems, equations, and inequalities!
Such fascinating people who dedicated their research to numbers, just for the sake of science!
I collected many results and tried to write a handbook of mathematicians and their results.

As a child I stayed in bed, leaning my back against the wall, with some papers and a pen in my hands, thinking and scribbling with numbers!

As skilled for arithmetic I was remarked from the beginning by my first grade teacher Elena Bălaşa and especially my second to fourth grade teacher Elena Mişcoci, both at Primary School in Bălceşti (district of Vâlcea), who organized in class calculation competitions among students: "who computes the fastest this multiplication" [at that time there were no pocket calculators, we had to do everything by hand.]
Our elementary math teacher from fifth to eighth grade, Ion Bălaşa, an excellent and very passionate educator, asked us the students to subscribe to "Gazeta Matematică" [Mathematical Gazette] and submit solutions to its proposed problem of algebra, geometry, and trigonometry for our knowledgeable level.
We had a very serious, strong, rigid, and complex scientific education at that time.
In High Schools, at Craiova for the first three years, with the instructor Larisa Bistriceanu, and afterwards at Rm. Vâlcea for the next two years with instructor Nicolae Vlădescu, I participated every school year in students' Mathematical Olympiads winning various awards.

This Number Theory notebook is a compilation of known results about numbers, and it also includes a short list of some renowned mathematicians. Unfortunately, only a part of it was recovered when I came back to Romania from my volunteer exile in Turkey and USA. Most of this notebook was damaged by dust, mould, humidity, cobwebs, and mouse from my parents' house garret in Bălceşti, or simply lost. Other manuscripts, not only of science, but also of poetry, novels, diaries were confiscated by the secret police (Securitate) and never returned, although they are mentioned in the 4 folders' about 880 pages police secret report about me that I got copies from the CNSAS (Consiliul Naţional

de Studiere a Arhivelor Securităţii = National Council for Studying the Archives of the Secret Police).

2. <u>Short List of Mathematicians</u>.

***Waclaw Sierpinski** (1882-1970) – Polish - Set Theory (on Transfinite Numbers), Analytic Number Theory.
***Popoviciu Tiberiu** - Number Theory.
***Henri Poincarè** (1854-1912) – Integer Function Theory – Poincarè inequality.
***Giuseppe Peano** (1858-1932) – The founder of the Axiomatic Arithmetic (Peano Axioms).
***Blaise Pascal** (1623-1662) The Pascal's Arithmetic Triangle.
***Alexandru Myller** (1879-1965) – Romanian – Mathematics History.
***Andrei Andreevici Markov** (1856-1892) - Number Theory.
***Adrian Marie Legendre** (1752-1833) – Number Theory. He was the first who formulated the problem of the asymptotic distribution of the prime numbers.
***Gottfried Wilhelm Leibnitz** (1646-1716) – Binary Arithmetic.
***Joseph Louis Lagrange** (1736-1813) – Number Theory.
***Ion Ionescu** (1870-1946) – Mathematical Gazette B – Arithmetic problems, Mathematics History.
***Muhammed ibn Musa Horezmi** (c. 780 – c. 850) – Arab – Book about addition and subtraction – Indian system of numeration.
***David Hilbert** (1862-1943) – German – Algebraic Number Theory. In 1900 he proposed 23 problems at the International Congress in Paris.
***Jacques Hadamard** (1865-1963)- French – Number Theory.
***Sophie Germain** (1776-1831) – French - Number Theory.
***Friederich Karl Gauss** (1777-1831) – "*Disquitiones arithmeticae*"; The Quadratic Reciprocity Law, The Method of Least Squares.
***Pierre de Fermat** (1601-1665) – French – "*Varia opera Mathematica*", Number Theory. In 1637 – The great Fermat Theorem.
***Leonhard Euler** (1707-1783) – Swiss – Number Theory. He introduced:
- The notion of general and particular solution in Differential Equation Theory,
- The congruency notion and its notation "≡", in 1801.

***Pafnuti Lvovici Chebyshev** (1821-1894) – Russian – Number Theory. He gave the formula for numerical approximation of the prime numbers less or equal to a given number.
***Georg Cantor** (1845-1918) German – Number Theory:
- The analytic numeric theory,
- The geometric numeric theory.

***Herman Minkowschi** (1866-1909) – The geometry of numbers theory.
***Johann Peter Gustav Lejeune Dirichlet** (1805-1859) – The Analytical Number Theory.
***Dan Barbilian (Ion Barbu)** (1895-1961) – Number Theory.
***Gabriel Sudan** (1899-1977) – Computational Theory.
Grigore Moisil: when he was 28 he became doctor docent in mathematics having had published already 274 paper works.

3. **Mathematical Results.**

Observation: 0 is divisible by 0!

Property:

$\varphi(0) = 2$

$\varphi(\pm 1) = 1$,

where φ is Euler's function.

Observation: There exists $\sqrt[1]{s}$, and $\sqrt[1]{s} = s$.

Property: $(n+1)(n+2)\cdots(n+n) = 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

Definition: $\sum a_1 ... a_m$ is the sum of all possible circular permutations of m numbers from N.

Giuseppe PEANO (1858-1932).

The Peano axioms: to create an axiomatic arithmetic (the natural number axiomatization).

1. There is the number 1 which does not follow after any other number.
2. Any natural number n has a successor n' and only one, therefore
   From $a = b$ results $a' = b'$.
3. From $a' = b'$ results $a = b$ (i.e. a natural number cannot be the successor of multiple numbers.
4. The induction axiom: If a sentence P is referring to any natural number n and if
   - 1 is proved for n=1,
   - 2 from the hypothesis of its validity for $n = m+1$, it results that P is true for any n.

(Some mathematicians and philosophers contest its validity, because he numbers the axioms 1, 2, 3, 4.)

Two journals: MATHESIS - 1898, and "Mathematical Review" – 1958.

Property (Brocard): The numbers whose square end in two equal digits are those numbers that end in 0, 12, 62, 38, 88.

Property (I. Ionescu): The numbers that multiplied by 9 that give as product the same numbers flipped are: $N = n_1 n_2 n_3 ... n_3 n_2 n_1$, where n_i are numbers (solutions) of the property's statement (explanation $n_i = 0...0$, or 1809,…).

Property: $(a+b)(b+c)(c+a) \geq 8abc$, $a, b, c \geq 0$.

(SF- generalization)

Property: $a^2 + b^2 + c^2 \geq ab + bc + ca$; it can be generalized.

Property (E. Cesaro): $a^p + (a+1)^p + ... + (a+9)^p$ ends as follows:

in 5, if $p \neq \mathcal{M}_4$,

in 3, if $p = \mathcal{M}_4$.

Property: If $a+b=$ constant , then $a\cdot b$ is maxim, when $a=b$ or $a=b-1$ (it depends if $a+b$ is divisible by 2).

Property: If $a_1+...+a_n=k>n$, where k constant, then $a_1...a_n$ is maximum for any a_ia_j or $a_i=a_j$ or $a_i=a_j-1$, ($a_j=a_i-1$).

Property: The sum of the squares of n natural numbers of a given sum is a minimum when any of the following numbers are equal or they differ by a unit:

Notations:

$N=\{0,1,2,...\}$, $N^*=\{1,2,...\}$.

Integer Numbers:

- Rational: $a,\ a\in Z$;
- Complex: $a+ib,\ a,b\in Z$.

Property: The number of solutions in N of equation $x_1+x_2+...+x_{p+1}=n$ is

$$\frac{(n+1)(n+2)...(n+p)}{1\cdot 2\cdot ....\cdot p}$$

Definition: Magic squares are the squares filled with natural numbers with the property that the sum of the numbers on each line, each column, and each diagonal is the same.
Albrecht Dürer (1514) – painter and mathematician, introduced the magic square notion.
Bachet de Meziriac (1612) – wrote the "Mathematics for fun".

Property: There exist an infinity of prime numbers of the form $4k-1;\ \ 6k-1$.

Property: If three prime numbers are in arithmetic progression then the ratio is $\mathcal{M}_6$ (except for 3, 5, 7).

Observation: If $p,8p-1=prime$, then $8p+1=$ is a composite number.

Property: If $(a,b)=1$, then $(a+b,a-b)=1\ or\ 2$.

Property: If $(a,b)=1$, then $(11a+2b,18a+5b)=1\ or\ 19$.

Property: If $2^n+1=prime\ number$, then $n=2^\alpha$.

Property: If $A^m+B^n=prime\ number$, then $(m,n)=2^\alpha$.

Property: If $n=impar$, then a^n+1 are not prime numbers.

Property: There are n consecutive numbers non prime.

Proof: $\left((n+1)!+2,...,(n+1!+(n+1)\right)$

Definition: The Fermat's numbers $2^{2^n}+1$ prime.

Property (Gauss): A regular polygon with p sides can be designed with only the ruler and the compass only when $p=2^{2^n}+1$ and $p=2^{2^n}+1$ is a prime number.

Property: **The Euler's polynomial** x^2+x+41, for $x=\frac{1}{0,39}$ gives different prime numbers.

Property: $P(n)=n^2+n+17$ is a prime number for $n=0,1,2,3,...,15$.

Property: There does not exist a polynomial (excluding the identical polynomial) with coefficients in Z such that for any $x\in Z$, $P(x)$ is a prime number.

Property: The expression $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+...$ gives the exponent of the prime number p when n is decomposed in prime factors!

Property: $\left[\frac{a_1+\cdots+a_n}{b}\right]\geq\left[\frac{a_1}{b}\right]+\cdots+\left[\frac{a_n}{b}\right]$.

Property: If $(a,b)=1$, then

$$\left[\frac{a}{b}\right]+\left[\frac{2a}{b}\right]+...+\left[\frac{(b-1)a}{b}\right]=\left[\frac{b}{a}\right]+\left[\frac{2b}{a}\right]+...+\left[\frac{(a-1)b}{a}\right]=\left[\frac{1}{2}\right](a-1)(b-1)$$

Property: $\tau_1+...+\tau_n=\left[\frac{n}{1}\right]+\left[\frac{n}{2}\right]+...+\left[\frac{n}{n}\right]$, where τ_i is the sum of the divisors of i.

Property (Iacobi): – An arithmetic progression, in which the ratio and the first term are co-prime (relative prime) numbers, contains an infinity of members that are prime with any given number.

Definition: Perfect Number is a number for which the sum of all positive divisors, strictly smaller than itself, is equal with the number itself. (Example: 6, 28, 496, 8128).

Property: Even perfect numbers have the general form: $N=2^t(2^{t+1}-1)$, where $t\in N$, $2^{t+1}-1$ is a prime number.

Theorem: If N is odd, perfect, then N has at least four different prime factors ($N>10^{20}$).

Property: $\sum_{k=1}^{n}\frac{1}{k(k+1)}=1-\frac{1}{n+1}$

Property: $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{2n-1}-\frac{1}{2n}=\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}$

Property: $\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}=\ln 2=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}$

Property: For $n>1$, we have $\frac{1}{2}<\frac{1}{n+1}+...+\frac{1}{2n}<\frac{3}{4}$

Property (Hermite): $[x]+\left[x+\frac{1}{n}\right]+...+\left[x+\frac{n-1}{n}\right]=[nx]$

Theorem: Given an irreducible fraction $\frac{a}{b}$ we have:

1. If $2\nmid b$ and $5\nmid b$, then $\frac{a}{b}$ transforms in a simple periodical decimal fraction.
2. If $b=2^{\alpha}5^{\beta}p_1...p_n$, where $n\geq 1$ and $\alpha\neq 0$ or $\beta\neq 0$, then $\frac{a}{b}$ transforms in a mix periodical function with the non-periodical part being of $\max\{\alpha,\beta\}$ digits.

Definition: Continued fraction is an expression of the following form:

$$a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+...}},$$

$[a_1, a_2, a_3, ...]$, where $a_2, a_3, ... \in N$, $a_i \in Z$, a_i are called the elements of the continued fraction, or incomplete quotients.
The continued fractions can be:

1. Limited
2. Unlimited
    a. Periodically simple $[a_1, a_2, ..., a_n; a_1, a_2, ..., a_n; ...]$
    b. Periodically mixed $[b_1, ...b_k; a_1, ..., a_n; a_1, ..., a_n; ...]$

Definition: Fibonacci sequence.

A recursive sequence:

$$u_1 = u_2 = 1$$
$$u_n = u_{n-1} + u_{n-12}$$;

It results:

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Property: $\frac{n}{\frac{1}{a_1}+...+\frac{1}{a_n}} \le \sqrt[n]{a_1...a_n} \le \frac{a_1+...+a_n}{n}$

Harmonic mean ≤ Geometric mean ≤ Arithmetic mean.

Property: $\frac{a_1}{a_2}+\frac{a_2}{a_3}+...+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1} \ge n$

Property: If $x_1 + x_2 + ... + x_n = a$, where a is a constant, then $x_1^{P_1}...x_1^{P_n}$ for $p_i \ge 0$, is maxim when $\frac{x_1}{p_1} = ... = \frac{x_n}{p_n}$.

Property: $a_i \ge 0$, $\left(\frac{a_1+...+a_n}{n}\right)^k \le \frac{a_1^k+...+a_n^k}{n}$, $k \ge 0$.

The Cauchy-Buniakovski Inequality:

$$\left(a_1b_1 + ... + a_nb_n\right)^2 \le \left(a_1^2 + ... + a_n^2\right)\left(b_1^2 + ... + b_n^2\right)$$

We have equality when: $\frac{b_1}{a_1} = ... = \frac{b_n}{a_n}$

Property: $\left(h_1 + ... + h_n\right)^p \equiv h_1^p + ... + h_n^p \pmod p$

Property: $\varphi(a \cdot b) = \frac{\varphi(a) \cdot \varphi(b).(a,b)}{\varphi\left((a,b)\right)}$

Property: Let's consider N an odd number, then among the smaller numbers than N and prime with N there exist as many even numbers as odd numbers.

Property: Let's consider $S = 1^n + 2^n + ... + (p-1)^n$

1. If $n = \mathcal{M}_{(p-1)}$ then $S \equiv -1 \pmod p$
2. If $n \ne \mathcal{M}_{(p-1)}$ then $S \equiv 0 \pmod p$

Property (Gauss): The product of all primitive solutions is congruent to $1(\text{mod } p)$; $p \neq 3$

Property: $a^1 \cdot a^2 \cdots a^\delta \equiv (-1)^{\delta+1} (\text{mod } p)$, where $a^i \not\equiv a^j (\text{mod } p)$, for $\forall i \neq j$, and $a^1, \cdots, a^\delta$ constitute all the residues modulo p.

Property (Gauss): $a^1 + a^2 + ... + a^\delta \equiv 0(\text{mod } p)$, where $a^i \not\equiv a^j (\text{mod } p)$, for $\forall i \neq j$ and $a^1, \cdots, a^\delta$ constitute all the residues modulo p.

Property: $x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y^1 + ... + x^1 y^{n-2} + y^{n-1}\right)$

Property: If n is odd, then

$$x^n + y^n = (x+y)\left(x^{n-1} - x^{n-2}y^1 + x^{n-3}y^2 - ... + y^{n-1}\right)$$

Definition: a is a square residue in rapport to the prime modulo p if the congruence $x^2 \equiv a(\text{mod } p)$ has solutions.

Theorem: The congruence $x^2 \equiv a(\text{mod } p)$ has:

1. Two solutions: x_0 and $p - x_0$ for a taking $\frac{p-1}{2}$ values
2. No solutions for a taking $\frac{p-1}{2}$ values

The Euler criterion:

1. If $a^{\frac{p-1}{2}} \equiv 1(\text{mod } p)$, then a is a squared residue
2. If $a^{\frac{p-1}{2}} \equiv -1(\text{mod } p)$, then a is a squared non-residue

Legendre's symbol:

$$\frac{a}{p} = \begin{cases} -1, & \text{if } a \text{ is squared non-residue in rapport with modulo } p \\ +1, & \text{if } a \text{ is squared residue in rapport with modulo } p \end{cases}$$

Property: $\left(\frac{a_1 ... a_k}{p}\right) = \frac{a_1}{p} ... \frac{a_k}{p}$

Definition: The minimal absolute residue is the residue r for which $r \leq \frac{p-1}{2}$. If the residue $r > \frac{p-1}{2}$, will take $p - r$.

The reciprocity law: If p, q are prime, then $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$.

Theorem: Any natural number can be represented as a sum of at most four squares.

Dirichlet' s Theorem: If $(a,b) = 1$, then there exist an infinity of numbers of the form $a + b \cdot k$.

Observation: A pair of prime twin large numbers is:

10016957, 10016959.

Bertrand's Theorem: (proved by Chebyshev (1821-1894)): Between n and $2n$, $n > 1$, there exists at least one prime number.

Euclid's Theorem: There exist an infinity of prime numbers.

Observation: The prime numbers' density diminishes while advancing in the natural numbers' sequence.

Property (Euler): The series $\frac{1}{2^a}+\frac{1}{3^a}+\frac{1}{5^a}+...+\frac{1}{p^a}+...$ is divergent.

Property (Hogatt): Any natural number is the sum of some distinct terms of the Fibonacci' sequence.

4. Philosophy.

The mathematics cannot be axiomatically created; it cannot be reduced to a formal logic. Its notions are created in contact with the reality (although some mathematical domains can be made axiomatic).
The axiomatic is just a superior phase of abstract; it is a transcription in the logical mold of known processes and directly tested or examined.

Learn, teaching others. (Seneca)
The wisdom comes with ages. (Ovidius)
The experience is gained through diligence. (Shakespeare)
The forest cannot be seen because of the trees. (*Proverb*)
The art is the highest expression of an interior arithmetic. (Leibnitz)
Repeated things are pleasant. (Horatio)

5. Series.

$$\sum_{k=1}^{n} k\cdot k! = (n+1)!-1$$

$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$$

$$\sum_{k=1}^{n} k^2 = \frac{2n+1}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{n} k^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$\sum_{k=1}^{n} k^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$$

$$\sum_{k=0}^{n} (k+1)x^k = \frac{(n+1)x^{n+2}-(n+2)x^{n+1}+1}{(x-1)^2}$$ (it is proved using with derivatives)

$$\sum_{k=1}^{n} 2^k(tg2^k x) = 2ctg2x - 2^{n+1}ctg2^{n+1}x$$

6. Inequalities.

1. $2^n n! < (n+1)^n$, for $n > 1$.
2. $2!4!...(2n)! > \left[(n+1)!\right]^n$, for $n > 1$.
3. $n^n < \left(n!\right)^2 < 2^{n(n-1)}$, for $n > 2$.
4. $\sqrt[n]{n} < \dfrac{2+\sqrt{n}}{\sqrt{n}}$
5. If $a_i, b_i \geq 0$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, then $a_1 b_1 + ... + a_n b_n \leq \left(a_1^p + ... + a_n^p\right)^{\frac{1}{p}} \left(b_1^q + ... + b_n^q\right)^{\frac{1}{q}}$
6. **The Stirling's Inequality**: $\sqrt{2\pi n}\left[\dfrac{e}{n}\right]^n < n! < \sqrt{2\pi n}\left[\dfrac{e}{n}\right]^n \cdot e^{\frac{1}{mn}}$
7. $\dfrac{1}{n^2+1}+\dfrac{1}{n^2+2}+...+\dfrac{1}{n^4} > 2\left(\dfrac{1}{2}+\dfrac{1}{3}+...+\dfrac{1}{n}\right)$
8. $\dfrac{1}{5n+1}+\dfrac{1}{5n+2}+...+\dfrac{1}{25n} > \dfrac{7}{6}$, where $n \in N$.
9. **The Jensen's Inequality**:
   Let's $f: I \to R$ and $epi(f) = \left\{\mathcal{M}(x,y) : x \in I, y \geq f(x)\right\}$ the epigraph (super graph) of f, then the $epi(f)$ is the a convex set $\Leftrightarrow$ $x_1, x_2 \in I$ and $t \in [0,1]$, we have $(1-t)x_1 + tx_2 \in I$ and
   $f\left((1-t)x_1 + tx_2\right) \leq (1-t) f(x_1) + tf(x_2)$.
10. **The Young-Fenchel Inequality**: Let's consider f convex, defined on an interval I, then:
    $\sup\left\{ax - f(x) : x \in I\right\} + f(x) \geq ax$, for $\forall x \in I$.

7. More Properties.

Property: $f(x) = ax^2 + bx + c$; $f(x) \in Z$, $\forall x \in Z$, if and only if $2a$, $a+b$, $c \in Z$.

Property (Erdös): $\forall k \in Z$, $k = \pm 1^2 \pm 2^2 \pm ... \pm m^2$, where m is dependent of k, and we can select the corresponding signs.

Property: $(n+1)(n+2)...(pn)$ is divisible by p^n.

Property: $(m_1 + ... + m_n)! = \mathcal{M}_{m_1! m_2! ... m_n!}$

Property (Cantor): If n prime numbers form an arithmetic progression, then the progression's ratio is divisible by every prim number $p < n$.

Observation: An arithmetic infinite progression of different natural numbers cannot have all its terms prime numbers.

Liouville's Theorem: The equation: $(p-1)! + 1 = p^m$, for p prime and greater than 5, $m \in N$, does not have any solution.

Cucurezeanu's Theorem: If p prime and greater than 7, $k, m \in N$, and $1 \leq k \leq p$, then the equation: $(k-1)!(p-k)! + (-1)^{k+1} = p^m$ does not have any solution.

Property: There exist an infinity of prime numbers q with the property $q \mid (n-1)!+1$ for $n<2$.

Chebyshev's Theorem: Between n and $2n$, $n>1$, there exist at least a prime number (Bertrand's postulate).

Chebyshev's Theorem: Between n and $2n$, $n>3$, there exist at least a prime number.

Theorem: Between n and $2n$, $(n>5)$, there exist at least 2 prime numbers.

Cucurezeanu's Theorem: Between $2n$ and $3n$, $n>1$, there exist at least one prime number.

Property (Cucurezeanu): Between n and $\frac{3}{2}n$, there exist at least one prime number.

Property: If p_n is the nth prime number, then $p_n^2 < 2^n$ for $n \geq 10$.

Property: Between n and $3n$, $n>1$, there exist at least 2 prime numbers.

Property (Sierpinski): $\forall a,b \in N$, $a \neq 1$ or $b \neq 1$, there exist an infinity of n natural numbers with the property: $n \mid a^n + b^n$.

Property: The exponent of the prime number p, from the following canonic decomposition $1 \cdot 3 \cdot 5 \cdots (2m+1)$ is:

$$\left(\left[\frac{2m+1}{p}\right]+\left[\frac{2m+1}{p^2}\right]+\ldots\right)-\left(\left[\frac{m}{p}\right]+\left[\frac{m}{p^2}\right]-\ldots\right)$$

Property: The number of the multiples of n smaller than x, is $\left[\frac{x}{n}\right]$.

Property: $\tau(n)=\sum_{m\geq 1}\left(\left[\frac{n}{m}\right]-\left[\frac{n-1}{m}\right]\right)$

Property: $p^n \nmid \left[(p-1)n\right]!$, $n \in N$.

Canonic decomposition = decomposition in prime factors

Definition: Fermat numbers. $2^{2^n}+1=F_n$,

$F_0 = 3$

$F_1 = 5$

$F_2 = 17$

$F_3 = 257$

$F_4 = 65537$

F_0, F_1, F_2, F_3, F_4 are prime numbers;

$F_5 = 4294967297 = 641 \cdot 6700417$

$F_6 = 18446744073709551617 = 274177 \cdot 67280421310721$

F_{1945} is divided by $5 \cdot 2^{1947}+1$ (which has 587 digits).

F_{12} is divisible with 114689.

F_{25} is divisible with 167772161.

F_{36} is divisible with 27487790694411 and it has 20 billion digits (Seelhof from Bremen).

F_{16} is divisible by $2^{18}\times 3150+1$.

F_{17} are 39457 digits, it is unknown if it is prime.

Property (Gauss): Using a ruler and a compass we can design polygons for which the number of sides is a number from the Fermat's sequence (which are prime numbers).

Property: $a^n > \frac{(a-1)^2}{4}\cdot n^2$, for $a>1$, $n\geq 2$.

Property: If $S_n=\sum_{k=1}^{n}\frac{x^k}{\left(1+x^{k+1}\right)\left(1+x^{k+1}\right)}$, $x\neq 1$, then

$$S_n=\frac{1}{x(1-x)}\left[\frac{1}{1+x^{n+2}}-\frac{1}{1+x^2}\right].$$

Observation: $2^{127}-1$ is prim number (the bigger known in 1934). It has 39 digits.

$180\left(2^{127}-1\right)^2-1$ is a prime number (1950).

$2^{2281}-1$ is a prime number (Prof. Lehmer, 1956).

$2^{4423}-1$ is a prime number with 1332 digits (Hurwitz in 1961, Selfridge, IBM, 7090 digits)

$2^{11273}-1$ the largest known prime number (computer generated).

$2^{257}-1$ is a composite number; $\underbrace{1\ldots1}_{23ori}$, is a prime number (M. Kraitchik).

Property (I. M. Vinogradov): There are values for a and p such that $a^p\equiv a(\text{mod } p^2)$.

Property: $e=2.7182818284...$ is an irrational number.

Property: The last digit non zero of

$$10^n!=\begin{cases}8,\ if\ n=1\\4,\ if\ n=2\\6,\ if\ n\geq 3\end{cases}$$

Property: If $p\geq 3$ is a prime number, $a,n\in N$, if $p^n=1+a^1+\ldots+a^p$, then $a=n=0$.

Theorem: Let's consider $x_i\geq 0$. If $x_1\cdots x_n=1$, then $x_1+\ldots+x_n\geq n$.

Property: If $a_i\geq 0$, $\alpha<0<\beta$, then

$$\left(\frac{a_1^\alpha+\ldots+a_n^\alpha}{n}\right)^{\frac{1}{\alpha}}\leq\sqrt[n]{a_1\cdots a_n}\leq\left(\frac{a_1^\beta+\ldots+a_n^\beta}{n}\right)^{\frac{1}{\beta}}$$

Property: $\sum_{n=1}^{\infty}\frac{n}{P_{p_n}}$ and $\prod_{n=4}^{\infty}\frac{1}{1-\frac{n}{P_{p_n}}}$ are divergent (p_n is the n th prime number).

Property (Waclaw Sierpinski): $\lim_{n\to\infty}\frac{p_n}{n\ln n}=1$, ($p_n$ is the n th prime number).

Property: $\sum_{n=1}^{\infty}\frac{1}{p_n}$ and $\prod_{n=2}^{\infty}\frac{1}{1-\frac{1}{p_n}}$ are divisible, (p_n is the n th prime number).

Property: $n\varphi(n!)\le n!\varphi(n)$, φ being Euler's function.

Property: The sum of the prime numbers with A and smaller than A is $s(A)=\frac{1}{2}A\varphi(A)$ where φ is Euler's function.

Property: $B^2s(A)\le A^2s(B)$, $A>B$, B contains only A factors, $s(A)$ is the sum of all prime numbers with A and smaller than A.

Property (Tiberiu Popoviciu): $\varphi(a,b)\le\sqrt{\varphi(a^2)\varphi(b^2)}$, $a,b\in N$, where φ is Euler's function.

Property: If $N=p_1^{\alpha_1}...p_n^{\alpha_n}$, $\alpha_i>1$, then $\varphi(N)$ determines uniquely N.

8. Criteria for Prime Numbers.

Wilson's Theorem: If $p>1$, then p is prime if and only if $(p-1)!\equiv -1(\text{mod } p)$.

Leibnitz's Theorem: If $p>2$, then p is prime if and only if $(p-2)!\equiv +1(\text{mod } p)$

Smarandache Criterion: If $p>3$, then p is prime if and only if $(p-3)!\equiv\frac{p-1}{2}(\text{mod } p)$.

Smarandache Criterion: If $p=6h\pm1>4$, then p is prime if and only if $(p-4)!\equiv\pm h(\text{mod } p)$.

Smarandache Criterion: If $p=24h+r>5$, $0\le r\le 24$, then p is prime if and only if

$$(p-5)!\equiv r\cdot h+\frac{r^2-1}{24}.$$

Smarandache Criterion: If $p=(k-1)!h\pm1$, then p is prime if and only if

$$(p-k)!\equiv\pm(-1)^k h(\text{mod } p).$$

Simionov's Criterion: If $1\le k\le p$, then p is prime if and only if

$$(k-1)!(p-k)!\equiv(-1)^k(\text{mod } p).$$

Criterion: p is prime if and only if $k\cdot\left[\frac{p}{k}\right]\ne p$, $k\ge 2$, $k\ne\mathcal{M}_p$.

Criterion: p is prime if and only if $k\cdot\left[\frac{p}{k}\right]\ne p$, $\forall k$, $2\le k\le\left[\sqrt{p}\right]$.

Fermat's Theorem: p is prime, $a\ne\mathcal{M}_p$, then $a^{p-1}\equiv 1(\text{mod } p)$.

Fermat's Theorem: p is prime, $a\ne\mathcal{M}_p$, then $a^p\equiv a(\text{mod } p)$.

Euler's Theorem: $(a,m)=1$, then $a^{\varphi(m)}\equiv 1(\text{mod } m)$, φ is the Euler function.

Moser's Theorem: p is prime, $a \in Z$, then $(p-1)!a^p + a = \mathcal{M}_p$.

Sierpinski's Theorem: p is prime, $a \in Z$, then $a^p + (p-1)!a = \mathcal{M}_p$.

Clement's Theorem:

$$\left.\begin{array}{l} p = prime \\ p+2 = prime \end{array}\right\} \Leftrightarrow 4\left[(p-1)!+1\right] + p \equiv 0 \left(\mathrm{mod}\ p(p+2)\right).$$

Cucuruzeanu's Theorem: (a generalization of Clement's theorem):

$$\left.\begin{array}{l} (p,i) = 1; \quad p = prime \\ i = \overline{2, n-1}; \ p+2 = prime \end{array}\right\} \Leftrightarrow n \cdot n!\left[(p-1)!+1\right] + \left[n! - (-1)^n\right] p \equiv 0 \left(\mathrm{mod}\ p(p+n)\right)$$

Property: If p is prime and $1 \le k \le p-1$, then:

1. $C_{p+k}^k \equiv 1 (\mathrm{mod}\ p)$.
2. $C_{p-1}^k \equiv (-1)^k (\mathrm{mod}\ p)$.

Property: If $a \equiv b (\mathrm{mod}\, m^n)$, then $a^m \equiv b^m \left(\mathrm{mod}\, m^{n+1}\right)$ (Proof with the Newton's binomial).

Property: If p is prime and $a^p \equiv b^p \left(\mathrm{mod}\, p\right)$, then $a^p \equiv b^p \left(\mathrm{mod}\, p^2\right)$.

Property: If p is prime, $p > 3$, then $a^p \equiv a \left(\mathrm{mod}\, 6p\right)$.

Property: If p and q are prime, $p \ne q$, $a^p \equiv b^p \left(\mathrm{mod}\, p\right)$, $a^q \equiv b^q \left(\mathrm{mod}\, q\right)$, then

$$a \equiv b \left(\mathrm{mod}\, pq\right).$$

Observation: If $a \equiv b \left(\mathrm{mod}\, m_1\right)$, $a \equiv b \left(\mathrm{mod}\, m_2\right)$, and $\left(m_1, m_2\right) = 1$, then

$$a \equiv b \left(\mathrm{mod}\, m_1 m_2\right).$$

Property: If p is prime, $(a, p) = 1$, and $a^{p-1} + b^{p-1} \equiv 0 (\mathrm{mod}\ p)$, then

$a^{p-1} + b^{p-1} \equiv 0 (\mathrm{mod}\ p^{p-1})$, (proof for $p = 2$).

Property: If p is prime, $p > 5$, then any number formed of $p-1$ equal digits, will be divisible by p.

Property: If p is prime, $p \ne 2$ and $a^p + b^p \equiv 0 (\mathrm{mod}\ p)$, then $a^p + b^p \equiv 0 (\mathrm{mod}\ p^2)$.

Property: If $m = p_1 \cdots p_s$, $p_i \ne p_j$, p_i and p_j prime numbers, and $a^m \equiv b^m \left(\mathrm{mod}\, m\right)$, then $a^m \equiv b^m \left(\mathrm{mod}\, m^2\right)$.

Property: The last 3 digits of $N = n^{100}$ are:

$$\begin{cases} 000, \ if \ n = 10k \\ 001, \ if \ n = 10k \pm 1, \ n = 10k \pm 3 \\ 376, \ if \ n = 10k \pm 2, \ n = 10k \pm 4 \\ 625, \ if \ n = 10k + 5 \end{cases}$$

Property: If $(a, m) = 1$, then the congruence $ax \equiv b (\mathrm{mod}\, m)$ solution is

$x \equiv b a^{\varphi(m)-1} (\mathrm{mod}\, m)$, φ is Euler's function.

Property: If $(a, b) = 1$, then $a^{\varphi(b)} + b^{\varphi(a)} \equiv 1 (\mathrm{mod}\, ab)$.

Property: If $p \neq q$, p, q are prime, then $p^{q-1} + q^{p-1} \equiv 1 (\text{mod } pq)$; (consequence of the previous property).

Property: If $(m,10) = 1$, then there exist a multiple of m of the form $\overline{\underbrace{a \cdots a}}$, with $a \in \overline{1,9}$

Property (W. Sierpinski): Let $s \in N$, then there exist $n \in N$ such that $s \mid n$ and the sum of the digits of n is equal to s.

Proof: $s = 2^{\alpha} 5^{\beta} t$, $(10,t) = 1$, then $n = 10^{\alpha+\beta} \left[10^{\varphi(t)} + 10^{2\varphi(t)} + ... + 10^{s\varphi(t)} \right]$.

Property: If $(a,n) = 1$, then $n \mid a^{(n-1)!} - 1$.

Property: If n is an even number, then $\left(n^2 - 1\right) \mid 2^{n!} - 1$.

Property: There does not exist $n > 1$ such that $n \mid (a+1)^n - a^n$, for $\forall a \in Z$.

Theorem (I. Moser): If p is prime, $\forall a$, then $(p-1)!a^p + a = \mathcal{M}_p$ (Fermat and Wilson theorem put together).

Theorem: If p is prime, $\forall a$, then $a^p + (p-1)!a = \mathcal{M}_p$.

Property: There is an infinity of composite numbers of the form: $(n!)^2 + 1$.

Property: If $n < p$, then p is prime if and only if $\frac{(n+1)...(n+p)}{p} + 1 = \mathcal{M}_p$ (Gh. Zapan).

Property: If p is prime; $A = a_0 x^n + ... + a_n$, $a_n \neq db_p$, $a_i \in Z$, if there exist $x_0 \in Z$ such that $p \mid A$, then there exist an infinity of $y \in Z$ such that $p \mid B = a_n y^n + ... + a_0$.

Proof: y has the property $x_0 y \equiv 1 (\text{mod } p)$.

Property: If n is odd, then $n \mid 1^n + 2^n + ... + (n-1)^n$.

Theorem: For $n > 2$, between n and $n!$ there exist at least a prime number.

Theorem: Any natural number greater or equal to 2 has at least a prime divisor.

Theorem: There exist at least 3 prime numbers each containing s digits ($\forall s \in N^*$).

The Eratosthenes Sieve.

Observation: There are**:**

4 prime numbers of 1 digit;
21 prime numbers of 2 digits;
163 prime numbers of 3 digits.

Statistics: 6,000,000 prime numbers. $P_{6,000,000} = 104,395,301$.

American scientists have a computer that will store in its memory the first 500,000,000 consecutive prime numbers.

There are 152,892 pairs of twin prime numbers until 30,000,000.

Property (Cucurezeanu): If $\forall x \in R$, $\prod_{\substack{p \leq x \\ p \text{ prime}}} p < 3,3^x$;

if $\forall x > 29$, $\prod_{p \leq x} p \geq 2^x$.

Property: $\frac{3}{4}\cdot\frac{x}{\ln x}<\prod(x)<\frac{3}{2}\frac{x}{\ln x}$, where $\prod(x)$ is the number of the prime numbers $\le x$.

Property (A. Schinzel): If $\min\{x,y\}\le 146$, then

$$\prod(x+y)\le\prod(x)+\prod(y).$$

Property: The exponent of the prime number p from $\frac{a!}{b!c!}$ is

$$\left(\left[\frac{a}{p}\right]-\left[\frac{b}{p}\right]-\left[\frac{c}{p}\right]\right)+\left(\left[\frac{a}{p^2}\right]-\left[\frac{b}{p^2}\right]-\left[\frac{c}{p^2}\right]\right)+\ldots.$$

Property (Vinogradov): $\forall n>3^{3^{16}}$, n odd, ca be written as the sum of 3 different odd prime numbers. (Until the number $3^{3^{16}}$ the property was unknown).

Observation: The $\underbrace{1\ldots1}_{37times}$ is a composite number; the number $\underbrace{1\ldots1}_{641times}$ is divisible by 1283.

Observation: *There exist prime numbers that remain prime after any permutation of their digits.

Examples: 13 and 31; 17 and 71; 37 and 73; 79 and 97;
113 and 131, 311; 199, 919, 991; 337, 373, 733.

Property (H. E. Richert): For $3<n<6\cdot10^{175}$ there does not exist prime numbers with the property (*), except of those that are formed with only digit 1.

F. Smarandache: A prime number of the form (*) is formed only with digits: 1, 3, 7, 9.

Proof: If the number would have also the digits: 0, 2, 4, 5, 6, 8 by permutations these digits will take the last position and, therefore, they'll be divisible by 2 or 5.

Theorem (W. Sierpinski): Let's consider $a_1,\ldots,a_m$ and $b_1,\ldots,b_n$ with $b_1,\ldots,b_n\in\{1,3,7,9\}$, then there exist an infinity of prime numbers of the form: $p=\overline{a_1\cdots a_m r_1\cdots r_s b_1\cdots b_n}$ (that start with $a_1\cdots a_m$ and end with $b_1\cdots b_n$).

Observation: It is not known if there exist an infinity of prime numbers formed only with the digit 1.

Statistics: L. Moser found all the prime numbers smaller than 100,000 if the digits from which are formed are written in an inverse order. (There are 102 prime numbers of this kind which are less than 100,000.)

Examples: The numbers of this gen less than 1,000 are:
101, 131, 151, 181, 313, 353, 373, 383, 727, 757, 787, 797, 919, 929.

Observation: It is not known if there exist an infinity of this type of numbers.

Property (Sierpinski): $\forall a,b\in N^*$, there exist p,q prime numbers such that $a<\frac{p}{q}<b$.

Property: $\lim_{n\to\infty}\frac{p_{\Pi(nx)}}{n}=x$, $\forall x\in N$, p_h is the h-th prime number

Property: There exist an infinite set of prime numbers such that $p_n>\frac{p_{n-1}+p_{n+1}}{2}$.

Property: There exist an infinite set of prime numbers such that $p_n < \frac{p_{n-1} + p_{n+1}}{2}$.

Hypotheses: There exist an infinite set of prime numbers such that $p_n = \frac{p_{n-1} + p_{n+1}}{2}$.

Example: for $n = 16, 37, 40, 47, 55, 56, 240, 273$.

Theorem (Erdös, P. Turan): There is an infinity of prime numbers such that $p_n^2 > p_{n-1}p_{n+1}$.

There is an infinity of prime numbers such that $p_n^2 < p_{n-1}p_{n+1}$.

Observation: (quadruple) $p,\ p+2,\ p+6,\ p+8$ are prime.

Example: $p = 5, 101, 191, 821, 1481, 3251$.

Statistics: Among the first 10,000,000 numbers there are 899 quadruples (Golubev).
Among the first 15,000,000 numbers there are 1209 quadruples.
The largest known quadruple is $p = 2{,}863{,}308{,}731$ (A. Ferrier).

Property (B. M. Bredihin): There is an infinity of prime numbers of the form $x^2 + y^2 + 1$.

Property: There is an infinity of prime numbers of the form $x^2 + y^2$.

Hypothesis: there exists the polynomial: $P(n)$ such that for $n \in N$ will give an infinite of prime numbers? (not all values to be prime).

Example: for first degree there is $P(x) = 2x + 1$.

Hypothesis: Does $P(x) = x^2 + 1$ give an infinite of prime numbers?

Property (Van der Corput): There is an infinity of arithmetic progressions that are formed from 3 different prime numbers.

Example: 3, 7, 11; 3, 11, 19; 3, 43, 83; … .

The Chinese wrong theorem: If $n \mid 2^n - 2$, then n is a prime number. (This is true for $1 < n \le 300$).

Property (N. G. W. H. Beeger, 1951): There exist an infinity of even numbers n such that $n \mid 2^n - 2$.

Property: There exist an infinity of pairs of different prime numbers p, q such that $pq \mid 2^{pq} - 2$.

Theorem (A Schinzel): For $\forall a \in Z,\ \forall m \in N$, there exist p, q different such that $pq \mid a^{pq} - a$.

Definition: n is a pseudo prime number if n is a composite number such that $n \mid 2^n - 2$.

Definition: n is an absolute pseudo prime number if n is a composite number such that $\forall a \in Z$, $n \mid a^n - a$. The smallest is $561 = 3 \cdot 11 \cdot 17$.

Other examples are: $7 \cdot 17 \cdot 31, ..., 5 \cdot 17 \cdot 29 \cdot 113 \cdot 337 \cdot 673 \cdot 2689$.

Hypothesis: There exist an infinity of such numbers (not proved).

Property: If p is a prime number, then $1^{p-1} + 2^{p-1} + ... + (p-1)^{p-1} + 1$ is divisible by p.

Hypothesis (G. Giuca - 1950): If $1^{p-1} + 2^{p-1} + ... + (p-1)^{p-1} + 1$ is divisible by p, then p is a prime number (not proved). It has been verified for $n \le 10^{1000}$.

Property (Littlewood): Let's consider $\Pi_1(x)=$ the number of prime numbers of the form: $4k+1$, which are $\le x$, then exists an infinity of natural numbers x such that $\Pi_1(x)>\Pi_3(x)$.

Let's consider $\Pi_3(x)=$ the number of prime numbers of the form: $4k+3$, which are $\le x$, then exists an infinity of natural numbers x such that $\Pi_1(x)<\Pi_3(x)$.

Example: $\Pi_1(26862)=1473>1472=\Pi_3(x)$.

Property: Any natural number of the form $4k+3,\ 6k+5$ contains at least a prim divisor of the same form ($4k+3$, respectively $6k+5$).

F. Smarandache: Analogously for $3k+2$.

F. Smarandache: for $n=4k+1$ or $n=6k+5$ it is not true.

Property (Ingham): Between m $(m+1)^3$, there exists an arbitrary large number of prime numbers.

Theorem: If $(a,m)=1$, a is the primitive root modulo m iff a does not satisfy none of the congruencies: $a^{\frac{\varphi(m)}{p_1}}\equiv 1(\mathrm{mod}\, m),...,a^{\frac{\varphi(m)}{p_r}}\equiv 1(\mathrm{mod}\, m)$, where p_i are all prime positive divisors of $\varphi(m)$.

Theorem: a is the primitive root p, $t\in Z$ such that $(a+pt)^{p-1}=1+pu$, $p\nmid u$, then $a+pt$ is a primitive root modulo $\pm p^{\beta}$.

Theorem: If a is a primitive root modulo p^{β}, then the odd number between a and $a+p^{\beta}$ is the primitive root modulo $\pm 2p^{\beta}$.

Definition: $(a,m)=1$, $g=$ primitive root modulo m is called the index of a modulo m in rapport to the base g**.** The number γ with the property $a\equiv g^{\gamma}(\mathrm{mod}\, m)$; is noted $\gamma=ind_g a$ or $\gamma=inda$.

Observation: $ind_g a$ has a similar property with the algorithm.

Property: The number of primitive roots modulo m is $\varphi(\varphi(m))$.

The number of the residue classes modulo m, prime with m, of order δ is $\varphi(\delta)$.

Definition: $\frac{\sqrt{5}+1}{2}$ is called "*the golden number*"**.**

The principle of inclusion and exclusion:

$$Card\left(\bigcup_{i=1}^{q}A_i\right)=\sum_{i=1}^{9}CardA_i-\sum_{1\le i<j\le 9}Card\left(A_i\cap A_j\right)+...+(-1)^{9+1}Card\left(\bigcap_{i=1}^{9}A_i\right)$$

The formula of a multinomial:

$$\left(a_1+a_2+...+a_p\right)^n=\sum_{\substack{n_1+...+n_p=n\\ n_1,...,n_p\ge 0}}\binom{n}{n_1,n_2,...,n_p}a_1^{n_1}a_2^{n_2}...a_p^{n_p}\text{, where}$$

$$\binom{n}{n_1,n_2,...,n_p}=\frac{n!}{n_1!n_2!...n_p!}$$

Van der Waerden's theorem: For $\forall k,t$ positive integer numbers, there exists a natural number denoted $w(k,t)$ which is the smallest integer number with the following property: If the set $\{1,2,...,w(k,t)\}$ is partitioned in k classes, there exists a class of the partition which contains an arithmetic progression with $t+1$ terms. $w(k,t)$ is called the Van der Waerden number.

9. Diophantine Equations.

1. $x^2+2xy+y^2-x-3y-2z+2=0$, $x,y,z\in N$ (Ilie Iliescu).
   Solutions: $(x_1,y_{1,}z_1)=(1,1,1)$

$$(x_{n+1},y_{n+1,}z_{n+1})=\begin{cases}(x_{n+1},y_{n+1,}z_{n+1}), & if\ y_n\neq 1\\ (1,x_{n+1},z_{n+1}), & if\ y_n=1\end{cases}$$

   The function $f:N\times N\to N$, $f(x,y)=\frac{(x+y-1)(x+y-2)}{2}+x$ is bijective.
2. $x^2+x+y^2=0$ does not have solutions in N.
3. If $x,y,z\in N: x^2+y^2+1=xyz$, then $z=3$ (Ilie Iliescu).
4. $x^2+x-2y^2=0$; Solutions: $(x_1,y_1)=(1,1)$; $(3x_n+4y_n+1,2x_n+3y_n+1)$, $n\in N$ (Ilie Iliescu).
   Proof: $g:E\setminus\{(1,1)\}\to E$, $g(x,y)=(3x-4y+1,3y-2x-1)$ is bijective;
   $E=\{(x,y)\in N\times N / x^2+x-2y^2=0\}$.
5. $x^2+1-7y^2=0$ does not have a solution in Z. (Ilie Iliescu)
6. $x^2-2y^n=1$ has an infinity of solutions in Z. (Ilie Iliescu)
   Proof: $G=\{z\in R/z=x+y\sqrt{2},\ x,y\in Z,\ x^2-2y^n=1\}$, G multinomial,
   $M=\left\{\begin{pmatrix}a & b\\ 2b & a\end{pmatrix}|\ a,b\in Z\ \ a^2-2b^2=1\right\}$, M is multiplicative group.
   $G\approx M$.
   It results that if $z_1=x_1+y_1\sqrt{2}$ is a solution (x_1,y_1) of the equation, then $z_1^n=x_n+y_n\sqrt{2}$ it is also a solution (x_n,y_n).
7. (Gelfond) The equation $x^2+2y^n=z^2$ has the solutions:

$$\begin{cases}x=\pm(a^2-2b^2)\\ y=2ab\\ z=a^2+2b^2\end{cases},$$

where $a,b\in Z_+$, $(a,b)=1$, b odd.

10. Euclid's Algorithm extended.

$Z : a, b \in N^*$
$E : d, h, k \in Z$ such that $d(a,b)$; $d = ah + bk$
M :

1. $(u, v, x) \leftarrow (1, 0, a)$
   $(s, t, y) \leftarrow (0, 1, b)$
2. z is the residue of the division of x by y
3. If $z = 0$, then 6)
4. $q \leftarrow \left[\frac{x}{y}\right]$
   $(\xi, \eta, z) \leftarrow (u, v, x) - (s, t, y)q$
5. $(u, v, x) \leftarrow (s, t, y)$ and $(s, t, y) \leftarrow (\xi, \eta, z)$, goes to 3)
6. $(h, k, d) = (s, t, y)$

11. Binary Algorithm.

The calculation of the LCD (Least Common Denominator):
$I : a, b \in N^*$
$E : d = (a, b)$
M :

1. $x \leftarrow a, y \leftarrow b, k \leftarrow 0$
2. If $2 \nmid x$ and $2 \nmid y$ then 4)
3. $x \leftarrow \frac{x}{2}$, $y \leftarrow \frac{y}{2}$, $k \leftarrow k + 1$
4. If $2 \nmid x$, then 6)
5. $x \leftarrow \frac{x}{2}$, then 4)
6. If $2 \nmid y$, then 10)
7. $y \leftarrow \frac{y}{2}$, then 6)
8. If $x \leq y$, then 10)
9. $x \leftarrow x - y$, then 5)
10. If $x \geq y$, then 12)
11. If $y \leftarrow y - x$, then 7)
12. $d = 2^k x$

12. **Conclusion**.

My intention was to thinking at writing a handbook of elliptic function theory applied in number theory, but also a volume of amusing/amazing (!) (recreational) problems, that require fantasy thinking, deviation from the rational, and scientific tricks.

PARADOXES

Neutrosophic Degree of a Paradoxicity

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1. Definition of a **Paradox**.

A paradox is called a statement <P> which is true and false in the same time.
Therefore, if we suppose that statement <P> is true, it results that <P> is false; and reciprocally, if we suppose that <P> is false, it results that <P> is true.

2. But there are statements that do not completely obey this definition.

We call a **Semi-Paradox** a statement <SP> such that either supposing that <SP> is true it results that <SP> is false (but not reciprocally), or supposing that <SP> is false it results that <SP> is true (but not reciprocally).
So, the statement has a degree of 0.50 (50%) of a paradox, and 0.50 of a non-paradox.

3. Three-Quarters Paradox.

3.1. Definition.

There are cases when a statement <QP> can be between a paradox and a semi-paradox. For example:

a) If we suppose that the statement <QP> is true, it results that <QP> is false, but reciprocally if we suppose that the statement <QP> is false, it may be possible resulting that <QP> is true. Therefore, the second implication (conditional) does not always occur.
b) Or, if we suppose that the statement <QP> is false, it results that <QP> is true, but reciprocally if we suppose that the statement <QP> is true, it may be possible resulting that <QP> is false. Therefore, the second implication (conditional) does not always occur.

In this case we may have a degree of paradoxicity in between 0.50 and 1, actually in a neighborhood of 0.75.
These types of fuzzy and especially neutrosophic implications are derived from the fuzzy or neutrosophic logic connectives.

3.2.See some **Examples of Three-Quarters Paradoxes**

Social Three-Quarters Paradox:
In a democracy should the non-democratic ideas be allowed?
a) If no, i.e. other ideas are not allowed - even those non-democratic -, then one not has a democracy, because the freedom of speech is restricted.
b) If yes, i.e. the non-democratic ideas are allowed, then one might end up to a non-democracy (because the non-democratic ideas could overthrow the democracy as, for example, it happened in Nazi Germany, in totalitarian countries, etc.).

Three-Quarters Paradox of Freedom of Speech & Religion (I):

As a freedom of speech do we have the right to insult religion?

a) If not, then we don't have freedom of speech.

b) If yes, i. e. we have the right to insult religion, then we don't respect the freedom of faith.

Devine Three-Quarters Paradox (II):

Can God prove He can commit suicide?

a) If not, then it appears that there is something God cannot do, therefore God is not omnipotent.

b) If God can prove He can commit suicide, then God dies - because He has to prove it, therefore God is not immortal.

Devine Three-Quarters Paradox (III):

Can God prove He can be atheist, governed by scientific laws?

a) If God cannot, then again He's not omnipotent.

b) If God can prove He can be atheist, then God doesn't believe in Himself, therefore why should we believe in Him?

Devine Three-Quarters Devine Paradox (IV):

Can God prove He can do bad things?

a) If He cannot, then He is not omnipotent, therefore He is not God.

b) If He can prove He can do bad things, again He's not God, because He doesn't suppose to do bad things.

Devine Three-Quarters Paradox (V):

Can God create a man who is stronger than him?

a) If not, then God is not omnipotent, therefore He is not God.

b) If yes, i. e. He can create someone who is stronger than Him, then God is not God any longer since such creation is not supposed to be possible, God should always be the strongest.

{God was egocentric because he didn't create beings stronger than Him.}

Devine Three-Quarters Paradox (VI):

Can God transform Himself in his opposite, the Devil?

a) If not, then God is not omnipotent, therefore He is not God.

b) If yes, then God is not God anymore since He has a dark side: the possibility of transforming Himself into the Devil [God doesn't suppose to be able to do that].

4. In general we have the following **Degree of a Paradox**:

Let's consider a statement <DP>.

(α) If we suppose that the statement <DP> is true it may result that <DP> is false, and reciprocally *(β)* if we suppose that the statement <DP> is false it may result that <DP> is true. Therefore, both implications (conditionals) depend on other factors in order to occur or not, or

partially they are true, partially they are false, and partially indeterminate (as in neutrosophic logic).

5. **Discussion**.

This is the general definition of a statement with some degree of paradoxicity.

a) If both implications (α) and (β) are true 100%, i.e. the possibility "it may result" is replaced by the certitude "it results" we have a 100% paradox.
b) If one implication is 100% and the other is 100% false, we have a semiparadox (50% of a paradox).
c) If both implications are false 100%, then we have a non-paradox (normal logical statement).
d) If one condition is p% true and the other condition q% true (truth values measured with the fuzzy logic connectives or neutrosophic logic connectives), then the **degree of paradoxicity** of the statement is the average $\frac{p+q}{2}$ %.
e) Even more general from the viewpoint of the neutrosophic logic, where a statement is *T%* true, *I%* indeterminate, and *F%* false, where *T, I, F* are standard or non-standard subsets of the non-standard unit interval $]^{-}0, 1^{+}[$.
If one condition has the truth value (T_1, I_1, F_1) and the other condition the truth value (T_2, I_2, F_2), then the **neutrosophic degree of paradoxicity** of the statement is the average of the component triplets:

$$(\frac{T_1+T_2}{2}, \frac{I_1+I_2}{2}, \frac{F_1+F_2}{2}),$$

where the addition of two sets A and B (in the case when T, I, or F are sets) is simply defined as:

$$A + B = \{x \mid x = a + b \text{ with } a \in A \text{ and } b \in B\}.$$

6. **Comment.**

When T, I, F are crisp numbers in the interval [0, 1], and I = 0, while T + F = 1, then the neutrosophic degree of paradoxicity coincides with the (fuzzy) degree of paradoxicity from d).

Reference:

Smarandache, Florentin, "*Neutrosophy. / Neutrosophic Probability, Set, and Logic*", American Research Press, Rehoboth, 1998.

Neutrosophic Diagram and Classes of Neutrosophic Paradoxes or To The Outer-Limits of Science

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Abstract.
These paradoxes are called "neutrosophic" since they are based on indeterminacy (or neutrality, i.e. neither true nor false), which is the third component in neutrosophic logic. We generalize the Venn Diagram to a Neutrosophic Diagram, which deals with vague, inexact, ambiguous, ill-defined ideas, statements, notions, entities with unclear borders. We define the neutrosophic truth table and introduce two neutrosophic operators (*neuterization* and *antonymization* operators) give many classes of neutrosophic paradoxes.

1. Introduction to the Neutrosophics.

Let <*A*> be an idea, or proposition, statement, attribute, theory, event, concept, entity, and <*nonA*> what is not <*A*>.
Let <*antiA*> be the opposite of <*A*>. We have introduced a new notation [1998], <*neutA*>, which is neither <*A*> nor <*antiA*> but in between.
<*neutA*> is related with <*A*> and <*antiA*>.

Let's see an example for vague (not exact) concepts: if <*A*> is "*tall*" (an attribute), then <*antiA*> is "*short*", and <*neutA*> is "*medium*", while <*nonA*> is "*not tall*" (which can be "*medium or short*"). Similarly for other <*A*>, <*neutA*>, <*antiA*> such as: <*good*>, <*so so*>, <*bad*>, or <*perfect*>, <*average*>, <*imperfect*>, or <*high*>, <*medium*>, <*small*>, or respectively <*possible*>, <*sometimes possible and other times impossible*>, <*impossible*>, etc.

Now, let's take an exact concept / statement: if <*A*> is the statement "*1+1=2 in base 10*", then <*antiA*> is "*1+1≠2 in base 10*", while <*neutA*> is undefined (doesn't exist) since it is not possible to have a statement in between "*1+1=2 in base 10*" and "*1+1≠2 in base 10*" because in base 10 we have 1+1 is either equal to 2 or 1+1 is different from 2. <*nonA*> coincides with <*antiA*> in this case, <nonA> is "*1+1≠2 in base 10*".

Neutrosophy is a theory the author developed since 1995 as a generalization of dialectics. This theory considers every notion or idea <*A*> together with its opposite or negation <*antiA*>, and the spectrum of "neutralities" in between them and related to them, noted by <*neutA*>.
The Neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.
Its Fundamental Thesis:
Any idea <*A*> is *T%* true, *I%* indeterminate (i.e. neither true nor false, but neutral, unknown), and *F%* false.
Its Fundamental Theory:

Every idea <*A*> tends to be neutralized, diminished, balanced by <*nonA*> ideas (not only by <*antiA*> as Hegel asserted) - as a state of equilibrium.
In between <*A*> and <*antiA*> there may be a continuous spectrum of particular <*neutA*> ideas, which can balance <*A*> and <*antiA*>.
To neuter an idea one must discover all its three sides: of sense (truth), of nonsense (falsity), and of undecidability (indeterminacy) - then reverse/combine them. Afterwards, the idea will be classified as neutrality.

There exists a Principle of Attraction not only between the opposites <*A*> and <*antiA*> (as in dialectics), but also between them and their neutralities <*neutA*> related to them, since <*neutA*> contributes to the Completeness of Knowledge.
Hence, neutrosophy is based not only on analysis of oppositional propositions as dialectic does, but on analysis of these contradictions together with the neutralities related to them.

Neutrosophy was extended to Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Neutrosophic Statistics, which are used in technical applications.

In the Neutrosophic Logic (which is a generalization of fuzzy logic, especially of intuitionistic fuzzy logic) every logical variable x is described by an ordered triple $x = (T, I, F)$, where T is the degree of truth, F is the degree of falsehood, and I the degree of indeterminacy (or neutrality, i.e. neither true nor false, but vague, unknown, imprecise), with T, I, F standard or non-standard subsets of the non-standard unit interval $]^{-}0, 1^{+}[$. In addition, these values may vary over time, space, hidden parameters, etc.

Neutrosophic Probability (as a generalization of the classical probability and imprecise probability) studies the chance that a particular event <*A*> will occur, where that chance is represented by three coordinates (variables): T% chance the event will occur, I% indeterminate (unknown) chance, and F% chance the event will not occur.

Neutrosophic Statistics is the analysis of neutrosophic probabilistic events.

Neutrosophic Set (as a generalization of the fuzzy set, and especially of intuitionistic fuzzy set) is a set such that an element belongs to the set with a neutrosophic probability, i.e. T degree of appurtenance (membership) to the set, I degree of indeterminacy (unknown if it is appurtenance or non-appurtenance to the set), and F degree of non-appurtenance (non-membership) to the set.

There exist, for each particular idea: PRO parameters, CONTRA parameters, and NEUTER parameters which influence the above values.
Indeterminacy results from any hazard which may occur, from unknown parameters, or from new arising conditions. This resulted from practice.

2. **Applications of Neutrosophics**:

Neutrosophic logic/set/probability/statistics are useful in artificial intelligence, neural networks, evolutionary programming, neutrosophic dynamic systems, and quantum mechanics.

3. Examples of Neutrosophy used in Arabic philosophy (F. Smarandache & S. Osman):

- While Avicenna promotes the idea that the world is contingent if it is necessitated by its causes, Averroes rejects it, and both of them are right from their point of view.

Hence <*A*> and <*antiA*> have common parts.

- Islamic dialectical theology (*kalam*) promoting creationism was connected by Avicenna in an extraordinary way with the opposite Aristotelian-Neoplatonic tradition.

Actually a lot of work by Avicenna falls into the frame of neutrosophy.

- Averroes's religious judges (*qadis*) can be connected with atheists' believes.

- al-Farabi's metaphysics and general theory of emanation vs. al-Ghazali's Sufi writings and mystical treatises [we may think about a coherence of al-Ghazali's "Incoherence of the Incoherence" book].

- al-Kindi's combination of Koranic doctrines with Greek philosophy.

- Islamic Neoplatonism + Western Neoplatonism.

- Ibn – Khaldun's statements in his theory on the cyclic sequence of civilizations, says that:

Luxury leads to the raising of civilization (because the people seek for comforts of life)

but also Luxury leads to the decay of civilization (because its correlation with the corruption of ethics).

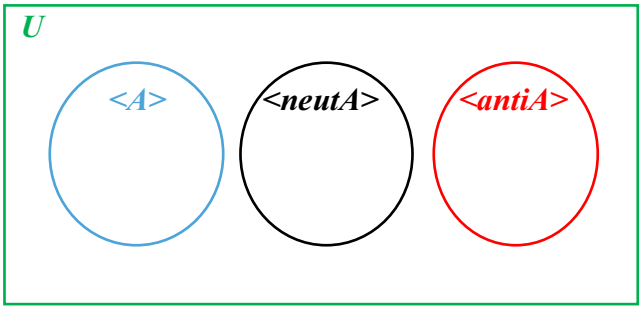
- On the other hand, there's the method of absent–by–present syllogism in jurisprudence, in which we find the same principles and laws of neutrosophy.

- In fact, we can also function a lot of Arabic aphorisms, maxims, Koranic miracles (Ayat Al-Qur'ăn) and Sunna of the prophet, to support the theory of neutrosophy.

Take the colloquial proverb that "The continuance of state is impossible" too, or "Everything, if it's increased over its extreme, it will turn over to its opposite"!

4. The Venn Diagram.

In a Venn Diagram we have with respect to a universal set *U* the following:

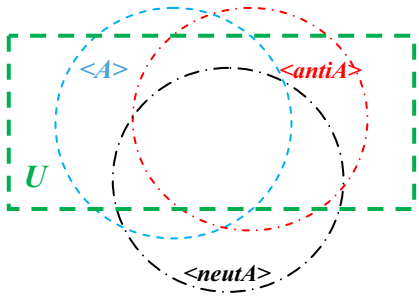


Therefore, there are no common parts amongst <*A*>, <*neutA*>, and <*antiA*>, and all three of them are (completely) contained by the universal set *U*. Also, all borders of these sets <*A*>, <*neutA*>, <*antiA*>, and *U* are clear, exact. All these four sets are well-defined.
While <*neutA*> means neutralities related to <*A*> and <*antiA*>, what is outside of <*A*> ∪ <*neutA*> ∪ <*antiA*> but inside of *U* are other neutralities, not related to <*A*> or to <*antiA*>.
Given <*A*>, there are two types of neutralities: those related to <*A*> (and implicitly related to <*antiA*>), and those not related to <*A*> (and implicitly not related to <*antiA*>).

5. The Neutrosophic Diagram, as extension of the Venn Diagram.

Yet, for ambiguous, vague, not-well-known (or even unknown) imprecise ideas / notions / statements / entities with unclear frontiers amongst them the below relationships may occur because between an approximate idea noted by <*A*> and its opposite <*antiA*> and their neutralities <*neutA*> there are not clear delimitations, not clear borders to distinguish amongst what is <*A*> and what is not <*A*>. There are buffer zones in between <*A*> and <*antiA*> and <*neutA*>, and an element *x* from a buffer zone between <*A*> and <*antiA*> may or may not belong to both <*A*> and <*antiA*> simultaneously. And similarly for an element *y* in a buffer zone between <*A*> and <*neutA*>, or an element *z* in the buffer zone between <*neutA*> and <*antiA*>. We may have a buffer zone where the confusion of appurtenance to <*A*>, or to <*neutA*>, or to <*antiA*> is so high, that we can consider that an element *w* belongs to all of them simultaneously (or to none of them simultaneously).
We say that all four sets <*A*>, <*neutA*>, <*antiA*>, and the neutrosophic universal set *U* are ill-defined, inexact, unknown (especially if we deal with predictions; for example if <*A*> is a statement with some degree of chance of occurring, with another degree of change of not occurring, plus an unknown part). In the general case, none of the sets <*A*>, <*neutA*>, <*antiA*>, <*nonA*> are completely included in *U*, and neither *U* is completely known; for example, if *U* is the neutrosophic universal set of some specific given events, what about an unexpected event that might belong to *U*? That's why an approximate *U* (with vague borders) leaves room for expecting the unexpected.

The Neutrosophic Diagram in the general case is the following:

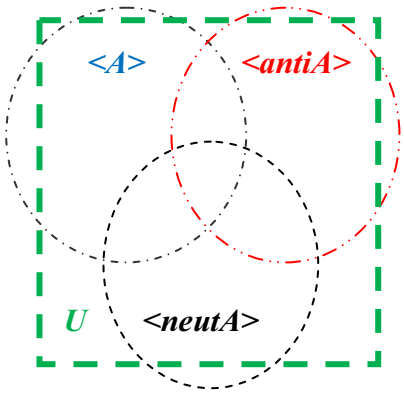


The borders of <*A*>, <*antiA*>, and <*neutA*> are dotted since they are unclear.

Similarly, the border of the neutrosophic universal set *U* is dotted, meaning also unclear, so *U* may not completely contain <*A*>, nor <*neutA*> or <*antiA*>, but *U* "approximately" contains each of them. Therefore, there are elements in <*A*> that may not belong to *U*, and the same thing for <*neutA*> and <*antiA*>. Or elements, in the most ambiguous case, there may be elements in <*A*> and in <*neutA*> and in <*antiA*> which are not contained in the universal set *U*.
Even the neutrosophic universal set is ambiguous, vague, and with unclear borders.

Of course, the intersections amongst <*A*>, <*neutA*>, <*antiA*>, and *U* may be smaller or bigger or even empty depending on each particular case.

See below an example of a particular neutrosophic diagram, when some intersections are contained by the neutrosophic universal set:



A neutrosophic diagram is different from a Venn diagram since the borders in a neutrosophic diagram are vague. When all borders are exact and all intersections among <*A*>, <*neutA*>, and <*antiA*> are empty, and all <*A*>, <*neutA*>, and <*antiA*> are included in the neutrosophic universal set *U*, then the neutrosophic diagram becomes a Venn diagram.

The neutrosophic diagram, which complies with the neutrosophic logic and neutrosophic set, is an extension of the Venn diagram.

6. Classes of Neutrosophic Paradoxes.

The below classes of neutrosophic paradoxes are not simply word puzzles. They may look absurd or unreal from the classical logic and classical set theory perspective.
If <*A*> is a precise / exact idea, with well-defined borders that delimit it from others, then of course the below relationships do not occur.

But let <*A*> be a vague, imprecise, ambiguous, not-well-known, not-clear-boundary entity, <*nonA*> means what is not <*A*>, and <*antiA*> means the opposite of <*A*>.
<*neutA*> means the neutralities related to <*A*> and <*antiA*>, neutralities which are in between them.

When *<A>, <neutA>,<antiA>,<nonA>, U* are uncertain, imprecise, they may be self-contradictory. Also, there are cases when the distinction between a set and its elements is not clear.
Although these neutrosophic paradoxes are based on 'pathological sets' (those whose properties are considered atypically counterintuitive), they are not referring to the theory of Meinongian objects (*Gegenstandstheorie*) such as round squares, unicorns, etc. Neutrosophic paradoxes are not reported to objects, but to vague, imprecise, unclear ideas or predictions or approximate notions or attributes from our everyday life.

7. Let's introduce for the first time two new **Neutrosophic Operators.**

1) An operator that "neuterizes" an idea. **To neuterize** [*neuter+ize*, transitive verb; from the Latin word *neuter* = neutral, in neither side], $n(.)$, means to map an entity to its neutral part. {We use the Segoe Print for "n(.)".}
"To neuterize" is different from "to neutralize" [from the French word *neutraliser*] which means to declare a territory neutral in war, or to make ineffective an enemy, or to destroy an enemy.
$n(<A>) = <neutA>$. By definition $n(<neutA>) = <neutA>$.
For example, if *<A>* is "*tall*", then $n(tall) = medium$, also $n(short) = medium$, $n(medium) = medium$.
But if *<A>* is "*1+1=2 in base 10*" then $n(<1+1=2 \text{ in base } 10>)$ is undefined (does not exist), and similarly $n(<1+1\neq 2 \text{ in base } 10>)$ is undefined.
2) And an operator that "antonymizes" an idea. **To antonymize** [*antonym+ize*, transitive verb; from the Greek work *antōnymia* = instead of, opposite], $a(.)$, means to map an entity to its opposite. {We use the Segoe Print for "a(.)".}
$a(<A>) = <antiA>$.
For example, if *<A>* is "*tall*", then $a(tall) = short$, also $a(short) = tall$, and $a(medium) = tall \text{ or } short$.
But if *<A>* is "*1+1=2 in base 10*" then $a(<1+1=2 \text{ in base } 10>) = <1+1\neq 2 \text{ in base } 10>$ and reciprocally $a(<1+1\neq 2 \text{ in base } 10>) = <1+1=2 \text{ in base } 10>$.

The classical operator for negation / complement in logics respectively in set theory, "*to negate*" ($\neg$), which is equivalent in neutrosophy with the operator "*to nonize*" (i.e. *to non+ize*) or *nonization (*i.e. *non+ization)*, means to map an idea to its neutral or to its opposite (a union of the previous two neutrosophic operators: *neuterization* and *antonymization*):
$\neg <A> = <nonA> = <neutA> \cup <antiA> = n(<A>) \cup a(<A>)$.

Neutrosophic Paradoxes result from the following neutrosophic logic / set connectives following all apparently impossibilities or semi-impossibilities of neutrosophically connecting *<A>, <antiA>, <neutA>, <nonA>,* and the neutrosophic universal set *U*.

8. **Neutrosophic Truth Tables**.

For $<A>$ = *"tall"*:

$<A>$	$a(<A>)$	$n(<A>)$	$\neg <A>$
tall	*short*	*medium*	*short or medium*
medium	*short or tall*	*medium*	*short or tall*
short	*tall*	*medium*	*tall or medium*

To remark that $n(<medium>) \triangleq medium$.
If $<A>$ = *tall,* then $<neutA>$ = *medium*, and $<neut(neutA)>=<neutA>$, or $n(<n(<A>)>)=n(<A>)$.

For $<A>$ = *"1+1=2 in base 10"* we have $<antiA> = <nonA>$ = *"1+1≠ 2 in base 10"*, while $<neutA>$ is undefined (N/A) - whence the neutrosophic truth table becomes:

$<A>$	$a(<A>)$	$n(<A>)$	$\neg <A>$
True	False	N/A	False
False	True	N/A	True

In the case when a statement is given by its neutrosophic logic components $<A> = (T, I, F)$, i.e. $<A>$ is *T%* true, *I%* indeterminate, and *F%* false, then the neutrosophic truth table depends on the defined neutrosophic operators for each application.

9. Neutrosophic Operators and Classes of Neutrosophic Paradoxes.

a) Complement/Negation

$\neg <A> \neq <nonA>$ and reciprocally $\neg <nonA> \neq <A>$.

$\neg(\neg <A>) \neq <A>$
$\neg(\neg <antiA>) \neq <antiA>$
$\neg(\neg <nonA>) \neq <nonA>$
$\neg(\neg <neutA>) \neq <neutA>$
$\neg(\neg U) \neq U$, where U is the neutrosophic universal set.
$\neg(\neg < \emptyset >) \neq < \emptyset >$, where $< \emptyset >$ is the neutrosophic empty set.

b) Neuterization.

$<n(<A>) \neq <neutA>$

$<n(<antiA>) \neq <neutA>$

$<\mathcal{n}(<nonA>) \neq <neutA>$

$<\mathcal{n}(\mathcal{n}(<A>)) \neq <A>$

c) Antonymization.

$<\mathcal{a}(<A>) \neq <antiA>$

$<\mathcal{a}(<antiA>) \neq <A>$

$<\mathcal{a}(<nonA>) \neq <A>$

$<\mathcal{a}(\mathcal{a}(<A>)) \neq <A>$

d) Intersection/Conjunction

$<A> \cap <nonA> \neq \emptyset$ (neutrosophic empty set) {symbolically $(\exists x)(x \in A \wedge x \in \neg A)$ },
or even more $<A> \cap <antiA> \neq \emptyset$ {symbolically $(\exists x)(x \in A \wedge x \in \mathcal{a}(A))$ } ,
similarly $<A> \cap <neutA> \neq \emptyset$ and $<antiA> \cap <neutA> \neq \emptyset$,
up to $<A> \cap <neutA> \cap <antiA> \neq \emptyset$.
The symbolic notations will be in a similar way.
This is Neutrosophic Transdisciplinarity, which means to find common features to uncommon entities.

For examples:
There are things which are good and bad in the same time.
There are things which are good and bad and medium in the same time (because from one point of view they may be god, from other point of view they may be bad, and from a third point of view they may be medium).

e) Union / Weak Disjunction

$<A> \cup <neutA> \cup <antiA> \neq U$.
$<antiA> \cup <neutA> \neq <nonA>$.
Etc.

f) Inclusion/Conditional

$<A> \subset <antiA>$

$(\forall x)(x \in A \rightarrow x \in \mathcal{a}(A))$

All is $<antiA>$, the $<A>$ too.
All *good* things are also *bad*.
All is *imperfect*, the *perfect* too.

$<antiA> \subset <A>$

$(\forall x)(x \in \mathcal{a}(A) \rightarrow x \in A)$

All is <*A*>, the <*antiA*> too.
All *bad* things have something *good* in them {this is rather a fuzzy paradox}.
All is *perfect* things are *imperfect* in some degree.

<*nonA*> ⊂ <*A*>
$(\forall x)(x \in \neg A \rightarrow x \in A)$
All is <*A*>, the <*nonA*> too.
All *bad* things have something *good* and something *medium* in them {this is a neutrosophic paradox, since it is based on good, bad, and medium}.
All is *perfect* things have some *imperfectness and mediocrity* in them at some degree.

<*A*> ⊂ <*neutA*>
$(\forall x)(x \in A \rightarrow x \in \mathcal{n}(A))$
All is <*neutA*>, the <*A*> too.

<*nonA*> ⊂ <*neutA*> {partial neutrosophic paradox of inclusion}
$(\forall x)(x \in \neg A \rightarrow x \in \mathcal{n}(A))$
All is <*neutA*>, the <*nonA*> too.

<*nonA*> ⊂ <*antiA*> {partial neutrosophic paradox of inclusion}
$(\forall x)(x \in \neg A \rightarrow x \in \mathcal{a}(A))$
All is <*antiA*>, the <*nonA*> too.

<*antiA*> ⊂ <*neutA*>
$(\forall x)(x \in \mathcal{a}(A) \rightarrow x \in \mathcal{n}(A))$
All is <*neutA*>, the <*antiA*> too.

<*A*> ∪ <*antiA*> ⊂ <*neutA*>
$(\forall x)((x \in A \vee x \in \mathcal{a}(A)) \rightarrow x \in \mathcal{n}(A))$
All is <*neutA*>, the <*A*> and <*antiA*> too.

Paradoxes of some Neutrosophic Arguments

<*A*> → <*B*>
<*B*> → <*antiA*>

∴ <*A*> → <*antiA*>
Example: too much work produces sickness; sickness produces less work (absences from work, low efficiency); therefore, too much work implies less work (this is a Law of Self-Equilibrium).

A> → <*B*>
<*B*> → <*nonA*>

∴ <*A*> → <*nonA*>

$<A> \rightarrow <B>$
$<B> \rightarrow <neutA>$

$\therefore <A> \rightarrow <neutA>$

g) Equality/Biconditional

Unequal Equalities
$<A> \neq <A>$
which symbolically becomes $(\exists x)(x \in A \leftrightarrow x \notin A)$
or even stronger inequality $(\forall x)(x \in A \leftrightarrow x \notin A)$.
Nothing is $<A>$, nor even $<A>$.

$<antiA> \neq <antiA>$
which symbolically becomes $(\exists x)(x \in \neg A \leftrightarrow x \notin \neg A)$
or even stronger inequality $(\forall x)(x \in \neg A \leftrightarrow x \notin \neg A)$

$<neutA> \neq <neutA>$
which symbolically becomes $(\exists x)(x \in \nu A \leftrightarrow x \notin \nu A)$
or even stronger inequality $(\forall x)(x \in \nu A \leftrightarrow x \notin \nu A)$

$<nonA> \neq <nonA>$
which symbolically becomes $(\exists x)(x \in \neg A \leftrightarrow x \notin \neg A)$
or even stronger inequality $(\forall x)(x \in \neg A \leftrightarrow x \notin \neg A)$

Equal Inequalities
$<A> = <antiA>$
$(\forall x)(x \in A \leftrightarrow x \in a(A))$
All is $<A>$, the $<antiA>$ *too;* and reciprocally, all is $<antiA>$, the $<A>$ too.
Or, both combined implications give: All is $<A>$ is equivalent to all is $<antiA>$.

And so on:
$<A> = <neutA>$
$<antiA> = <neutA>$
$<nonA> = <A>$

Dilations and Absorptions
$<antiA> = <nonA>$,
which means that $<antiA>$ is dilated to its neutrosophic superset $<nonA>$, or $<nonA>$ is absorbed to its neutrosophic subset $<antiA>$.
Similarly for:
$<neutA> = <nonA>$
$<A> = U$
$<neutA> = U$
$<antiA> = U$

$<nonA> = U$

h) Combinations of the previous single neutrosophic operator equalities and/or inequalities, resulting in more neutrosophic operators involved in the same expression.

For examples:
$<neutA> \cap (<A> \cup <antiA>) \neq \emptyset$ {two neutrosophic operators}.
$<A> \cup <antiA> \neq \neg <neutA>$ and reciprocally $\neg (<A> \cup <antiA>) \neq <neutA>$.
$<A> \cup <neutA> \neq \neg <antiA>$ and reciprocally.
$\neg (<A> \cup <neutA> \cup <antiA>) \neq \emptyset$ and reciprocally.

Etc.

i) We can also take into consideration other logical connectors, such as *strong disjunction* (we previously used the *weak disjunction*), *Shaffer's connector, Peirce's connector*, and extend them to the neutrosophic form.
j) We may substitute $<A>$ by some entities, attributes, statements, ideas and get nice neutrosophic paradoxes, but not all substitutions will work properly.

10. Some particular paradoxes:

A Quantum Semi-Paradox.
Let's go back to 1931 Schrödinger's paper. Saul Youssef writes (flipping a *quantum coin*) in arXiv.org at *quant-ph/9509004*:
"The situation before the observation could be described by the distribution (1/2,1/2) and after observing heads our description would be adjusted to (1,0) . The problem is, what would you say to a student who then asks: "Yes, but what causes (1/2,1/2) to evolve into (1,0)? How does it happen?" {http://god-does-not-play-dice.net/Adler.html}.
It is interesting.
Actually we can say the same for any probability different from 1: If at the beginning, the probability of a quantum event, $P(\text{quantum event}) = p$, with $0 < p < 1$, and if later the event occurs, we get to $P(\text{quantum event}) = 1$;
but if the event does not occur, then we get $P(\text{quantum event}) = 0$,
so still a kind of contradiction.

Torture's paradox.
An innocent person *P*, who is tortured, would say to the torturer *T* whatever the torturer wants to hear, even if *P* doesn't know anything.
So, *T* would receive incorrect information that will work against him/her.
Thus, the torture returns against the torturer.

Paradoxist psychological behavior.
Instead of being afraid of something, say <A>, try to be afraid of its opposite <antiA>, and thus – because of your fear – you'll end up with the <anti<antiA>>, which is <A>.

Paradoxically, negative publicity attracts better than positive one (enemies of those who do negative publicity against you will sympathize with you and become your friends).

Paradoxistically [word coming etymologically from *paradoxism, paradoxist*], to be in opposition is more poetical and interesting than being opportunistic.
At a sportive, literary, or scientific competition, or in a war, to be on the side of the weaker is more challenging but on the edge of chaos and, as in Complex Adoptive System, more potential to higher creation.

Law of Self-Equilibrium.
{Already cited above at the Neutrosophic Inclusion / Conditional Paradoxes.}
$<A> \rightarrow <B>$ and $<B> \rightarrow <antiA>$, therefore $<A> \rightarrow <antiA>$!
Example: too much work produces sickness; sickness produces less work (absences from work, low efficiency); therefore, too much work implies less work.

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S-DENYING A THEORY

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Abstract.

In this paper we introduce the operators of *validation* and *invalidation* of a proposition, and we extend the operator of *S-denying* a proposition, or an axiomatic system, from the geometric space to respectively any theory in any domain of knowledge, and show six examples in geometry, in mathematical analysis, and in topology.

1. Definitions.

Let T be a theory in any domain of knowledge, endowed with an ensemble of sentences E, on a given space M.

E can be for example an axiomatic system of this theory, or a set of primary propositions of this theory, or all valid logical formulas of this theory, etc. E should be closed under the logical implications, i.e. given any subset of propositions $P_1, P_2, \ldots$ in this theory, if Q is a logical consequence of them then Q must also belong to this theory.

A sentence is a logic formula whose each variable is quantified {i.e. inside the scope of a quantifier such as: $\exists$ (exist), $\forall$ (for all), modal logic quantifiers, and other various modern logics' quantifiers}.

With respect to this theory, let P be a proposition, or a sentence, or an axiom, or a theorem, or a lemma, or a logical formula, or a statement, etc. of E.

It is said that P is *S-denied*[1] on the space M if P is valid for some elements of M and invalid for other elements of M, or P is only invalid on M but in at least two different ways.

[1] The multispace operator S-denied (*Smarandachely-denied*) has been inherited from the previously published scientific literature (see for example Ref. [1] and [2]).

An ensemble of sentences E is considered *S-denied* if at least one of its propositions is *S-denied*.

And a theory T is *S-denied* if its ensemble of sentences is *S-denied*, which is equivalent to at least one of its propositions being *S-denied*.

The proposition P is partially or totally denied/negated on M. The proposition P can be simultaneously validated in one way and invalidated in (finitely or infinitely) many different ways on the same space M, or only invalidated in (finitely or infinitely) many different ways.

The invalidation can be done in many different ways.

For example the statement A = *"x≠5"* can be invalidated as *"x=5"* (total negation), but "$x \in \{5, 6\}$" (partial negation).

(Use a notation for *S-denying*, for invalidating in a way, for invalidating in another way a different notation; consider it as an operator: neutrosophic operator? A notation for invalidation as well.)

But the statement B = *"x > 3"* can be invalidated in many ways, such as *"x ≤ 3",* or *"x = 3",* or *"x < 3",* or *"x = -7",* or *"x = 2",* etc. A negation is an invalidation, but not reciprocally – since an invalidation signifies a (partial or total) degree of negation, so invalidation may not necessarily be a complete negation. The negation of B is $\neg B$ = *"x ≤ 3",* while *"x = -7"* is a partial negation (therefore an invalidation) of B.

Also, the statement C = "John's car is blue and Steve's car is red" can be invalidated in many ways, as: "John's car is yellow and Steve's car is red", or "John's car is blue and Steve's car is black", or "John's car is white and Steve's car is orange", or "John's car is not blue and Steve's car is not red", or "John's car is not blue and Steve's car is red", etc.

Therefore, we can *S-deny* a theory in finitely or infinitely many ways, giving birth to many partially or totally denied versions/deviations/alternatives theories: $T1$, $T2$, … . These new theories represent degrees of negations of the original theory T.

Some of them could be useful in future development of sciences.

Why do we study such *S-denying* operator? Because our reality is heterogeneous, composed of a multitude of spaces, each space with different structures. Therefore, in one space a statement may be valid, in another space it may be invalid, and invalidation can be done in various ways.

Or a proposition may be false in one space and true in another space or we may have a degree of truth and a degree of falsehood and a degree of indeterminacy. Yet, we live in this mosaic of distinct (even opposite structured) spaces put together.

S-denying involved the creation of the multi-space in geometry and of the *S-geometries* (1969).

It was spelt *multi-space*, or *multispace*, of *S-multispace*, or *mu-space*, and similarly for its: *multi-structure*, or *multistructure*, or *S-multistructure*, or *mu-structure.*

2. **Notations.**

Let $<A>$ be a statement (or proposition, axiom, theorem, etc.).

a) For the classical Boolean logic *negation* we use the same notation. The negation of $<A>$ is noted by $\neg A$ and $\neg A = <nonA>$.

An *invalidation* of $<A>$ is noted by $i(A)$, while a *validation* of $<A>$ is noted by $v(A)$:

$$i(A) \subseteq 2^{<nonA>} \setminus \{ \emptyset \} \text{ and } v(A) \subseteq 2^{<A>} \setminus \{ \emptyset \}$$

where 2^X means the power-set of X, or all subsets of X.

All possible invalidations of $<A>$ form a set of invalidations, notated by $I(A)$. Similarly for all possible validations of $<A>$ that form a set of validations, and noted by $V(A)$.

b) *S-denying* of $<A>$ is noted by $S_{\neg}(A)$. S-denying of $<A>$ means some validations of $<A>$ together with some invalidations of $<A>$ in the same space, or only invalidations of <A> in the same space but in many ways.

Therefore, $S_{\neg}(A) \subseteq V(A) \cup I(A)$ or $S_{\neg}(A) \subseteq I(A)^k$, for $k \geq 2$.

3. **Examples.** Let's see some models of *S-denying*, three in a geometrical space, and other three in mathematical analysis (calculus) and topology.

3.1. The first *S-denying* model was constructed in 1969. This section is a compilation of ideas from paper [1].

An axiom is said *Smarandachely denied* if the axiom behaves in at least two different ways within the same space (i.e., validated and invalided, or only invalidated but in multiple distinct ways).

A *Smarandache Geometry* [SG] is a geometry which has at least one Smarandachely denied axiom.
Let's note any point, line, plane, space, triangle, etc. in such geometry by s-point, s-line, s-plane, s-space, s-triangle respectively in order to distinguish them from other geometries.
Why these hybrid geometries? Because in reality there does not exist isolated homogeneous spaces, but a mixture of them, interconnected, and each having a different structure.
These geometries are becoming very important now since they combine many spaces into one, because our world is not formed by perfect homogeneous spaces as in pure mathematics, but by non-homogeneous spaces. Also, SG introduce the degree of negation in geometry for the first time [for example an axiom is denied 40% and accepted 60% of the space] that's why they can become revolutionary in science and it thanks to the idea of partial denying/accepting of axioms/propositions in a space (making multi-spaces, i.e. a space formed by combination of many different other spaces), as in fuzzy logic the degree of truth (40% false and 60% true).
They are starting to have applications in physics and engineering because of dealing with non-homogeneous spaces.
The first model of S-denying and of SG was the following:
The axiom that through a point exterior to a given line there is only one parallel passing through it [Euclid's Fifth Postulate], was S-denied by having in the same space: no parallel, one parallel only, and many parallels.
In the Euclidean geometry, also called parabolic geometry, the fifth Euclidean postulate that there is only one parallel to a given line passing through an exterior point, is kept or validated.
In the Lobachevsky-Bolyai-Gauss geometry, called hyperbolic geometry, this fifth Euclidean postulate is invalidated in the following way: there are infinitely many lines parallels to a given line passing through an exterior point.
While in the Riemannian geometry, called elliptic geometry, the fifth Euclidean postulate is also invalidated as follows: there is no parallel to a given line passing through an exterior point.
Thus, as a particular case, Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries may be united altogether, in the same space, by some SG's. These last geometries can be partially Euclidean and partially Non-Euclidean simultaneously.

3.2. **Geometric Model (particular case of SG).**

Suppose we have a rectangle *ABCD*.

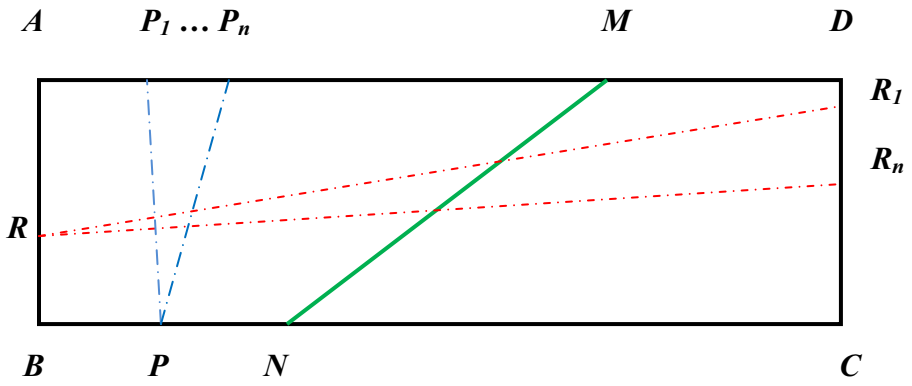

Fig. 1.

In this model we define as:
Point = any point inside or on the sides of this rectangle;
Line = a segment of line that connects two points of opposite sides of the rectangle;
Parallel lines = lines that do not have any common point (do not intersect);
Concurrent lines = lines that have a common point.

Let's take the line *MN*, where *M* lies on side *AD* and *N* on side *BC* as in the above *Fig. 1*. Let *P* be a point on side *BC,* and *R* a point on side *AB*.

Through *P* there are passing infinitely many parallels (PP_1, ..., PP_n, …) to the line *MN*, but through *R* there is no parallel to the line *MN* (the lines RR_1, ..., RR_n cut line *MN*). Therefore, the Fifth Postulate of Euclid (that though a point exterior to a line, in a given plane, there is only one parallel to that line) in *S-denied* on the space of the rectangle *ABCD* since it is invalidated in two distinct ways.

3.3. Another Geometric Model (another particular case of SG).

We change a little the Geometric Model 1 such that:

The rectangle *ABCD* is such that side *AB* is smaller than side *BC*. And we define as *line* the arc of circle inside (and on the borders) of *ABCD*, centered in the rectangle's vertices *A, B, C,* or *D.*

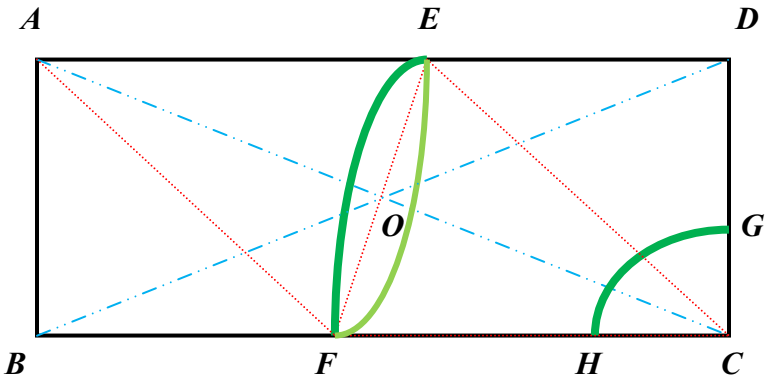

Fig. 2.

The axiom that: through two distinct points there exist only one line that passes through is S-denied (in three different ways):

a) Through the points *A* and *B* there is no passing line in this model, since there is no arc of circle centered in *A, B, C,* or *D* that passes through both points. See *Fig. 2*.
b) We construct the perpendicular $EF \perp AC$ that passes through the point of intersection of the diagonals *AC* and *BD*. Through the points *E* and *F* there are two distinct lines the dark green (left side) arc of circle centered in *C* since $CE \equiv FC$, and the light green (right side) arc of circle centered in *A* since $AE \equiv AF$. And because the right triangles $\triangle COE$, $\triangle COF$, $\triangle AOE$, and $\triangle AOF$ are all four congruent, we get $CE \equiv FC \equiv AE \equiv AF$.

c) Through the points G and H {such that $CG \equiv CH$ (their lengths are equal)} there is only one passing line (the dark green arc of circle GH, centered in C) since $AG \neq AH$ (their lengths are different), and similarly $BG \neq BH$ and $DG \neq DH$.

3.4. **Example for the Axiom of Separation.**

The Axiom of Separation of Hausdorff is the following:

$\forall\ x, y \in M,\quad \exists N(x), N(y):\ N(x) \cap N(y) = \emptyset,$

where $N(x)$ is a neighborhood of x, and respectively $N(y)$ is a neighborhood of y.

We can *S-deny* this axiom on a space M in the following way:

a) $\exists\ x_1, y_1 \in M:\ \exists N_1(x_1), N_1(y_1):\ \ N_1(x_1) \cap N_1(y_1) = \emptyset,$
where $N_1(x_1)$ is a neighborhood of x_1, and respectively $N_1(y_1)$ is a neighborhood of y_1; [validated].

b) $\exists\ x_2, y_2 \in M:\ \forall N_2(x_2), N_2(y_2):\ \ N_2(x_2) \cap N_2(y_2) \neq \emptyset;$
where $N_2(x_2)$ is a neighborhood of x_2, and respectively $N_2(y_2)$ is a neighborhood of y_2; [invalidated].

Therefore we have two categories of points in M: some points that verify The Axiom of Separation of Hausdorff and other points that do not verify it. So M becomes a partially separable and partially inseparable space, or we can see that M has some degrees of separation.

3.5. Example for the Norm.

If we remove one or more axioms (or properties) from the definition of a notion $<A>$ we get a pseudo-notion $<pseudoA>$.

For example, if we remove the third axiom (inequality of the triangle) from the definition of the $<norm>$ we get a $<pseudonorm>$.

The axioms of a **norm** on a real or complex vectorial space V over a field F, $x \mapsto || \ . \ ||$, are the following:

a) $||x|| = 0 \Leftrightarrow x = 0.$
b) $\forall x \in V,\ \forall \alpha \in F,\ || \alpha \cdot x \ || = | \alpha | \cdot ||x \ ||.$
c) $\forall x, y \in V,\ ||x+y|| \leq ||x|| \cdot ||y||$ (inequality of the triangle).

For example, a **pseudo-norm** on a real or complex vectorial space V over a field F, $x \mapsto_p || \ . \ ||$, may verify only the first two above axioms of the norm.

A pseudo-norm is a particular case of an *S-denied norm* since we may have vectorial spaces over some given scalar fields where there are some vectors and scalars that satisfy the third axiom

[validation], but others that do not satisfy [invalidation]; or for all vectors and scalars we may have either $||x+y|| = 5\cdot||x||\cdot||y||$ or $||x+y|| = 6\cdot||x||\cdot||y||$, so invalidation (since we get $||x+y|| > ||x||\cdot||y||$) in two different ways.

Let's consider the complex vectorial space $\mathscr{C} = \{a+b\cdot i$, where $a, b \in R$, $i = \sqrt{-1}\}$ over the field of real numbers R.

If $z = a+b\cdot i \in \mathscr{C}$ then its pseudo-norm is $\| z \| = \sqrt{a^2 + b^2}$. This verifies the first two axioms of the norm, but do not satisfy the third axiom of the norm since:

For x = 0 + $b\cdot i$ and $y = a + 0\cdot i$ we get:

$||x+y|| = ||a+b\cdot i|| = \sqrt{a^2 + b^2} \leq ||x||\cdot||y|| = ||0+b\cdot i||\cdot||a+0\cdot i|| = |a\cdot b|$, or $a^2 + b^2 \leq a^2b^2$;

But this is true for example when $a = b \geq \sqrt{2}$ (validation), and false if one of a or b is zero and the other is strictly positive (invalidation).

Pseudo-norms are already in use in today's scientific research, because for some applications the norms are considered too restrictive.

Similarly one can define a pseudo-manifold (relaxing some properties of the manifold), etc.

3.6. **Example in Topology**.

A topology $\mathscr{O}$ on a given set E is the ensemble of all parts of E verifying the following properties:

a) E and the empty set $\emptyset$ belong to $\mathscr{O}$.
b) Intersection of any two elements of $\mathscr{O}$ belongs to $\mathscr{O}$ too.
c) Union of any family of elements of $\mathscr{O}$ belongs to $\mathscr{O}$ too.

Let's go backwards. Suppose we have a topology $\mathscr{O}_1$ on a given set E_1, and the second or third (or both) previous axioms have been *S-denied*, resulting an <u>*S-denied topology*</u> $\mathcal{S}_\neg(\mathscr{O}_1)$ on the given set E_1.

In general, we can go back and "recover" (reconstruct) the original topology $\mathscr{O}_1$ from $\mathcal{S}_\neg(\mathscr{O}_1)$ by recurrence: if two elements belong to $\mathcal{S}_\neg(\mathscr{O}_1)$ then we set these elements and their intersection to belong to $\mathscr{O}_1$, and if a family of elements belong to $\mathcal{S}_\neg(\mathscr{O}_1)$ then we set these family elements

and their union to belong to $\mathcal{C}_1$; and so on: we continue this recurrent process until it does not bring any new element to $\mathcal{C}_1$.

Conclusion.

Decidability changes in an *S-denied theory*, i.e. a defined sentence in an S-denied theory can be partially deducible and partially undeducible (we talk about degrees of deducibility of a sentence in an *S-denied theory*).

Since in classical deducible research, a theory $\mathcal{T}$ of language $\mathcal{L}$ is said complete if any sentence of $\mathcal{L}$ is decidable in $\mathcal{T}$, we can say that an *S-denied theory* is partially complete (or has some degrees of completeness and degrees of incompleteness).

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Five Paradoxes and a General Question on Time Traveling

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1) Traveling to the Past.

Joe40, who is 40 years old, travels 10 years back to the past when he was 30 years old. He meets himself when he was 30 years old, let's call this Joe30.

Joe40 kills Joe30.

If so, we mean if Joe died at age 30 (because Joe30 was killed), how could he live up to age 40?

2) Traveling to the Future.

Joe30, who is 30 years old, travels 10 years in the future and meats himself when he will be 40 years old, let's call this Joe40.

Joe40 kills Joe30.

At what age did Joe die, at 30 or 40?

If Joe30 died, then Joe40 would not exist.

3) Traveling Pregnant Woman.

a) A 3-month pregnant woman, Jane3, travels 6 months to the future where she gives birth to a child Johnny3.

b) Then she returns with the child back, and after 1 month she travels 5 months to the future exactly at the same time as before.

Then how is it possible to have at exactly the same time two different situations: first only the pregnant woman, and second the pregnant woman and her child?

4) Traveling in the Past before Birth.

Joe30, who is 30 years old, travels 40 years in the past, therefore 10 years before he was born.

How is it possible for him to be in the time when he did not exist?

5) Traveling in the Future after Death.

Joe30, who is 30 years old, travels 40 years in the future, 10 years after his death. He has died when he was 60 years old, as Joe60.

How is it possible for him to be in the time when he did not exist any longer?

A General Question about Time Traveling:

When traveling say 50 years in the past [let's say from year 2010 to year 1960] or 50 years in the future [respectively from year 2010 to year 2060], how long does the traveling itself last?

a) If it's an instantaneous traveling in the past, is the time traveler jumping from year 2010 directly to year 1960, or is he continuously passing through all years in between 2010 and 1960? Similar question for traveling in the future.
b) If the traveling lasts longer {say, a few units (seconds, minutes, etc.) of time}, where will be the traveler at the second unit or third unit of time? I mean, suppose it takes 5 seconds to travel from year 2010 back to year 1960; then in the 1st second is he in year 2000, in the 2nd second in year 1990, in the 3rd second in year 1980, in the 4st second in year 1970, and in the 5st second in year 1960? So, his speed is 10 years per second?
Similar question for traveling in the future.

PHILOSOPHY OF SCIENCE

On the Relation between Mathematics, Natural Sciences, And Scientific Inquiry

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In this article, we will shortly review a few old thoughts and recent thoughts on the relation between Mathematics and the Natural Sciences. Of course, the classic references to this open problem will include Wigner's paper (1964); a more recent review article is Darvas (2008). But it appears that this issue is partly on the domain of natural philosophy and also philosophy of inquiry. Therefore we will begin with a review on some known thoughts of Kant, Bacon, Popper, etc.

Our hope here is to find out clues to reveal the hidden structure of Nature, just as what Planck did a century ago. (An early note to our scientific colleagues: In writing this article we choose to switch off our role as 'practical scientist' and switch on the 'free thinker' mode, therefore you can sit back and relax, because chance is what we write here is not related to what you're doing; this is more on science as a whole. But of course if you're interested in this kind of article, you can read on.)

In the meantime, we've written a rather serious article on this issue, but after midnight our thought becomes twisted, and now we are going to rewrite it again in the style of Scott Adams' Dilbert comics. This belongs to our favorite comic strips. If at certain point you feel like we're going too far (probably saying to yourself: *Heck, what kind of tablets these guys have swallowed?*), perhaps you should stop reading or send this file to recycle bin. Otherwise, you can continue reading and make up your mind later on.

The Hidden Structure of Nature: What it is, what it was

It appears as a fair guess to say that the greatest Natural philosopher was Kant. One of his most cited remark is perhaps the distinction between '*phenomena*' and '*noumena*' (from 'nous'). To put this idea a bit simpler, we can say that phenomena refer to processes or symptoms that appear to the eyes, while noumena refer to the hidden configuration or inner structures which are beyond what meet the eyes.

But that notion of 'noumena' is quite problematic, because it does not clarify how 'hidden' or which deeper level that we're looking for. If for instance, we discuss here the elementary particles, then does it mean that present hadron physics theories or strong forces already reflect the noumena, or shall we find out hidden structure beyond the hadrons, perhaps something like sub-quark or Planck scale models?

In this regards, some physicists already mention that there is a scale invariance character of elementary particles, which suggests that we can always reveal new structure at deeper and deeper scale. Perhaps it is quite safe to say that the restriction here is not on theoretical side, but more on the precision of measurement apparatus.

If in accordance with Kant the phenomena are qualitatively distinct from the '*noumena*' (the hidden structure of Nature), then problem of finding <noumena> will be more adverse if we ask not only what Nature is today, but also what Nature was in the past. In this regards, it is quite apparent that the uncertainties of the problem become twofold, one concerns the deep structure of Nature itself, and the next concerns the premise of the smooth continuation of time.

Most evolution theories apparently are based on this premise of a smooth progression of things (some modern models are based on dynamical equation like Lotka-Volterra equation, but how to define time itself remains an open issue). On the other side, there are new theories based on possibility of 'sudden changes' happening on large scales, for instance the concept *self-organized criticality* introduced by Per Bak *et al*., emergence theory, spontaneous symmetry breaking, etc. (Darvas 2008).

In such a model based on the self-organized criticality, sudden changes can happen after a long period of stasis. For example, consider a pile of sand: initially it can pile up almost vertically, until sometime it will change such that a slope will form what is known as 'critical angle' (see Figure 1).

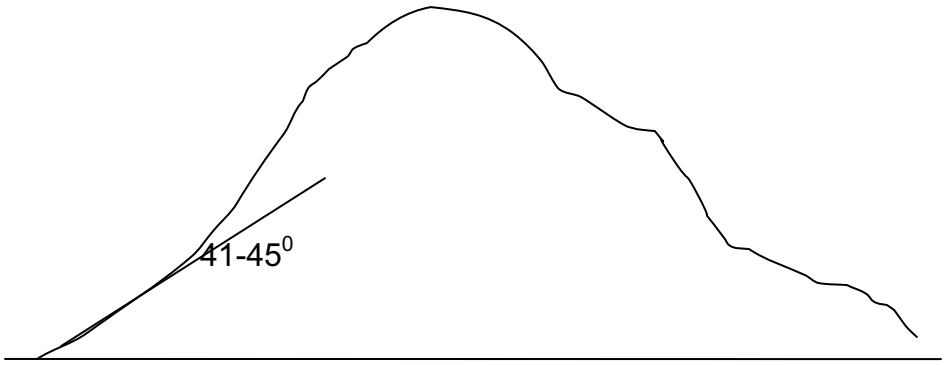

Figure 1. Sudden change to form critical angle.

Hopefully the above example can give illustration how the sudden changes can happen during such critical phenomena. Various other critical phenomena can be related to this self-organized criticality, so that it is quite problematic to conceive how smooth

continuation of changes can take place 'naturally'. Another well-known example is the geological layering formation near Yukatan area, Mexico. As reported by Alvarez *et al.,* they indicate some kind of periodic changes in the past at the order of thousand years.

Nowadays, the self-organized criticality phenomena have been studied extensively in various context; for a quick look see for instance Boldyrev (arXiv:hep-th/9610080), and Ambjørn, Jurkiewics & Loll (arXiv: hep-th/0712.2485, gr-qc/0711.0273). One can also find that introducing the discontinuous progression of 'time' will lead to a quite different Galilean law of motion, and so forth.

One can also note here that in some ancient thinking, large natural changes can take place in the same time with large social upheavals. From the viewpoint of modern dynamics theory, whether such a large climate or environmental change really can affect social upheaval remains mystery, although there has been study on the relation between human/population evolution processes and their environment (by T. Barnosky from Berkeley Univ.). From a viewpoint, this may explain why some people feel that they can predict anything except to predict when the sky would fall upon them (remember the Asterix comics, for instance).

Nonetheless, we should limit our discussions here on self-organized criticality only in the domain of Natural phenomena. Meanwhile, other people may find that those sudden changes may also be related to Kuhn's idea of '*paradigm changes'* in the history of science (Gholson & Barker, 1985). For example, one can notice from history of modern science that the long-stagnant period between Planck's blackbody radiation (1901) until 1921 was a precursor to the rapid development in short period (1922-1928), where the modern Physics began. As Weinberg once remarked such a rapid change is so remarkable in history such that scientists nowadays refer to 'Classical Physics' for all things happened before this era. Nonetheless, in this article we don't discuss such a possible parallelism any further.

On the Methods of Inquiry: From Bacon, Popper, to Habermas

As the night goes very late, now we will continue this rambling note on how scientists may possibly discover something new in their fields. In this section, it is safe to say that you can forget all what you already learned on scientific methods, because this section is not about that classic teaching on science. This section is more about where to begin the scientific process itself. As we all know, how to invent and how to discover are perhaps one of the most fundamental questions for all living scientists (including physicists). In writing this section we would rely solely on a few irrelevant experiences with pets and also to a crystal ball which tells nothing.

It seems worth to mention here Darvas' (2008) note on the distinctive standpoints between Bacon, Polanyi and Popper. Bacon emphasized methodological processes which should be given attention in science; somewhat a more philosophical part of what Galileo did experimentally.

In the mean time, Polanyi gave emphasize on the 'personal knowledge.' By personal here he meant human mind which consists of things he/she learnt (objective knowledge), things he/she thought (tacit knowledge), and also things he/she perceived (subjective knowledge). In other words, according to Polanyi, one's personal knowledge does not necessarily mean to be always subjective, though it may include subjective knowledge.

Karl Popper who wrote his seminal book '*Objective Knowledge'* apparently as a response to Polanyi's book '*Tacit Knowledge'* disagreed strongly with this idea of personal knowledge. Popper himself apparently emphasized the role of knowing 'episteme', via continuing criticism. In his model, validation of theory is not possible to achieve via experiments, they can only support or reject a hypothesis. For further discussion on this issue, see Darvas' review (2008).

We can make further remark here that Polanyi's assertion of personal knowledge and tacit knowledge today has begun to be implemented in the so-called Knowledge Management. This is a modern method to organize the unstructured parts of human knowledge, for instance see the *OneNote* feature in recent version of MS Office.

In the context of Knowledge Management, one can predict that in the future our present methods of file management will be improved to enable people organize better their tacit knowledge. For instance, scientists perhaps would prefer to organize their files according to their specific 'mind-mapping' diagram, instead of standard 'vertical' folder systems. The distinction is shown in Figure 2 and Figure 3 below.

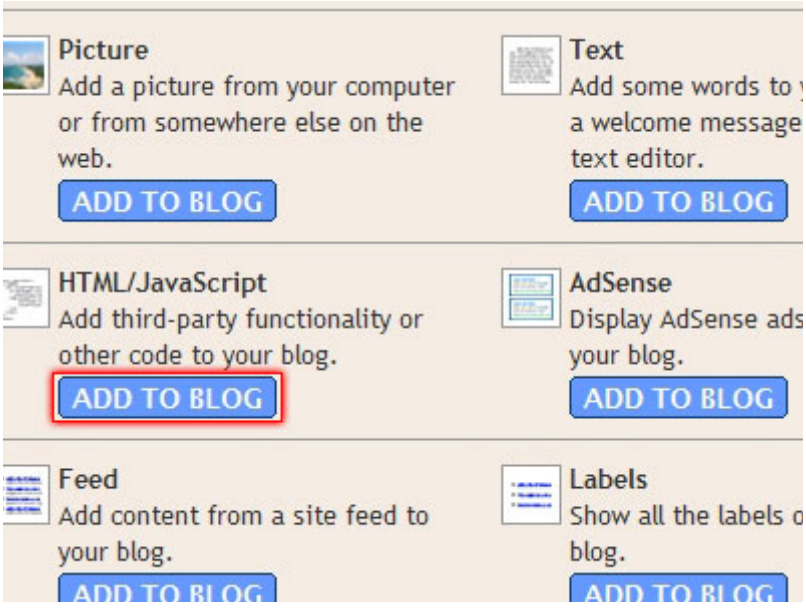

Figure 2. The common File/Information structure.

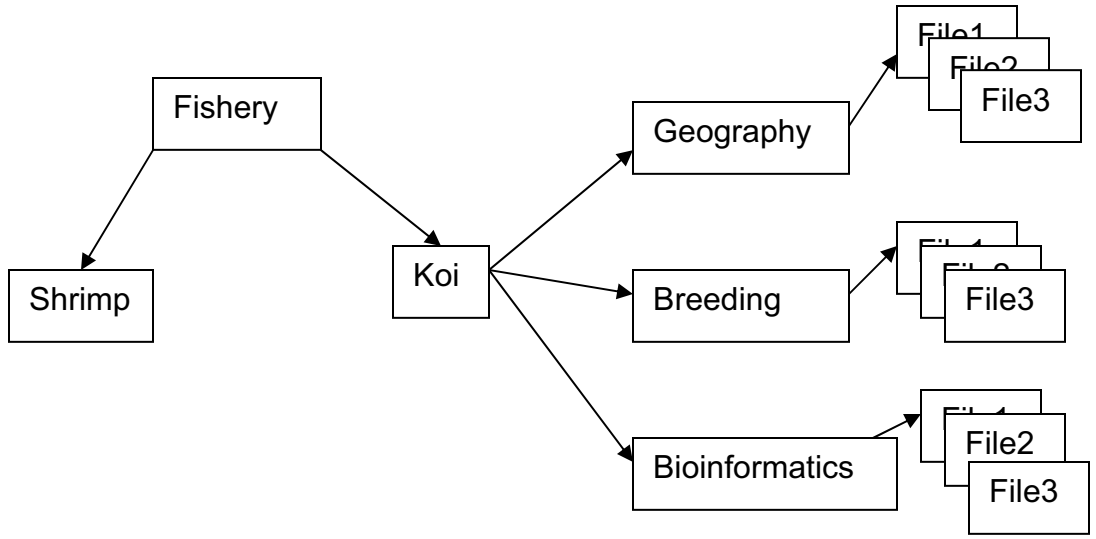

Figure 3. Mind-map diagram and file management

It is clear that mind-map diagram enables the users to track his/her files according to his/her interests, because most people think visually. Of course, the present method to organize files (folders etc), which is based on cabinet system around 1950s, can be retained, provided they can be integrated with the visual/mindmap approach.

Now we discuss some recent thoughts on scientific programs. Meanwhile, Jurgen Habermas, a leading philosopher from the Frankfurt School, opened a whole new can of worms outside of this traditional debate on the 'objectivity of science.' He suggests a quite different argument compared to Popper's objective science. First, we can refer to Lakatos' idea (see Gholson & Barker, 1985) of '*scientific program*,' i.e. the progress of scientific development was actually determined by a group of respective scientists in each area, who also would write recommendations to the governments. While this standard practice is quite common in the most developed countries, especially after the WWII, it has been pointed out by Habermas (1968) that in this respect the scientific development programs themselves are not free of interests, ranging from industrial interests, a country's economics preservation, energy interests, and so on. This is not to say that this practice is wrong by itself, but it is to indicate that it becomes quite difficult to describe these programs as 'objective knowledge', at least in the sense of those ancient Greek scholars who seek knowledge as part of their effort to understand the Logos.

In Habermas' view, it is impossible to perceive that modern sciences follow the same path of these ancient Greek scholars, because in today's modern world, the Logos disappears and it is replaced with 'scientific programs.' To put in other words, what we

study in modern days are not BioLogos, but perhaps BioPrograms, not ZooLogos but ZooPrograms and so on. For example, in Bacon's worldview one can sense that the ultimate 'program' of science is to conquer the world surrounding human. As shown by Fritjof Capra (*The Turning Point*) this kind of philosophy of science led to environmental degradation, etc. See Figure 4 & Figure 5.

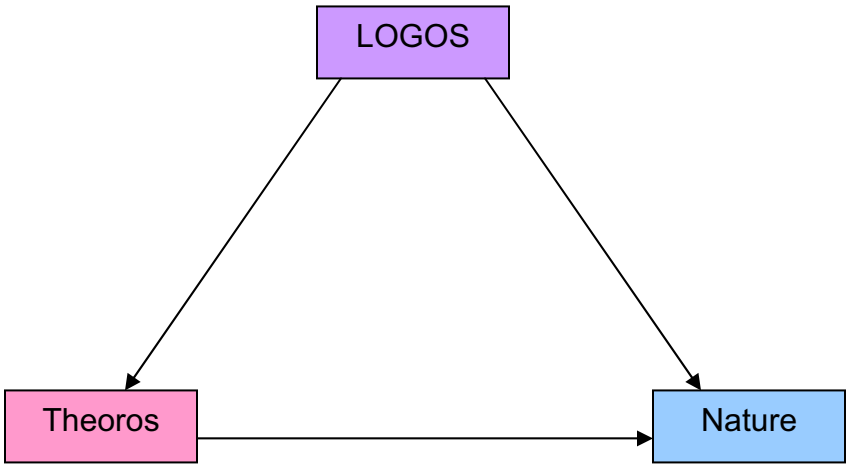

Figure 4. Scheme of method of Inquiry by Ancient Greek scholars

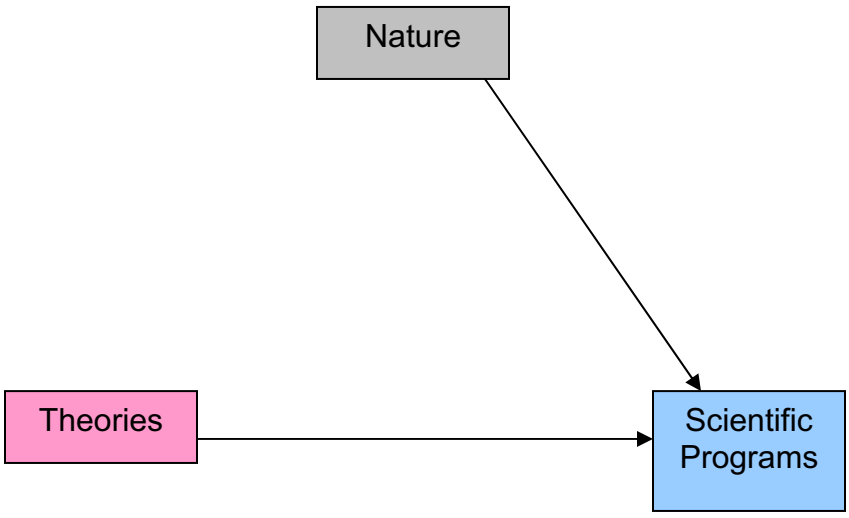

Figure 5. Scheme according to Habermas

To summarize, the methods of Inquiry in our modern times have been influenced by the so-called scientific programs. Of course, at this point one can ask whether is it possible to do research which meet the scientific programs but at the same time meet the ideals of those ancient Greek scholars? And also which is the best possible methods of Inquiry, which can lead one into a new invention or scientific discovery? As we pointed out in the beginning of this section, this question apparently belongs to the most fundamental questions for a scientist.

There are actually a few well-known methods of Inquiry, depending on one's preference:

(a) **Einfuhlung**: this may be a favorite method for Einstein, because he wrote that a physicist should sense something subtle in Nature before he works on the formalism itself.

(b) **Generalizing Math**: this method may be called as Dirac's trick, i.e. consider one equation and try to generalize its math. Thereafter you can look for its plausible implications: Does the new equation imply new physics? At least this method works for Dirac equation, and yield prediction of positron. Another example here is that one can recognize that possible breakthroughs in mathematics come from relaxing Euclid's axioms one by one, for instance by relaxing the fifth axiom (there is only one parallel through a given point to a given line) one can find geometries which go beyond flat surface: i.e. hyperbolic geometry (Lobachevsky-Bolyai-Gauss) – there are many parallels through a given point to a given line, and elliptic geometry (Riemann) – there is no parallel through a given point to a given line. Then go further and combine these geometries, since our universe is not homogeneous but heterogeneous, and consider the Smarandache's multispace, which is formed by a space which can be Euclidean and another space non-Euclidean, or even many spaces put together such that an axiom is valid in one space and invalid in various ways in other spaces (Smarandache geometries).

(c) **Antithesis-Synthesis Dialectic**: this method apparently is more favored by Popper, who suggests that scientific efforts move step by step nearing the hidden truth. Dialectic method was introduced by Hegel, who says that things make progress via creating antithesis and synthesis of what already exist. In other words, one should find out what others have done in a field, and then move on with something that others have not done before.

(d) **Smarandache's Neutrosophic Method**: this method is a generalization of Hegel's dialectic, and suggests that scientific research will progress via studying the opposite ideas and the neutral ideas related to them in order to have a bigger picture.

(e) **Music** (sense of art): you may pick up violin or play flute, guitar or piano, and *voilà!* You discover another great thing like the next SuperDuper-General Relativity theory. Sounds a bit like exaggeration? Perhaps, but according to a study, students with musical ability tend to perceive mathematical principles better. This effect can be explained from the viewpoint that in traditional school, emphasis is given on the left side of the brain, while actually to maximize human brain's potential, one should use both sides in equal way (see also Darvas 2008).

(f) Irrelevant Fiction Stories/Books. You may have heard that Bohr likes Dickens, Einstein and others also liked fiction stories such as Sherlock Holmes. If those books are not available near you, perhaps you can begin doing permutation on random words taken from a dictionary, and find out possible meaning of their combination.

(g) **Climbing or Going to Mountain**: at least this method worked for Heisenberg and plenty of other physicists who sense better grasp on their problems while they were going to mountain. Some people say that going to mountain will give you a sense of unity with the entire Universe, see for example J. Redfield's book (*The Celestine Prophecy*). Not a bad thing to try, at least.

(h) **Lateral Thinking**: if you think that the traditional scientific method is a bit too methodical for you, then perhaps you can try DeBono's lateral thinking. Another way may be called as '*diagonal thinking*', i.e. start with a known premise from one field of science, and then derive conclusions in other field. For example, you start with quantum principles and then derive conclusions for cosmology (i.e. quantum cosmology). Or start with antimatter/antihydrogen and find conclusions in Newtonian mechanics (e.g. is there classical antimatter?) or in Smarandache's unmatter.
And so on. See Figure 6.

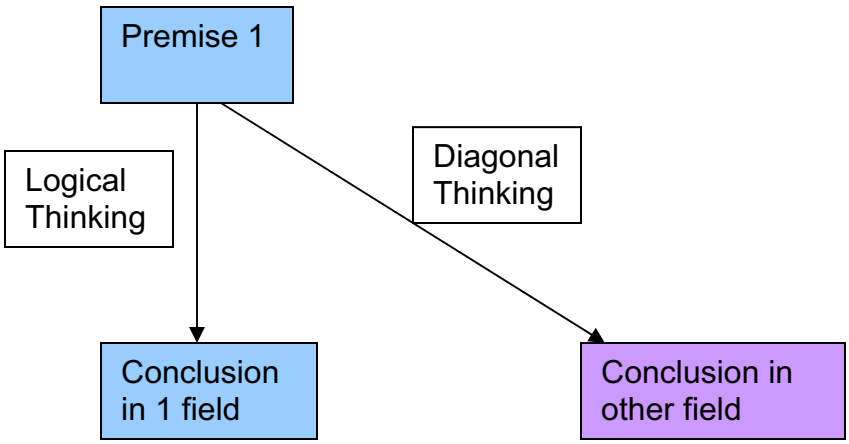

Figure 6. Scheme of diagonal thinking

If this method doesn't sound good to you, perhaps you can try to extend it a bit further, i.e. do diagonal thinking twice and you may call it 'zigzag thinking' (See Figure 7). For instance, to put quantum principles to cosmology is one thing, but you can also find relation from cosmology and particle physics, which is a very active field nowadays, called 'cosmo-particle physics'. And so on, you can also invent your own thinking way which enables you to adapt your specific abilities to your fields of interest.

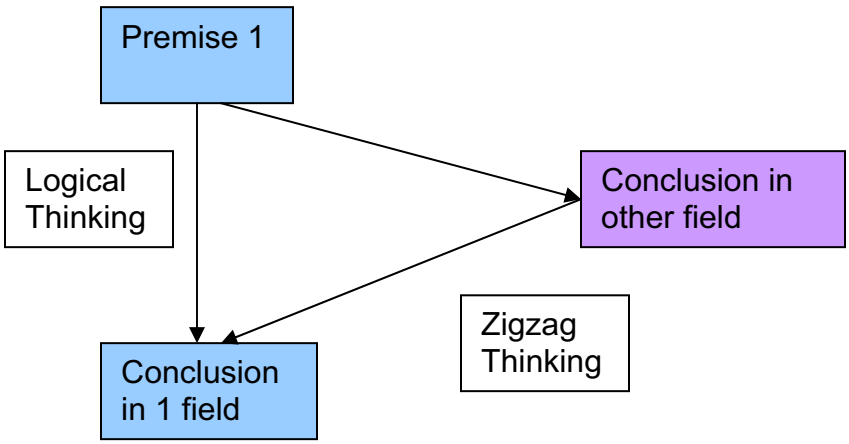

Figure 7. Scheme of zigzag thinking

After citing some of those possible methods of Inquiry, now we're going to discuss on the relation between mathematics and symmetries behind the Nature itself.

Planck and the Symmetries of Nature

In his note on Planck, his favorite figure, Einstein wrote in 1932 (P.M. Robitaille, "Max Planck," *Progress in Physics* vol. 4, Oct. 2007):

> "*Many kinds of men devote themselves to science, and not all for the sake of science herself. There are some who come into her temple because it offers them the opportunity to display their particular talents. To this kind of men, science is a kind of sport in the practice of which they exult, just as an athlete exults in the exercise of his muscular prowess.*
> *There is another class of men who came into the temple to make an offering of their brain pulp in the hope of securing a profitable return. These men are scientists only by the chance of some circumstance which offered itself when making a choice of career.*
> *Should an angle of God descends and drive from the temple of science all those who belong to the categories I have mentioned, I fear the temple would be nearly emptied. But a few worshippers would still remain – some from former times and some from ours. To this latter belongs our Planck. And that is why we love him…"*

According to the above very interesting remark on Planck, Einstein pointed out 3 distinctive attitudes on science which someone (or some groups of scientists) may display: sport, expected return, and true believers. If we wish to find out some parallels between this note and Habermas' viewpoint as discussed above, then perhaps it is quite appropriate to compare those interests and 'expected return' motives; and also between 'true believers' and what Habermas called as '*liberative knowledge.*'

In this regards, it is also worth to mention here that Max Planck's greatest achievement, i.e. the discovery of the true statistical description of blackbody radiation is a good example on how mathematics derivation (with a fair number of premises) can lead scientists to a new and unexpected kind of knowledge.

As discussed by Darvas (2008), mathematics role in science is unavoidable, but how actually mathematics correspond to the Nature itself remains unexplainable, or in Wigner's word "unreasonable effective". In other words, we can accept the role of mathematics to describe Nature because of its effectiveness, although it is unreasonable. By doing so, of course we don't refer here to the ancient belief that Nature itself **is** inherently mathematical (as Pythagoras would say: "The whole thing is a number."). What we refer here is just another saying of Pythagoras: "Mathematics is the way to

understand the Universe." (Darvas 2008) In other words, it is because simply mathematics is the only consistent and effective tool that humankind can use to analyze the world surrounding us.

By mathematics here we do not only refer to the *symmetries, invariance, and transformation principles* that scientists ought to use in order to find the pattern of Nature, but we can also use Wigner's definition:

> "*Mathematics is the science of skillful operations with concepts and rules invented just for this purpose. The principal emphasis is on the invention of concepts*." (Darvas 2008, p. 10)

Or in other words we can find a somewhat simplistic description of mathematics: "A symbolic and formal formulation to express concepts." For instance, there are plenty of formulations to describe logic without introducing '*the principle of excluded middle'*, from Lukasiewicz, until Zadeh's fuzzy logic. Recently, a new kind of logic is developed by F. Smarandache, called 'Neutrosophic Logic', in his effort to unify the mathematical logic, statistics, and philosophy in one theoretical footing. Further implication of this new model of triple-infinite valued logic can be found in mathematical domain known as '*Information Fusion Theory*' (see for example Dezert-Smarandache Theory on paradoxist information).

Therefore, apparently we can say that for the practical (pure) mathematicians, to conceive new mathematics, one does not have to care of its implications. It is task for physical sciences to think of these implications in real world. In other words, we can write the following scheme to describe how the set of mathematical theories can intersect with the set of physical theories, and also intersect with the set of observables. See Figure 8.

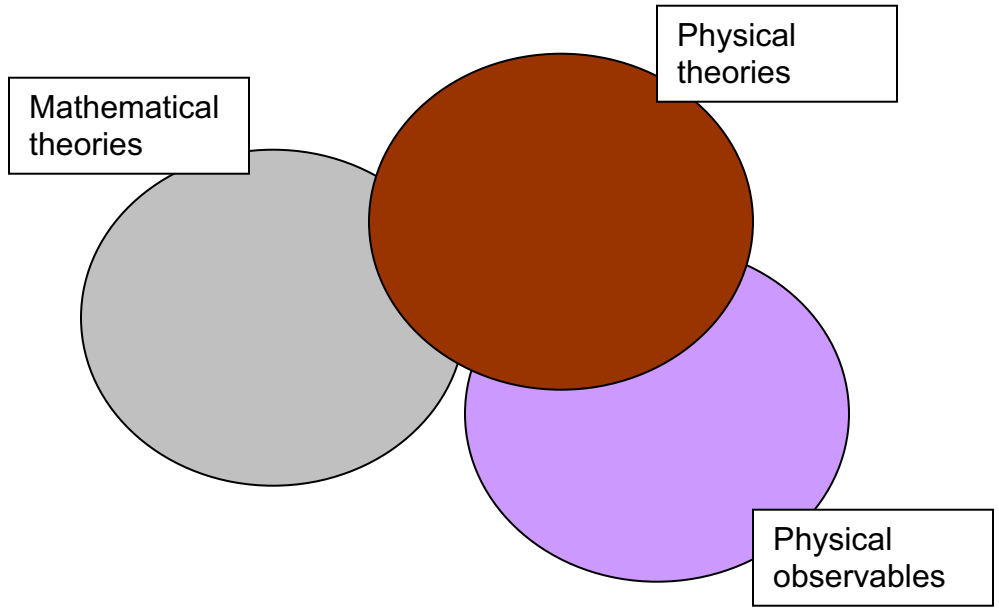

Figure 8. The set of mathematical, physical theories, and observables

One last remark on this section is that some physicists may not agree with what we discuss above, especially those who belong or call themselves 'positivist.' For example, Hawking (arXiv:hep-th/9409195, p.1, 1994) once noted that:

> "*a physical theory is just a mathematical model, and it is meaningless to ask whether it corresponds to reality*."

Bohr himself was also widely reputable as one of the most positivists among others, which led to his famous debate with Einstein on the interpretation of Quantum Mechanics. Einstein of course belongs to 'traditional' physicists who somehow believe that one should find out the deep physics behind Nature, that is why he sought for some kind of 'field' structure to explain the quantum effects, and therefore he considered that Quantum Theory is incomplete. (This open problem has been discussed at length in the *Solvay Conference on Physics* XXII held in Brussels, 2001).

However, from a viewpoint the positivists may be useful, because it should be apparent that in practice we can only speak of the physical observables (measured by some kind of apparatus setting), therefore we don't know what reality is. In particular, if we put Kant's word 'noumena' instead of '*reality*', then Hawking's quote above becomes more make-sense:

> "*a physical theory is just a mathematical model, and it is meaningless to ask whether it corresponds to noumena*."

Therefore a better scheme to represent the classic dichotomy between positivists and the so-called 'realists' is as follows (Figure 9):

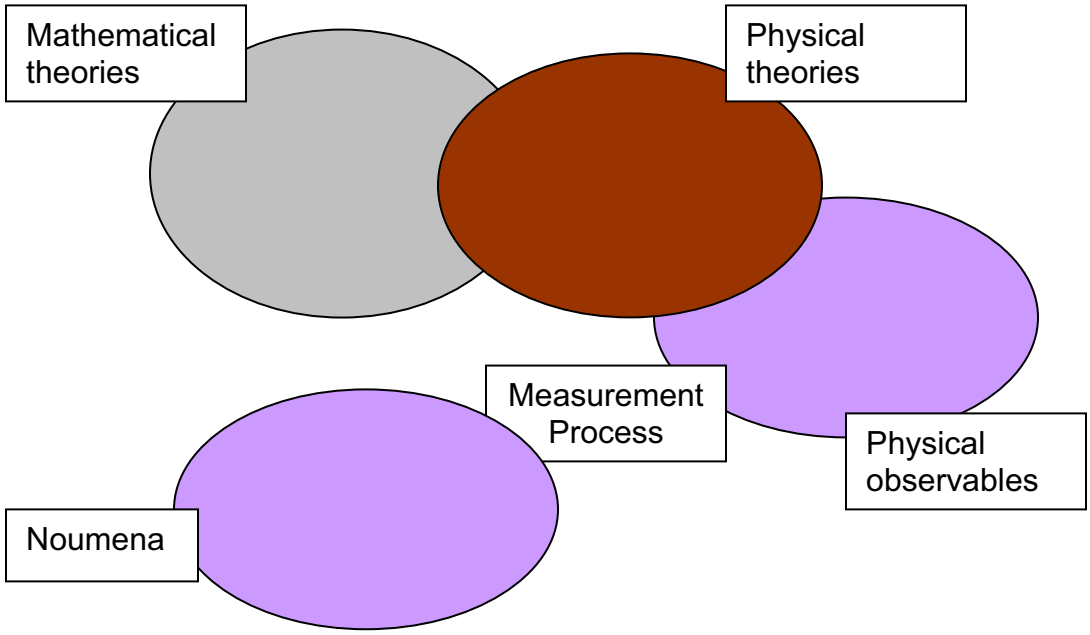

Figure 9. The set of mathematical, physical theories, and observables

From the scheme shown in Figure 6 it should be clearer why the positivists assert that one can only know (speaking of physical theories) the physical observables via measurement process, but not what Nature really is.

We can also conclude from Figure 9 that it is possible that the Noumena does not necessarily fit into our Mathematical knowledge, which seems quite a contradiction with Pythagorean's belief. (Of course, it is also possible to suppose that the set of Noumena inherently correspond to the Mathematical theories.)

Extracting Knowledge from Geometry: Some possible routes

Now if you feel some relief from reading in preceding section that at least in physical theories one can expect the 'gluing' part between mathematical ideas and physical observables, you will never know how physical theories can become so weird, depending on the mathematical notions where they have started from.

For example, one common problem in physical sciences is how to 'extract' knowledge in geometry, for instance a planet's motion or trajectory of satellites. And a fundamental

mathematical concept behind this geometry is the definition of 'distance.' For the beginners, traditionally we use the Cartesian coordinates as follows:

$$ds^2 = dx^2 + dy^2 + dz^2 . \qquad (1)$$

But then physicists began to include time as the fourth component of the metric, to become Minkowski metric:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2.dt^2 , \qquad (2)$$

which is known as the basis of the Special Relativity theory (1905). In the meantime, General Relativity theory uses the non-flat metric with constant curvature, which was introduced by Gauss, Riemann etc.:

$$g = g_{ab}.dx_a \otimes dx^b . \qquad (3)$$

By virtue of the equivalence principle, this pseudo-Riemann metric with constant curvature corresponds to the gravitation phenomena.

While it's instructive to study this pseudo-Riemann metric in order to understand the General Relativity theory, one can consider another dimension(s) to become 5-dimensional or 6-dimensional metrics and so on. There are also some new theories with extra-dimensions, including higher-dimensional gravities, multidimensional gravity theories, and also Smarandache's multispace theories.

This kind of metrics with extra-dimension(s) has become so advanced in the so-called superstring theories, where the most recent theory is so-called 26-dimensional Bosonic string.

It is less mentioned in literature that Riemann himself in one of his talk did make a deliberate remark, mentioning that even the concept of distance and metric are merely construction of human mind. He also suggested possibility to study metric where the metric interval is expressed as the fourth exponent of the distance, i.e. for Minkowski metric it can be written:

$$ds = \sqrt[4]{dx^4 + dy^4 + dz^4 + (ic\ .dt)^4} . \qquad (4)$$

While this metric looks quite awkward, it may be useful for studying gravitation theories, in particular in the context of generalization of pseudo-Riemann metric, for example using the Finsler geometries (e.g. the so-called Berwald-Moor metric), or Smarandache geometries endowed with semi-metrics. (Rabounski, 2010 [9])

It is worth to note here, that one can also consider an extra dimension to Minkowski metric in terms of velocity component, so the metric becomes:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2.dt^2 - \tau^2.dv^2, \qquad (5)$$

Which is the so-called Carmeli metric, and the velocity component corresponds to the galaxy's velocity.

Now, it is possible to find out the symmetries, invariance, and transformation laws corresponding to the above (1)-(5) metrics, and also to figure out their implications to the physical world. Symmetry itself can be defined as "*invariance with respect to a transformation group*" (Esposito & Marmo, 2005).

Another way is to introduce the Hausdorff dimension into the metric, which one allows to consider non-integer dimension. This seems to correspond to the fact that this Earth and other planetary surfaces are far from smooth; therefore the metric of smooth surface is only an approximation. The problem then is how to express the differential geometry principles for this non-smooth metric. Now we face quite a paradox because a surface with Hausdorff non-integer dimension can be non-differentiable or non-integrable, just like Weirstrass function. Then how can one define differential geometry for non-differentiable surfaces? A particularly noteworthy example in this regards is perhaps Nottale's Scale Relativity Theory which defines differentiation on such non-differentiable geometry. Another model of universe based on the non-integrable geometry has been presented by Maciejewski et al. (2002), while Ronchetti & de Sabbata (2002) discussed a quantum gravity model based on the notion of Hausdorff dimension. These novel approaches are mentioned here as mere examples on how different theories can emerge from different assumption of the non-smooth geometry.

At this point, perhaps it is not appropriate to speak of mathematics as the inherent properties of Nature anymore (as Pythagoreans would say), we can only guess what is the most consistent geometry corresponding to a given set of Natural phenomena (known to these days). We can only guess it and hopefully will find the true geometrical structure of Nature, possibly via studying the most generalized type of the metric.

The same principles apparently also apply to the physics of elementary particles or bioinformatics. Without reiterating here what Darvas (2008) has described, especially concerning the role of complex numbers in describing *codon*, or C. M.Yang's method using *quasi-28-gon*, one can note that in elementary particles or bioinformatics, the role of metric in standard physics has been replaced with the 'symmetry principles' of certain groups.

And with respect to the group theories, then it appears that these symmetry principles can be used to extract new knowledge, just as the role that symmetry consideration may have played during formulation of Newton's equations or Maxwell's equations (Darvas 2008).

Nonetheless, there are other types of governing dynamics, for instance the spontaneous symmetry breaking, which can lead to another type or new symmetry principle. How exactly this approach will affect our perception of bioinformatics or the structure of life

itself, remains an open question. For example, does life come from some phenomena related to the spontaneous symmetry breaking of some chemical compounds?

Concluding Note

We have shortly discuss in this article, how scientists including physicists, mathematicians and bioinformatics specialists etc., rely on some special properties in mathematics (via symmetries, transformation and invariance principles) to reveal new kinds of knowledge. These properties are *supposed* to be able to give some clues of the dynamics of the Nature (or better perhaps, of the dynamics of some given observable phenomena).

Nonetheless, as with the *choice* of the groups or the metric to be used, it remains an open question to the scientists themselves. In this regard, one should not force his/her own conception to the Nature. Instead, one can begin to learn and respect the Nature.

Concerning how far the contradiction between these approaches can be, one can re-phrase an old saying reflecting the (quite antagonistic) Baconian world view: "If you torture the data long enough, Nature will confess." The modern version of the same 'attitude' toward Nature perhaps can be written: "If you torture geometry long enough, Nature will confess."

Returning to the Einstein's note on Plank as cited above, the somewhat protagonist view of scientists would learn from Nature and seek to understand it, instead of just forcing Nature to "behave" just as what he/she commend. In other words, apparently it would be better if the physical explanation can be extracted directly from the metric itself plus some new concepts, instead of retaining the same concept but having to "torture" the geometry. In this sense, perhaps one can understand why the General Relativity theory is so fascinating, because it just reinterprets the pseudo-Riemann metric and gives it new physical meaning.

Kaluza-Klein theory also remains beautiful because it only introduces minimal modification to GTR, by including a fifth-component into the metric. But at this point, we don't want to make early remark on other modern theories including supersymmetry, string theories, etc.

Last but not least, by making this quite strong wording on 'torture' of geometry, of course we do not mean that only a handful of approaches are plausible, and other theories shall be forbidden. With regards to mathematical theories, one is free to conceive any kind of idea he/she had, nonetheless at the same time when one develops physical theories, it should be better if they can explain or predict some phenomena where the theories can be put compared with observation. Or if we are allowed to quote what Prof. Gell-Mann once remarked: physicists should find a balance between abstraction and phenomena, just like in Odyssey story one should sail between Scylla and Charibdis.

As for this end of this article, allow me to repeat here a great wisdom saying: *May the force be with you.*

Acknowledgement

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Of intent, citation game, and scale-free networks: A heuristic argument

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A heuristic argument was presented in favor of hypothesis that scientific communication corresponds to a process known as scale-free network. As a result, it is argued that scientific referencing through citation follows the same process, therefore it could be expected that this shall also exhibit fractality as observed in various phenomena associated with scale-free networks. This argument appears conceivable because the process of citation involves a decision-making, coined here as 'citation game.' In this regard, it is recommended to conduct citation analysis to measure the fractality of this process. While at present this heuristic argument cannot be considered as conclusive, further research is recommended to verify or refute this hypothesis.

Keywords: scientific communication, citation analysis, scale-free networks

Introduction

Nowadays, there is a vast amount of scientific publications in this world, with a strong tendency towards more specialized subjects. This phenomenon seems to support Kuhn and Polanyi's viewpoint

that scientific progress is advanced via communication within various scientific societies. Furthermore, scientific communication is conducted in various forms, including: i) attending gathering (lectures, seminars, conferences etc.), ii) reading relevant text (periodicals, reports, textbooks etc.), iii) direct meeting (visiting each other), iv) sending emails; v) visiting online homepage (including arXiv.org). With the advancement and wide availability of TCP/IP-based networks, apparently method iv) and v) are growing fast in popularity among scientists prior to committing in a stronger form of participation in the other three methods of scientific communication. And it seems that this was the original intention of the proposal by Berners-Lee some decades ago.

This article was partly motivated by a recent communication with Prof. M. Pitkänen who happened to see that his TGD entry in Wikipedia has been categorized in 'delete list' [1], merely because some visitors argued that this is his own original research. Regardless of the content of TGD itself, this seemingly common practice by Wikipedia (and also other online publications) obviously raises a question whether such a 'silent voting' is actually acceptable, at least from theoretical viewpoint (i.e. science sociology). This author predicts that similar situation also happens to other scientists, who believe that their wholehearted research was 'deleted' from having a chance to be published or referred in various periodicals, because of the similar 'silent voting' happens.

It seems also worth noting that nowadays citation analysis has been widely used to measure the popularity rate of certain articles, known as 'impact factor' analysis. But it is known that popularity does not equal to the experimental verification required by a scientific theory (Popper). It is not surprising therefore that this process induces some critics, for it is quite similar to other types of mass communication, i.e. rating has replaced depthness, and

popularity has replaced reality. While such popularity-voting methods are generally acceptable in other popular culture, it seems that science demands more than this. However, according to this 'popularity' proponents, scientists' task does not include merely to spread the ultimate reality of Nature *per se*, but to produce *discourses*, i.e. to tell a conceivable story. There is other argument suggesting that science is merely consensus among the experts in a respective field, and therefore popularity could be a good indication of such consensus. Summarizing, it seems that we could expect that the popularity-discourse proponents will argue in favor of this phrase: '*in the land of the blind, the best storyteller will be the king*.' The storyteller therefore is not required to be not (so) blind, suffice it if he could produce good stories of how wonderful the world looks like.

After discussing this citation analysis from some considerations, some implications and plausible future direction of research is discussed.

Wheeler's game: to participate or not to participate

While surely there is other method to describe scientific citation process, for instance using imperfect information theory: "…*scientists may trade ideas to generate citations*," [2] apparently the present method is not conclusive for analytical purposes. Therefore in this article we use another route: the quantum mind hypothesis.

According to J.A. Wheeler, the Universe comes into reality through the participation of its observers; therefore completion of Quantum Mechanics can be viewed in this regard through integrating the role of observers. While this proposition could lead us to a paradoxical ontological question, i.e. whether there is reality without

consciousness (observer), nonetheless this imposes some interesting implications. There are also some recent theories developed around this line of thought, suggesting the role of Mind in Quantum Reality. For instance, Stapp puts forward this hypothesis by arguing that there is distinction between 'Attention', 'Intention' and 'Will' in Quantum Physics [3]. This is a starting premise in the present article.

In this regard, supposed we could accept that scientific progress is determined by dissemination of journals and other kind of scientific publications, then it seems reasonable to expect that science merely consists of accumulation of decision making. This scientific decision making process could be termed as: '*to participate or not to participate'* choice (in Wheeler's sense) of other scientists' viewpoint, which usually was represented in citation. From this reasoning a new term is coined: '*citation game'*, corresponding to quantum-like decision making [4][5][6]. It seems worth therefore to conduct a citation analysis to measure and track backward this process of decision making.

A plausible numerical test: scale-free network hypothesis

Supposed we could accept that various electronic communication methods have gained popularity in recent years among scientists (method iv and v as described above), then it could be expected that the fractality property as observed in scale-free networks of electronic communication [7][8][9][10] could also play a significant role in scientific communication, provided there is no other decisive factor. In other words, it is argued here that provided freedom to publish scientific articles are preserved without restriction, then citation analysis in the respective journals will reveal fractality pattern, because it corresponds to scale-free networks, and *vice versa.* This

method will enable us to conduct a precise analysis of the hypothesis as stated here.

In this article, citation analysis was recommended only in order to test a hypothesis of the scientific decision-making behind the citation process:

a. *Hypothesis*: Number of citation received by most popular scientific articles was attributed to natural phenomena (mutually exclusive events), i.e. it follows fractality property similar to other phenomena associated with scale-free networks.
b. *Null Hypothesis*: Number of citation received by most popular scientific articles doesn't follow fractality property similar to other scale-free network phenomena, because there are other factors involved in the decision-making process.

While at first glance this hypothesis offers nothing new, this is intended to provide a formal basis of such statistical data analysis where citation is the focus of attention.

Now the remaining question is how to provide a numerical test of the proposed hypothesis. As first step, it seems worth to mention here that there are some recent suggestions [11][12] that indeed we live in a fractal world (world inside world). This hypothesis subsequently implies that there are various phenomena, which exhibit fractality, from the scale of particle physics up to astrophysics scale. Accordingly, there are some recent reports suggesting that powerlaw function has a neat linkage to fractality property and also scale-invariance in Nature [13][14]. This could be a plausible basis of numerical test. Accordingly, it is hypothesized here that in scale-free network environment, the relationship between number of citation W and year Y could be expressed in a powerlaw function:

$$W = \alpha . Y^{\beta} \tag{1}$$

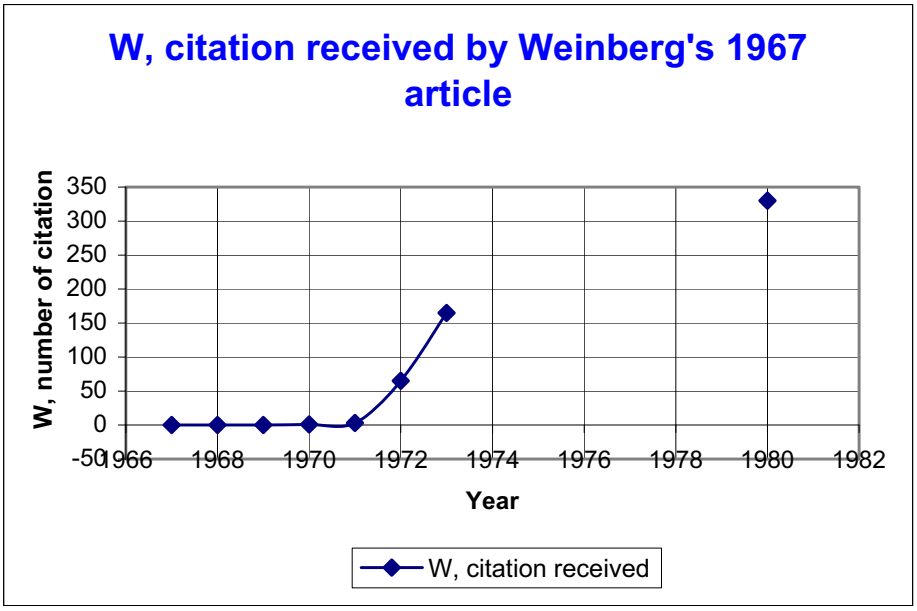
and the power coefficient β is neatly related to fractal dimension d in the form of [13][14]:

$$\beta = f(d) \qquad (2)$$

Table 1. Number of citation received by Weinberg's 1967 article

Year	W, number of citation received
1967	0
1968	0
1969	0
1970	1
1971	3
1972	65
1973	165
1980	330

Graph 1. Graph plot of citation received by Weinberg's 1967



Using the similar line of thought, we could make a simple analysis of citation received by a famous article of Weinberg in 1967. The data was obtained from Weinberg's own book [15], and it is shown in Table 1 and Graph 1. This data was selected merely based on some obvious reasons: i) it has been published in a book form, so it is directly accessible; ii) this article was referred to as "the most frequently cited article on elementary particle physics of the previous half century." [15]

It shall be noted here, that because of lack of data for year 1974-1979, then we do simple linear proportional 'filling' for these years (grey data). This 'normalized' data is presented in Table 2. The subsequent curve fitting method shows that linear and logarithmic regression gives a good correlation ratio of R^2=0.939. The graph plot of these methods is shown in Graph 2.

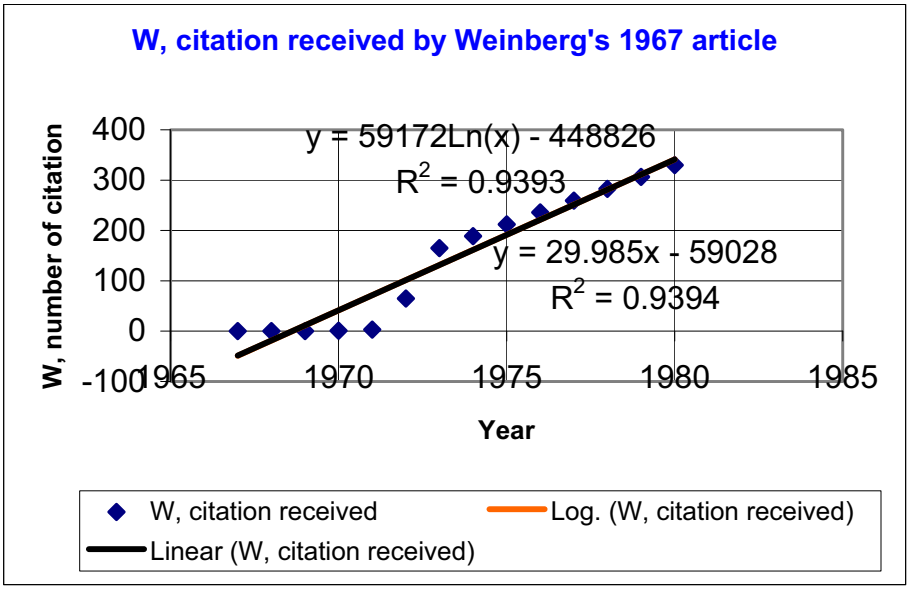
Converting the data to log-natural (ln) scale, we find that log-linear regression gives less correlation ratio R^2=0.6382. The result of this statistical regression is also shown in Table 2. The log-natural scale regression is shown in Graph 3, which obviously shows that parabolic regression at log-scale gives better curve fitting. This subsequently implies that citation received by this article apparently doesn't follow the proposed powerlaw/fractality hypothesis outlined above.

Table 2. Regression analysis of citation

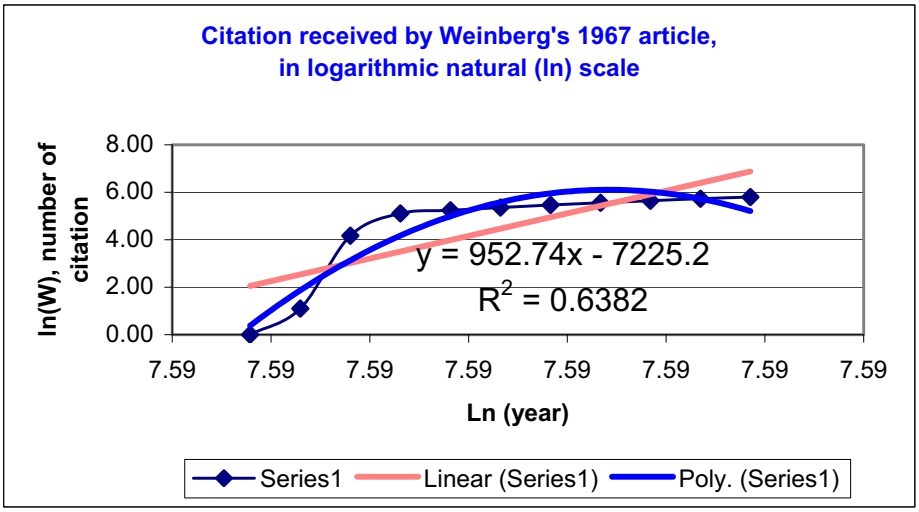
Actual		Logarith.	Linear	Lnscale	Lnscale	Loglinear
Y, Year	W	W'	W''	Ln(Y)	Ln(W)	W'''
1967	0	-49.88	-47.51			
1968	0	-19.81	-17.52			
1969	0	10.25	12.46			
1970	1	40.30	42.45	7.59	0.00	2.08
1971	3	70.33	72.43	7.59	1.10	2.57

1972	65	100.34	102.42	7.59	4.17	3.05
1973	165	130.34	132.40	7.59	5.11	3.53
1974	188.57	160.32	162.39	7.59	5.24	4.02
1975	212.14	190.29	192.38	7.59	5.36	4.50
1976	235.71	220.24	222.36	7.59	5.46	4.98
1977	259.29	250.18	252.35	7.59	5.56	5.46
1978	282.86	280.10	282.33	7.59	5.64	5.95
1979	306.43	310.01	312.32	7.59	5.72	6.43
1980	330	339.90	342.30	7.59	5.80	6.91
R^2		0.9393	0.9394			0.6382
St.dev		8.85	8.84			0.29

Graph 2. Graph plot of linear and logarithmic regression fitting



Graph 3. Graph plot of linear regression fitting in ln scale



The numerical expressions of the above regression lines are as follows:

a) Logarithmic regression at normal scale (W'):

$$W' = 29.985 x Year - 59028 \quad (3)$$

b) Linear regression at normal scale (W"):

$$W'' = 59172 x \ln(Year) - 448826 \quad (4)$$

c) Linear regression at natural logarithmic (ln) scale (W"'):

$$\ln(W''') = 952.74 x \ln(Year) - 7225.2 \quad (5)$$

Because linear regression at log-scale could be directly translated to powerlaw function (1), then we conclude from the data that citation received by Weinberg's article does not follow assumption of scale-free networks. This observation, however, could be attributed to the fact that prior to 1990 the electronic communication was not available to scientists, therefore its scale-free effects were not observed. While surely this statistical analysis is very simple, this method could be used as a preliminary citation analysis. For more extensive data, of

course more advanced statistical techniques are recommended. More extensive study of the scaling properties of journals is available elsewhere [16].

Thanks to the presence of citation database in various leading labs (SLAC, CERN, for instance), it seems possible to conduct such an extensive analysis, particularly using special analysis tools [17]. In turn, it could provide a quantitative picture of how good is the 'freedom to publish' principle has been kept in the real world of scientific journals. Even a negative result could be a good sign of the presence of other factors, which could play a role in the publishing policy of the journals. This method could also be plotted spatially or per journal basis to encourage further analysis of why in some countries people could expect less restriction to publish while this perhaps does not happen in other countries. It is also known that Bose-Einstein condensate with Hausdorff dimension $D_H \sim 2$ could exhibit fractality, so in the near future it could be expected that such scale-free network property could be observed in lab scale [18].

Further research is recommended to verify whether the scale-free network hypothesis of citation data as outlined here is conceivable, corresponding to the observed citation data of scientific articles, particularly in the fields of astrophysics and particle physics (in CERN or SLAC).

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Social archive and the role of new media in scientific dissemination: A viewpoint

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Introduction: the role of new media

Science is of course very far from the art, nonetheless there are some aspects of science which can be compared to art. For instance, there is *elitic art* who prefers that art is for art only. On the other side, there is *pop art*, which relates smoothly to industrialisation. And there is also *avant garde* art, which asserts that all things can be thought of as art (like mirror, glasses, broken windows etc). Similarly, in science some researchers believe that it is the best way to keep the 'ordinary people' outside of the traditional scientific communication (for example, arxiv.org declares that it is an exclusive scientific channel for scientists *only*), while on the other side people sometimes also wants to know what happens behind the wall of scientific labs, and so on.

Enter the social media. Various forms of electronic communication and publication have entered in recent years (2005 up to now) which are sometimes are dubbed as '*new media'* [1][2][4]. This includes, for instance, blogging, youtube, facebook, online directory, blog directory etc. Let's consider a simple example: a decade ago, a new paper in a science journal from a remote country will take some months to be noticed by scientists (in particular via notification by the scientist himself/herself). But today, at the same day the paper appears in electronic journals, there is high likelihood that it will be disseminated simultaneously in numerous forms of new media channels, like google, blogging directories and other indexing services.

The problem is that some scientists feel that a number of scientific works get plenty of publication coverage in this new media, while at the same time an equally 'worthy' paper get less publication coverage. In other words, does it mean that nowadays 'popularity' in new media has replaced what we called before as 'scientific value' of paper? This introduces confusion to already complicated situation of modern society where we listen and read numerous amount of news and information each day, so it is quite make sense that people wants a clarification of 'scientific worthiness' of certain news he/she reads.

Grade of scientification

In order to clarify the situation, we offer a simplified analysis based on the asynchronous/ synchronous communication and also 'grade of scientification', which is a new notion. This grade is defined simply to enable us to rank the channel of communication, which are 'more' serious and

which are less serious, at least from 'scientific worthiness' viewpoint. By 'synchronous' here we mean as method of communication which takes effect immediately (like telephone). See Table 1

Table 1. Grade of Scientific Communication

Type/Grade	Grade A	Grade B	Grade C
Asynchronous	Peer-reviewed Journals, Proceedings, Citation index	Scientific Books, Scientific Magazines, Preprint services, indexing	Popular Science books/magazines, online forum, emails, blogs, online directories, video
Synchronous	Scientific Conferences	Lectures, Public Seminars, semi-formal discussion	Yahoo messenger, Google Talk, other new media, informal talk

Implication of Table 1 would mean that perhaps scientific communication can accept or agree with the fast-growing social media to disseminate scientific works, if only we limit its role as '*Grade C'*, i.e. not to regard them as 'very serious' scientific channel. Furthermore, perhaps we can introduce a new word here 'social archive', in order to reflect both the method of 'social network' as the essence of new media, and the scientific archiving. In other words, we can simplify all these new developments as follows:

Social network + new media + online repository/preprint/indexing == Social archive

How to make Social Archive useful

Scientists improve their work not only by thinking by themselves, but also by receiving comments and suggestions from their peers. Such a method of review has been established in traditional scientific communication, called as 'peer reviewing'. But there are other forms of 'input' that scientists can receive from their 'outer world', for instance what indexing system now begin to call as '*impact factor'* (based on 'Citation'), whatever the bias it may introduce. Subsequently, there is a growing number of the so-called 'citation analysis', focusing on the 'social' influence of certain scientific works.

Another type of input, although not so 'serious' is of course from the public itself, those people which are enthusiastic on the science, either by email, blog posting, etc. Another way is perhaps to introduce some 'rating review' in those blogging, just like amazon.com enables potential book readers to see what others say. In this regards the administrator may enable the comment/rating review be sent to the scientists in order for them to see how their papers may get better response.

Of course, a scientist can always choose either to take care of the 'new media' response, or just get rid of them, and focus on more serious review by his/her peers. Nonetheless, a balanced view may be better, i.e. to consider both channels more or less equally. In this regards the 'periodic table of the social media elements' can be considered too [3].

SAIL: A hypothetical Social Archive Indexing Language

Considering the aforementioned line of thought, it becomes quite apparent that the present system that scientists often use to communicate their workds (indexing, preprint service etc), is not really compatible with the recent development of new media. Therefore, one can think of possibility to introduce a standard method to let the indexing database, let say in XML type (see [6][7][8]) to communicate with blog directories or with atomic feeder systems.

Let's call this hypothetical Indexing Language as SAIL (Social Archive Indexing Language), which perhaps may be compared with INCISO introduced in [5]. For a good comparison, we can start with the SPIRES-HEP's method to indexing entries (based on real data of these writers, see Appendix):

ASTR;
AUTHOR = Smarandache, Florentin;
AUTHOR = Christianto, V.;
AFFILIATION = New Mexico U.;
TITLE = Schrödinger Equation and the Quantization of Celestial Systems;
PUB-NOTE = Prog.Phys.2,2006;
DOI = ;
DATE = Apr 2006;
P = 5;
CITATION =Prog.Phys.4,2006;
EXP = ;
CNUM = ;

Now we can transform this data into XML-type format :

SAMPLE XML FRAGMENT

```
<archive num="gt000x">
  <author1>
        <firstname>George</firstname>
        <lastname>Burdell</lastname>
        <affiliation> New Mexico U.;</affiliation >
 <author1>
 <author2>
        <firstname>V.</firstname>
        <lastname>Christianto</lastname>
        < affiliation > New Mexico U.;</ affiliation >
 <author2>
 <author3>
        <firstname></firstname>
```

```
            <lastname></lastname>
            < affiliation > </ affiliation >
  <author3>
            <TITLE></TITLE>
            <PUB-NOTE></PUB_NOTE>
            <DOI > </ DOI>
            <Date > </ Date>
            <page > </ page>
            <Citation > </ Citation>
            <Exp > </ Exp>
            <CNUM > </ CNUM>
</archive num>
```

Of course, this is only an example, in order to give some illustration on how the existing indexing/database system can be extended slightly to enable them to communicate with the new media repository. The next step is to build communication with the atomic feeder for blogging directories, and so on.

Concluding remarks

The new media has begun to embrace the communication sphere of modern society, or perhaps better, a postmodern society. Therefore new ways to interact with the common people shall be considered by the scientific societies. After all, science moves on not only by making continuous progress in its own, but also because of its interaction with the public sphere...

This article was of course quite elementary, but hopefully would be found useful.

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(1st draft, 7Apr. 2009)

Appendix: Typical Reply from SPIRES administrator

Thank you for the update. We believe that this information has been corrected in SPIRES, and will be searchable within a day or two, depending on the site that you use. Please feel free to send us any further corrections or comments you may have about the databases, or let us know if this is not resolved to your satisfaction.

Best Regards,

SLAC Library
----------Your Initial Request----------

paperData=ADD;
DOC-TYPE = Published;
ASTR;
AUTHOR = Smarandache, Florentin;
AUTHOR = Christianto, V.;
AFFILIATION = New Mexico U.;
TITLE = Schrödinger Equation and the Quantization of Celestial Systems;
PUB-NOTE = Prog.Phys.2,2006;
DOI = ;
DATE = Apr 2006;
P = 5;
CITATION =Prog.Phys.4,2006;
EXP = ;
CNUM = ;
scl=S;hn=From author to SLAC Library (Official use only);
;

submit=Send

PSYCHOLOGY

IMPROVEMENT OF WEBER'S AND FECHNER'S LAWS ON SENSATIONS AND STIMULI

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Abstract.
In this paper one uses a mathematical modeling of psychological processes and one improves the Weber's Law and Fechner's Law on sensations and stimuli.

1991 MSC: 92C20, 92J45, 92J30

Introduction.

According to the neutrosophic theory, between an <*idea*> *(=spiritual)* and an <*object*> *(=material)* there are infinitely many states.

Then, how can we mix an <*idea*> with an <*object*> and obtain something in between: $s\%$ spiritual and $m\%$ material ($s+m=100\%$)? [kind of chemical alloy].

Or, as Boethius, a founder of scholasticism, urged to "join faith to reason" in order to reconcile the Christian judgment with the rational judgment.

Fechner's Law Improvement:

For example <*mind*> and <*body*> co-exist. Gustav Theodor Fechner, who inaugurated the experimental psychology, obsessed with this problem, advanced the theory that every object is both mental and physical (psychophysics).

Fechner's Law, $S = k \log R$, with S the sensation, R the stimulus, and k a constant, which is derived from Weber's Law,

$$\frac{\Delta R}{R} = k$$

with ΔR the increment of stimulus just detectable, should be improved, because the function $\log R$ is indefinitely increasing as $R \to \infty$ to:

$$S(R) = k \frac{\ln R}{\ln R_M},$$

for $R \in [R_m, R_M]$, and $S(R) = 0$ for $R \in [0, R_m) \cup (R_M, \infty)$, where k is a positive constant depending on three parameters: individual being, type of sensation, and the kind of stimulus, and R_m, R_M represent the minimum and maximum stimulus magnitude respectively perceptible by the subject, the second one bringing about the death of sensation.

Fechner's "functional relation", as well as later psychologists' power law

$$R = kS^n,$$

with n depending on the kind of stimulus, were upper unbounded, while the beings are surely limited in perception.

$$S:\ [0,\ \infty)\ \rightarrow\ \{0\}\cup\ [S_m,\ S_M],$$

with S_m, S_M the minimum and maximum perceptible sensations respectively.

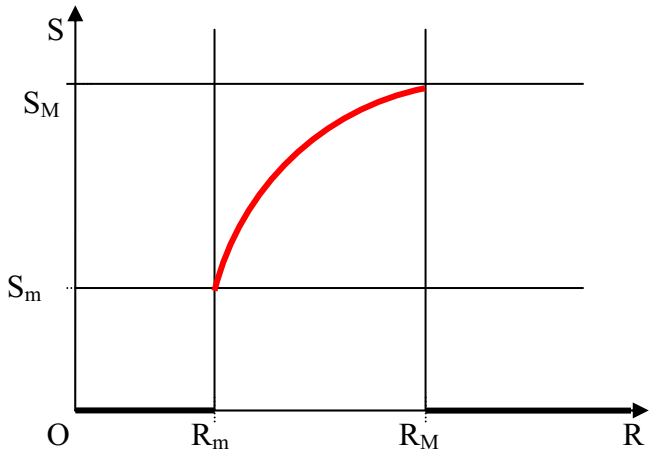
Of course $R_m > 1$, $S(R_m) = S_m$, and $S(R_M) = S_M = k$;

ln, increasing faster, replaces log because the sensation is more rapidly increasing at the beginning, and later going on much slower.

At $R = R_M$, S attaints its maximum, beyond whom it becomes flat again, falling to zero.

The beings have a low and high threshold respectively, a range where they may feel a sensation.

Graph of Fechner's Law Improvement



For example in acoustics: a sound is not heard at the beginning and, if it constantly keeps enlarging its intensity, at a given moment we hear it, and for a while its loudness increases in our ears, until the number of decibels - getting bigger than our possibility of hearing - breaks our eardrums… We would not hear anything anymore, our sensation died...

Now, if at a given moment t_0 the stimulus R remains constant and equal to R_0 (between the conscious limits of the being, for a long period of time t), and the sensation $S(R_0) = c$, then we get the following formulas:

a) In the case when the stimulus is not physically or physiologically damaging the individual being:

$$S_{dec}(t) = c \cdot \log_{1/e}\left(t + \frac{1}{e}\right) = -c\ \ln\left(t + \frac{1}{e}\right), \text{ for } 0 \le t \le \exp\left(-\frac{S_m}{c}\right) - \frac{1}{e},$$

and 0 otherwise;

which is a decreasing function;

b) In the case when the stimulus is hurting the individual being:

$$S_{inc}(t) = c\ \ln(t + e),$$

for $0 \leq t \leq \exp\left(\frac{S_M}{c}\right) - e$, and 0 otherwise;

which is an increasing function until the sensation reaches its upper bound; where c, as a constant, depends on individual being, type of sensation, and kind of stimulus.

Examples:

a) If a prisoner feels a constant smell in his closed room for days and days, isolated from the exterior, and he doesn't go outside to change the environment, he starts to feel it less and less and after a critical moment he becomes inured to the smell and do not feel it anymore - thus the sensation disappears under the low perceptible limit.

b) If a water drop licks constantly, at the same interval of time, with the same intensity, on the head of a prisoner tied to a pillar, the prisoner after a while will feel the water drop heavier and heavier, will mentally get ill and out of his mind, and will even physically die - therefore again disappears the sensation, but above the high limit. See how one can kill someone with a... water drop!

c) If one permanently plays the same song for days and days to a person enclosed in a room without any other noise from outside, that person will be driven crazy, even psychologically die, and the sensation will disappear.

Weber's Law Improvement.

Weber's Law can be improved to $\frac{\Delta R}{\ln R} = k$, with $R \in [R_m, R_M]$, where k is a constant depending on the individual being, type of sensation, and kind of stimulus, due to the fact that the relative threshold ΔR increases slower with respect to R.

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QUANTUM PHYSICS

Some Unsolved Problems, Questions, and Applications of The Brightsen Nucleon Cluster Model

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Abstract.
Brightsen Model is opposite to the Standard Model, and it was build on John Weeler's Resonating Group Structure Model and on Linus Pauling's Close-Packed Spheron Model.
Among Brightsen Model's predictions and applications we cite the fact that it derives the average number of prompt neutrons per fission event, it provides a theoretical way for understanding the low temperature / low energy reactions and for approaching the artificially induced fission, it predicts that forces within nucleon clusters are stronger than forces between such clusters within isotopes; it predicts the *unmatter* entities inside nuclei that result from stable and neutral union of matter and antimatter, and so on.
But these predictions have to be tested in the future at the new CERN laboratory.

Introduction.
According to the Brightsen Nucleon Cluster Model [1] all nuclides of beta stable isotopes can be described by three fundamental nucleon clusters {NPN,PNP,NP), with halo clusters (NN,PP,NNN) now experimentally observed. The Brightsen model builds on the early cluster models of the Resonating Group Structure of John Wheeler [2] and the Linus Pauling Close-Packed Spheron Model [3], which predict mathematically that the wave function of a composite nucleus can be viewed quantum mechanically as a combination of partial wave functions that correspond to the multiple ways nucleons (protons, neutrons) can be distributed into close-packed clusters, thus rejecting the standard model Hartree-Fock formalism of average field interactions between independent nucleons in nuclear shells. Presented in this section are a number of unsolved problems, questions, and future experimental pathways based on the Brightsen Nucleon Cluster Model formalism--many additional applications can be gleamed from careful study of the literature cited in the references provided:

Unsolved Problems, Questions, Applications.
1. The Brightsen Model derives the average number of prompt neutrons per fission event for many radioactive isotopes of human importance (U-235, U-233, Pu-239, Pu-241) as well as emission of light charged particles, suggesting that all modes of fission derive from a four step process [4]. Further study of these claims are warranted given the importance of understanding the fission of radioactive isotopes for energy production.

2. The Brightsen Model provides a theoretical pathway for experimentalists to understand the numerous laboratory results of low temperature transformation/low energy reactions, such as the well studied ^{104}Pd (p, alpha) ^{101}Rh reaction [5]. Application of the Brightsen Model to low energy fusion reactions as a possible result of interactions between nucleon clusters is of fundamental importance to human energy demands.

3. The Brightsen Model predicts the existence of “unmatter entities” inside nuclei [6], which result from stable and neutral union of matter and antimatter nucleon clusters. As a result, the

Brightsen Model predicts that antimatter has corresponding antigravity effects [7]. This prediction can be tested in the future at CERN beginning 2008 using antihydrogen. Once accurate measurements can be made of the gravitational acceleration of antihydrogen, and the results compared with matter hydrogen, if the two forms have opposite acceleration, then a major prediction of the Brightsen Model will be confirmed (e.g., that antimatter has both anti-gravity effect and anti-mass). If experimentally confirmed, then predictive equations will need to be developed using the Brightsen Model formalism of union of matter and antimatter clusters (e.g., the unsolved mathematical formation of unmatter entities inside nuclei). The importance of this aspect of the Brightsen Model links to the current problem in physics of the missing matter of the universe and possible unification of gravity at relativistic (macroscopic) and quantum (microscopic) states.

4. The Brightsen Model offers a theoretical approach for artificially induced fission of dangerous radioactive nuclei to produce relatively stable elements [5]. In theory, if externally produced electromagnetic radiation can be caused to resonate with the exact magnetic moment of a specific sub-nuclear nucleon cluster (e.g., NPN,PNP,NP nucleon clusters), than an individual nucleon cluster can in theory be excited to a energy such that it is expelled from the nucleus, resulting in transmutation of the parent isotope via fission and/or beta or alpha decay to less radioactive daughter structures. The applications of this process for nuclear energy production are clear and worthy of experimental test.

5. The Brightsen Model predicts that one sub-cluster isodyne [5] of the very stable Helium-4 isotope consists of two weakly stable deuteron [NP] clusters, each with their own distinct energy level, spin, magnetic moment, etc. Experimental tests are needed to confirm this fundamental model prediction. If confirmed, new physics mathematical description of shell structure of isotopes would follow.

6. The Brightsen Model predicts that forces “within” nucleon clusters (NPN,PNP,NP) are stronger that forces “between” such clusters within isotopes, a result of different combinations of the spin doublet and triplet clusters. It is predicted that research here would result in new measurable macroscopic properties of atomic nuclei including new fundamental force interactions.

7. The Brightsen Model predicts that the next “magic number” will be found at N = 172, Z = 106, A = 278 (Seaborgium-278). Experimental confirmation of this prediction would require a revised explanation of magic numbers in isotopes based on nucleon clusters as the fundamental building blocks of shell structure in atomic nuclei, as opposed to independent nucleons in an average field.

8. The Brightsen Model predicts that the large cross section of Boron-10 (as opposed to the small cross section of Boron-11) results from the presence of a stable and independent nucleon cluster structure [PNP], which coexists with two [NP] and one [NPN] clusters that maintain very small cross sections. Thus the vast majority of the cross section dynamics of Boron-10 is predicted by the Brightsen Model to derive from a strongly interacting [PNP] cluster. This four cluster formalism for Boron-10 (e.g., 1 PNP, 2 NP, 1 NPN) also correctly derives the I =3 spin experimentally observed.

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Introduction to biquaternion number, Schrödinger equation, and fractal graph

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1. Introduction

It is known that quaternion number has wide application in theoretical physics and engineering fields alike, in particular to describe Maxwell electrodynamics. In the meantime, recently this quaternion number has also been used to draw fractal graph. The present note is intended as an introduction to this very interesting study, i.e. to find linkage between quaternion/biquaternion number, quantum mechanical equation (Schrödinger equation) and fractal graph. Hopefully this note will be found useful for subsequent study.

2. An alternative derivation of Schrödinger-type equation

In this section we will make an attempt to re-derive a Schrödinger-type equation, but with a new definition of total energy.

In this regard, it seems worth noting here that it is more proper to use Noether's expression of total energy in lieu of standard derivation of Schrödinger's equation ($E = \vec{p}^2 / 2m$). According to Noether's theorem [4], the total energy of the system corresponding to the time translation invariance is given by:

$$E = mc^2 + (cw/2).\int_0^\infty \left(\gamma^2.4\pi r^2.dr\right) = k\mu c^2 \tag{1}$$

where k is *dimensionless* function. It could be shown, that for low-energy state the total energy could be far less than $E = mc^2$. In this regard, interestingly Bakhoum [5] has also argued in favor of using $E = mv^2$ for expression of total energy, which expression could be traced back to Leibniz. Therefore it seems possible to argue that expression $E = mv^2$ is more generalized than the standard expression of special relativity, in particular because the total energy now depends on actual velocity [4].

We start with Bakhoum's assertion $E = mv^2$, instead of more convenient form $E = mc_s^{\ 2}$. This notion would imply [5]:

$$H^2 = p^2.c^2 - m_o^{\ 2}.c^2.v^2 . \tag{2}$$

Therefore, for phonon speed in the limit p→0, we write [6]:

$$E(p) \equiv c_s.p \ . \tag{3}$$

In the first approximation, we could derive Klein-Gordon-type relativistic equation from equation (2), as follows. By introducing a new parameter:

$$\zeta = i(v/c) , \tag{4}$$

then we can rewrite equation (2) in the known procedure of Klein-Gordon equation:

$$E^2 = p^2.c^2 + \zeta^2 m_o^{\ 2}.c^4 , \tag{5}$$

where $E = mv^2$. [5] By using known substitution:

$$E = i\hbar.\partial/\partial t , \quad p = \hbar\nabla / i , \tag{6}$$

and dividing by $(\hbar c)^2$, we get Klein-Gordon-type relativistic equation:

$$-c^{-2}.\partial\Psi/\partial t + \nabla^2\Psi = k_o^{'2}.\Psi , \tag{7}$$

where

$$k_o^{'} = \zeta m_o c / \hbar \,. \tag{8}$$

One could derive Dirac-type equation using similar method. Nonetheless, the use of new parameter (4) seems to be indirect solution, albeit it simplifies the solution, because here we can use the same solution from Klein-Gordon equation.

Alternatively, one could derive a new quantum relativistic equation, by noting that expression of total energy $E = mv^2$ is already relativistic equation. We will derive here an alternative approach using Ulrych's [7] method to get relativistic wave function from this expression of total energy [4].

$$E = mv^2 = p.v \tag{9}$$

Taking square of this expression, we get:

$$E^2 = p^2.v^2 \tag{10}$$

or

$$p^2 = E^2 / v^2 \tag{11}$$

Now we use Ulrych's substitution [7]:

$$\left[(P - qA)_\mu (\overline{P} - qA)^\mu \right] = p^2 , \tag{12}$$

and introducing standard substitution in Quantum Mechanics (6), one gets:

$$\left[(P - qA)_\mu (\overline{P} - qA)^\mu \right]\Psi = v^{-2}.(i\hbar.\partial / \partial t)^2 \Psi , \tag{13}$$

or

$$\left[\left(-i\hbar\nabla_\mu - qA_\mu\right)\left(-i\hbar\nabla^\mu - qA^\mu\right) - (i\hbar / v.\partial / \partial t)^2\right]\Psi = 0 . \tag{14}$$

This equation is comparable to Schrödinger equation for a charged particle interacting with an external electromagnetic field [8]:

$$\left[\left(-i\hbar\nabla_\mu - qA_\mu\right)\left(-i\hbar\nabla^\mu - qA^\mu\right)\Psi\right] = \left[-i2m.\partial / \partial t + 2mU(x)\right]\Psi . \tag{15}$$

In other words, we could re-derive Schrödinger-type equation for a charged particle from Ulrych's approach [7].

Alternatively, one can use similar assertion as Schrödinger described in his original equation:

$$E = mv^2 = p^2 / m \tag{16}$$

Using the same method (equation 12), we get:

$$\left[\left(-i\hbar\nabla_\mu - qA_\mu\right)\left(-i\hbar\nabla^\mu - qA^\mu\right) - m.(i\hbar.\partial / \partial t)\right]\Psi = 0 . \tag{17}$$

For m→1, one recovers standard Schrödinger equation [8].

3. Introduction to Quaternion number

Let us begin with a few definitions of numbers. It is known that complex numbers are an extension to the real numbers. They can be seen as two dimensional vectors where also multiplication is defined. We define it as z=a+bi, where a real part, b imaginary part, and i^2= -1.

Complex: z=t+xi
Quaternion: z = t + xi + yj + zk
Octonion: z = t + xi + yj + zk+ aE + bI + cJ + wK

Where: z=(t,V), t=scalar, V=vector.

Furthermore, we can define conjugate complex of z:

$$\vec{z} = a - bi$$

which has properties

$$\vec{z}z = a^2 + b^2 = |z|^2$$

For application of these numbers in quantum physics, see [7][9][10].

4. Introduction to Biquaternion number

Biquaternion numbers are an extension of quaternion to four dimensions [11]. They can be seen as four dimensional vectors (with one scalar and a vector in three space). In physics they are also used in relativity; it is also very useful to describe Maxwell electrodynamics in its original form [7][12].

We could define:

$z = a + bi + cj + dk$ where $i^2=j^2=-1$ and $k^2=jk=1$

For those not familiar with the matrices of Biquaternion and quaternion algebra, here are the tables:

Biquaternion math table

	i	j	k
i	−1	k	−j
j	k	−1	−i
k	−j	−i	1

Quaternion math table

	i	j	k
i	−1	k	−j
j	−k	−1	i
k	j	−i	−1

In both quaternion and Biquaternion math i^2 = -1. The Biquaternion rules provide for one real variable, two complex variables (i and j) and one variable which Charles Muses refers to as countercomplex (k). In quaternion math there is one real variable and three complex variables. In Biquaternion math, unlike quaternion math, the commutative law holds, that is reversing the order of multiplication doesn't change the product.

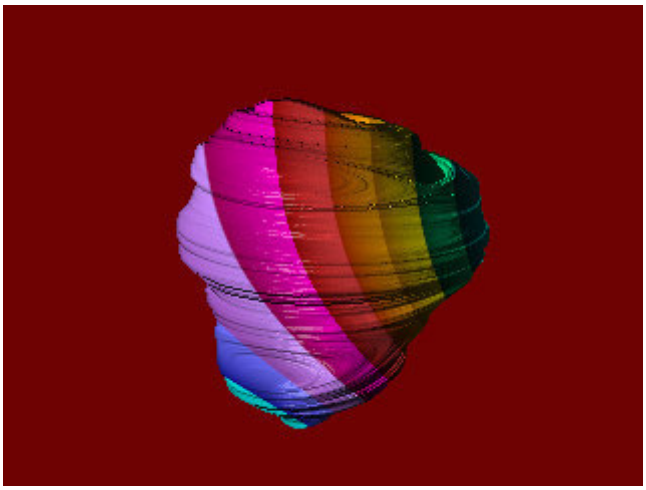
One other concept that mathematicians like to dwell on is the idea of a "ring". There is one ring in quaternion and Biquaternion math, "ijk". If you start anywhere in this ring and proceed to multiply three variables in a loop, backwards or forwards, you get the same number, 1 for Biquaternion, -1 for quaternion [13].

From this viewpoint, we can find further extension of Schrödinger type equation described above to biquaternion form [7][10].

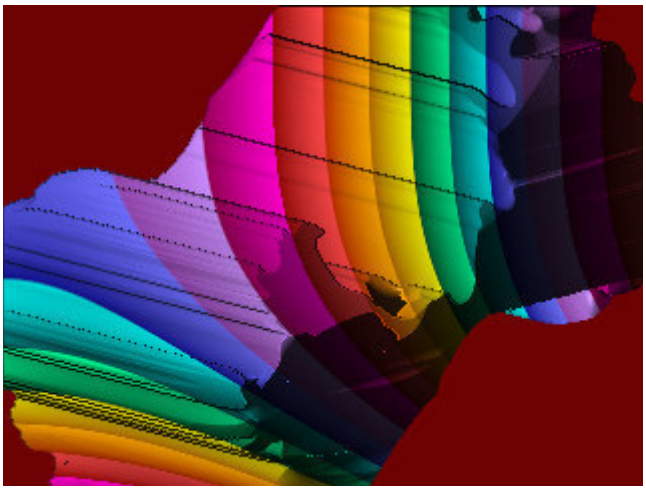
Now we're ready to find simulation of this number via fractal graph. [11]

5. Fractal graph (examples)

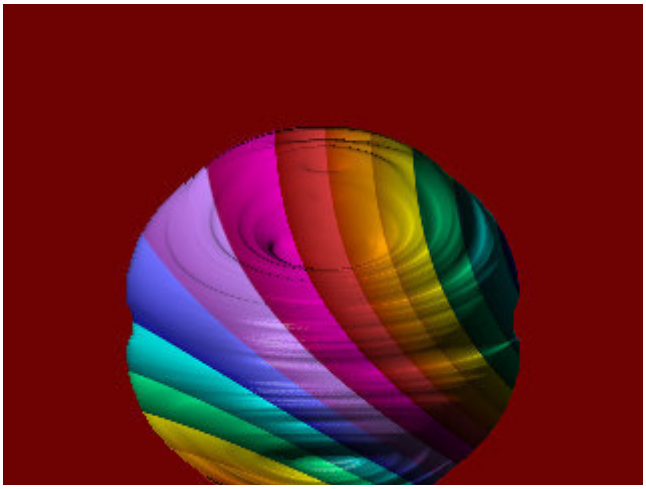
A few examples of fractal graph from quaternion number can be found at www.fraktalstudio.de, and www.bugman123.com. The following graphs were drawn with Dofo-Zon (www.mysticfractal.com), and FractalExplorer ().



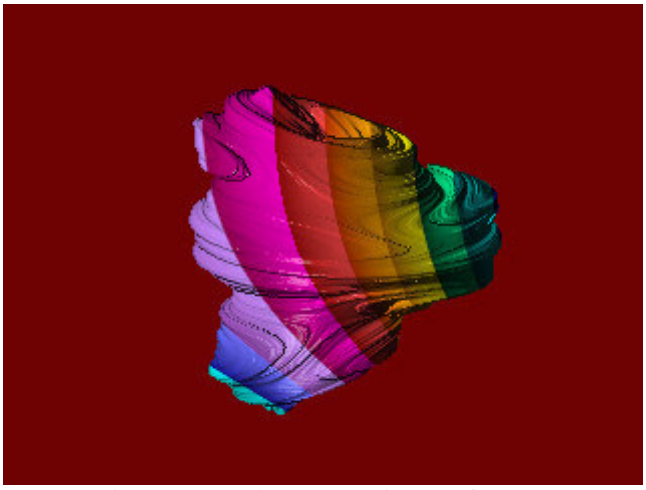
Picture 1. Random quaternion (Dofo-Zon)



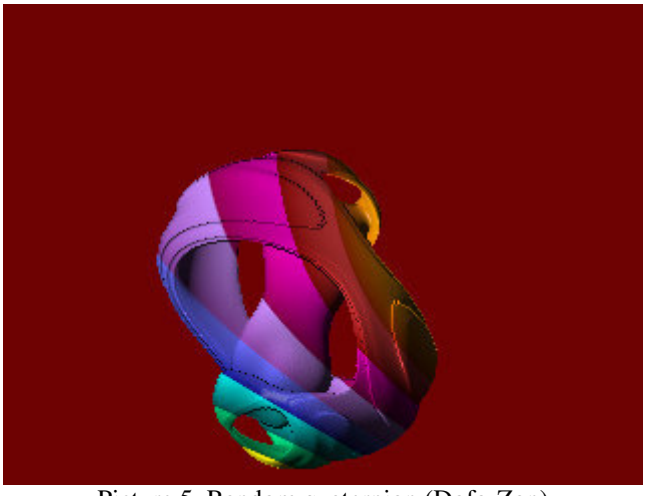
Picture 2. Random quaternion (Dofo-Zon)



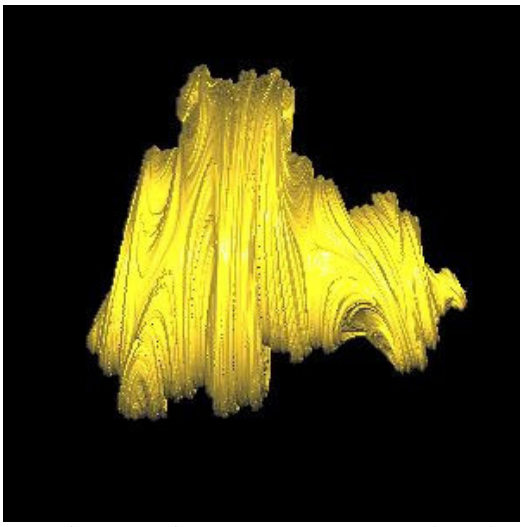
Picture 3. Random quaternion (Dofo-Zon)



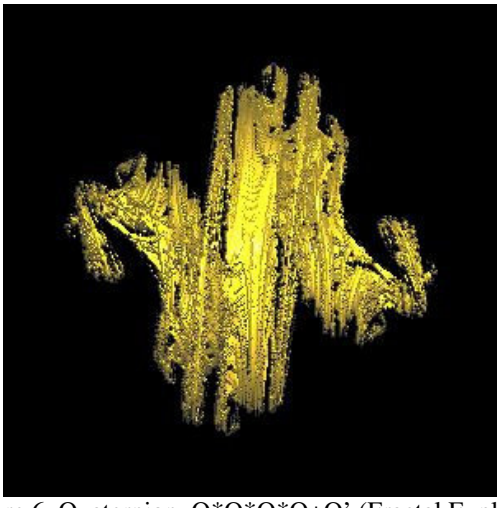
Picture 4. Random quaternion (Dofo-Zon)



Picture 5. Random quaternion (Dofo-Zon)



Picture 6. Quaternion, Q*Q*Q+Q' (Fractal Explorer)



Picture 6. Quaternion, Q*Q*Q*Q+Q' (Fractal Explorer)

Concluding remarks

We have explored some of those stunning images created using the notion of quaternion numbers to draw fractal graphs. This application of quaternion numbers in physics are known, therefore it could be expected that such quaternion/biquaternion fractal graphs will also be found useful in theoretical physics alike. This will be the subject of further exploration.

Dec. 14th, 2005

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Numerical Result of Supersymmetric Klein-Gordon Equation. Plausible Observation of Supersymmetric-Meson

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In the context of some recent papers suggesting CT-symmetric QM in order to generalize PT-symmetric QM, in this paper we present an idea that there is quite compelling reasoning to argue in favour of supersymmetric extension of Klein-Gordon equation. Its numerical solutions in some simplest conditions are presented. Since the potential corresponding to this supersymmetric KGE is neither Coulomb, Yukawa, nor Hulthen potential [2a], then one can expect to observe a new type of matter, which may be called 'supersymmetric-meson'. Its presence may be expected in particular in the process of breaking of Coulomb barrier in low energy schemes. Further observation is of course recommended in order to refute or verify this proposition.

Introduction

In recent years, there is growing interest on various paths of generalization of supersymmetric extension of Quantum Mechanics, for instance using PT-symmetry [2][6] and CT-symmetry [1]. Interestingly, it can be shown that this CT-symmetry or PT-symmetry yield real eigenvalues, and may also correspond to the zeroes of Riemann zeta function [1]. Therefore, it seems interesting to see whether implications of this new symmetry to some known equations in Quantum Mechanics could yield new observables.

In this context, one can argue that it is possible too to extend Klein-Gordon equation using the hypothesis of PT-symmetry. While this idea has been discussed generally in [2], to our present knowledge its solution has not been presented yet up to this time.

Therefore in the present paper, numerical solutions of this PT-symmetric Klein-Gordon equation in some simplest conditions are presented; in particular we consider solution of Klein-Gordon equation with complex valued time-differential operator. Apart from PT-symmetric considerations, our motivation to consider complex valued Klein-Gordon operator comes from the fact that modified Klein-Gordon correspond to quadratic Dirac equation [5]. Since the potential corresponding to this PT-symmetric KGE is neither Coulomb, Yukawa, nor Hulthen potential [2a], then one can expect to observe a new type of matter, which may be called 'supersymmetric-meson'.

First we will find out numerical solution of (known) standard Klein-Gordon equation, and thereafter we consider its PT-symmetric extension. All numerical computation was performed using Mathematica. [8]

Further observation is of course recommended in order to verify or refute the propositions outlined herein.

Numerical solution of Klein-Gordon equation

First we write down the standard Klein-Gordon equation [3, p.9]:

$$\left(\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - \hbar^2\nabla^2 + m^2c^2\right)\varphi(x,t) = 0\,. \tag{1}$$

Alternatively, one used to assign standard value c=1 and also $\hbar = 1$, therefore equation (1) may be written as:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\varphi(x,t) = 0\,, \tag{2}$$

Where the first two terms are often written in the form of square Nabla operator. One simplest version of this equation [3]:

$$-\left(\frac{\partial S_0}{\partial t}\right)^2 + m^2 = 0 \tag{3}$$

yields the known solution:

$$S_0 = \pm mt + cons\tan t \tag{4}$$

The equation (3) yields wave equation which describes a particle at rest with positive energy (lower sign) or with negative energy (upper sign). Radial solution of equation (3) yields Yukawa potential which predicts meson as observables.

It is interesting to note here, however, that numerical solution of equation (1), (2) and (3) yield slightly different result, as follows.

- For equation (1) we get.

```
-h^2D[#,x,x]+(m^2)(c^2)+(h^2/c^2)D[#,t,t])&[y[x,t]]==0
```

$$c^2 m^2 + \frac{h^2\, y^{(0,2)}[x,\,t]}{c^2} - h^2\, y^{(2,0)}[x,\,t] == 0$$

```
DSolve[%,y[x,t],{x,t}]
```

$$\left\{\left\{y[x,\,t] \to \frac{c^2 m^2 x^2}{2h^2} + C[1]\left[t - \frac{\sqrt{-c^2h^4}\,x}{c^2h^2}\right] + C[2]\left[t + \frac{\sqrt{-c^2h^4}\,x}{c^2h^2}\right]\right\}\right\}$$

- For equation (2) we get.

```
( -D[#,x,x]+m^2+D[#,t,t])&[y[x,t]]==0
```

$$m^2 + y^{(0,2)}[x,\,t] - y^{(2,0)}[x,\,t] == 0$$

```
DSolve[%,y[x,t],{x,t}]
```

$$\left\{\left\{y[x,\,t] \to \frac{m^2 x^2}{2} + C[1][t-x] + C[2][t+x]\right\}\right\}$$

- For equation (3) we get.

```
( m^2-D[#,t,t])&[y[x,t]]==0
```

$$m^2 - y^{(0,2)}[x, t] == 0$$

DSolve[%,y[x,t],{x,t}]

$$\left\{\left\{y[x, t] \to \frac{m^2 t^2}{2} + C[1][x] + t\, C[2][x]\right\}\right\}$$

One may note that this numerical solution is in quadratic form ($\frac{m^2 t^2}{2} + cons\tan t$), therefore it is rather different from equation (4).

Numerical solution of Klein-Gordon equation with complex valued time-differential operator.

As it has been discussed in the context of quaternion Quantum Mechanics [5], it may be useful to consider complex valued Klein-Gordon operator in lieu of standard Nabla operator in equation (2). Therefore, here we rewrite a plausible extension of equation (2) and (3) as follows:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) + i\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\right]\varphi(x,t) = -m^2, \qquad (5)$$

And for equation [3] we can write:

$$\left[\left(\frac{\partial S_0}{\partial t}\right)^2 + i\left(\frac{\partial S_0}{\partial t}\right)^2\right] = m^2. \qquad (6)$$

Numerical solutions for these equations were obtained in similar way with the previous equations:

- For equation (5) we get.

(-D[#,x,x]+D[#,t,t]-I*D[#,x,x]+I*D[#,t,t]+m^2)&[y[x,t]]==0

$$m^2 + (1 + i)\, y^{(0,2)}[x, t] - (1 + i)\, y^{(2,0)}[x, t] == 0$$

DSolve[%,y[x,t],{x,t}

$$\left\{\left\{y[x, t] \to \left(\frac{1}{4} - \frac{i}{4}\right) m^2 x^2 + C[1][t - x] + C[2][t + x]\right\}\right\}$$

- For equation (6) we get.

(- m^2+D[#,t,t]+I*D[#,t,t])&[y[x,t]]==0

$$-m^2 + (1 + i)\, y^{(0,2)}[x, t] == 0$$

DSolve[%,y[x,t],{x,t}]

$$\left\{\left\{y[x, t] \to \left(\frac{1}{4} - \frac{i}{4}\right) m^2 t^2 + C[1][x] + t\, C[2][x]\right\}\right\}$$

At this point one may note that supersymmetric extension of Klein-Gordon equation, in particular by introducing complex-valued differential operator yields quite different solutions compared to known standard solution of Klein-Gordon equation (4), i.e. in the form:

$$y(x,t) = \left(\frac{1}{4} - \frac{i}{4}\right) m^2 t^2 + cons \tan t \tag{7}$$

Since the potential corresponding to this PT-symmetric KGE is neither Coulomb, Yukawa, nor Hulthen potential [2a], then one can expect to observe a new type of matter, which may be called 'supersymmetric-meson'. If this new type of particles can be observed in near future, then it can be regarded as early support to the new hypothesis of PT-symmetric and CT-symmetric as considered by some preceding papers [1][2][6].

In our opinion, its presence may be expected in particular in the process of breaking of Coulomb barrier in low energy schemes.

Further observation is also recommended in order to verify and explore further this proposition.

Concluding remarks

In this paper we present an idea that there is quite compelling reasoning to argue in favour of supersymmetric extension of Klein-Gordon equation. Its numerical solutions in some simplest conditions are presented.

Since the potential corresponding to this supersymmetric KGE is neither Coulomb, Yukawa, or Hulthen potential, then one can expect to observe a new type of matter, which may be called 'supersymmetric-meson'. Its presence may be expected in particular in the process of breaking of Coulomb barrier in low energy schemes. Further observation is of course recommended in order to refute or verify this proposition.

It is recommended to conduct further observation in order to verify and also to explore various implications of our propositions as described herein.

Acknowledgment

The writers would like to thank to Profs. C. Castro & D. Rapoport for valuable discussions.

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Numerical solution of Schrödinger equation with PT-symmetric periodic potential, and its Gamow integral

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Abstract
In a number of preceding papers we introduced a new PT-symmetric periodic potential, derived from biquaternion radial Klein-Gordon equation. In the present paper we will review our preceding result, and continue with numerical solution of Gamow integral for that periodic potential. And then we also compare with other periodic potentials which are already known, such as Posch-Teller or Rosen-Morse potential. We also discuss a number of recent development in the context of condensed matter nuclear science, in particular those experiments which are carried out by Prof. A. Takahashi and his team from Kobe University. There is hint to describe his team's experiment as 'mesofusion' (or mesoscopic fusion). We then analyze possibility to enhance the performance of Takahashi's mesofusion experiment under external pulse field. Further experiments are of course recommended in order to verify or refute the propositions outlined herein.

a. Introduction

In a number of preceding papers we introduced a new PT-symmetric periodic potential, derived from biquaternion radial Klein-Gordon equation. [1][2] In the present paper we will review our preceding result, and continue with numerical solution of Gamow integral for that periodic potential. And then we also compare with other periodic potentials which are already known, such as Posch-Teller or Rosen-Morse potential [9][10][11].

We also discuss a number of recent development in the context of condensed matter nuclear science, in particular those experiments which are carried out by Prof. A. Takahashi and his team from Kobe University [6][7]. There is hint to describe his team's experiment as 'mesofusion' (from mesoscopic fusion). We then analyze possibility to enhance the performance of Takahashi's mesofusion experiment under external pulse field.

Further experiments are recommended in order to verify or refute the propositions outlined herein.

b. PT-symmetric periodic potential and its Gamow integral

In this section, first we will review our preceding result on the periodic potential based on radial Klein-Gordon equation, and then we discuss its numerical solution for Gamow integral.

There were some attempts in literature to introduce new type of symmetries in Quantum Mechanics, beyond the well-known CPT symmetry, chiral symmetry etc. In this regards, in recent years there are new interests on a special symmetry in physical systems, called PT-symmetry with various ramifications.

It has been argued elsewhere that it is plausible to derive a new PT-symmetric Quantum Mechanics (PT-QM) which is characterized by a PT-symmetric potential [3][4]:

$$V(x) = V(-x)\,. \tag{1}$$

One particular example of such PT-symmetric potential can be found in sinusoidal-form potential:

$$V = \sin\alpha\,. \tag{2}$$

PT-symmetric harmonic oscillator can be written accordingly [3]. Znojil has argued too [4] that condition (1) will yield Hulthen potential:

$$V(\xi) = \frac{A}{\left(1-e^{2i\xi}\right)^2} + \frac{B}{\left(1-e^{2i\xi}\right)}. \tag{3}$$

In our preceding paper [2][5], we argue that it is possible to write biquaternionic extension of Klein-Gordon equation as follows:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) + i\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\right]\varphi(x,t) = -m^2\varphi(x,t)\,, \tag{4}$$

Or this equation can be rewritten as:

$$\left(\Diamond\overline{\Diamond} + m^2\right)\varphi(x,t) = 0\,, \tag{5}$$

Provided we use this definition:

$$\begin{aligned}\Diamond = \nabla^q + i\nabla^q &= \left(-i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right)\\ &+ i\left(-i\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z}\right)\end{aligned} \tag{6}$$

Where e_1, e_2, e_3 are *quaternion imaginary units* obeying (with ordinary quaternion symbols: e_1=i, e_2=j , e_3 =k):

$$\begin{aligned}&i^2 = j^2 = k^2 = -1\,,\ ij = -ji = k\,,\\ &jk = -kj = i\,, ki = -ik = j\,.\end{aligned} \tag{7}$$

And quaternion *Nabla operator* is defined as [2][5]:

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} \tag{8}$$

Note that equation (8) already included partial time-differentiation.

Therefore one can expect to use the same method described above to find solution of radial biquaternion KGE [2][5].

First, the standard Klein-Gordon equation reads:

$$\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\varphi(x,t)=-m^2\varphi(x,t)\,. \tag{9}$$

At this point we can introduce polar coordinate by using the following transformation:

$$\nabla=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\ell^2}{r^2}\,. \tag{10}$$

Therefore by introducing this transformation (10) into (9) one gets (by setting $\ell=0$):

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+m^2\right)\varphi(x,t)=0\,. \tag{11}$$

Using similar method (10)-(11) applied to equation (5), then one gets radial solution of BQKGE for 1-dimensional condition [2][5]:

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right)-\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right)+m^2\right)\varphi(x,t)=0\,, \tag{12}$$

Using Maxima computer package we find solution of (12) as a new potential taking the form of sinusoidal potential:

$$y=k_1\sin\left(\frac{|m|r}{\sqrt{-i-1}}\right)+k_2\cos\left(\frac{|m|r}{\sqrt{-i-1}}\right), \tag{13}$$

Where k_1 and k_2 are parameters to be determined. Now if we set k_2 =0, then we obtain the potential function in the form of PT-symmetric periodic potential (2):

$$V=k_1\sin(\alpha)\,, \tag{14}$$

Where $\alpha=\left(\frac{|m|r}{\sqrt{-i-1}}\right)$.

In a recent paper [8], we interpret and compare this result from the viewpoint of EQPET/TSC model which has been suggested by Prof. Takahashi in order to explain some phenomena related to Condensed matter nuclear Science (CMNS).

c. Schrödinger equation and Gamow integral of PT-symmetric periodic potential

Now let us consider a PT-Symmetric potential of the form:

$$V=k_1.\sin(\beta.r)\,, \tag{15}$$

where

$$\beta=\frac{|m|}{\sqrt{-i-1}}\,. \tag{16}$$

Hence, the respective Schrödinger equation with this potential can be written as follows:

$$\Psi''(r) = -k^2(r).\Psi(r) \tag{17}$$

Where

$$k(r) = \frac{2m}{\hbar^2}[E - V(r)] = \frac{2m}{\hbar^2}[E - k_1.\sin(b.r)] \tag{18}$$

For the purpose of finding Gamow function, in area near x=a we can choose linear approximation for Coulomb potential, such that:

$$V(x) - E = -\alpha(x - a), \tag{19}$$

Substitution to Schrödinger equation yields:

$$\Psi'' + \frac{2m\alpha}{\hbar^2}(x - a)\Psi = 0 \tag{20}$$

which can be solved by virtue of Airy function.

In principle, the Gamow function can be derived as follows:

$$\frac{d^2y}{dx^2} + P(x)y = 0 \tag{21}$$

Separating the variables and integrating, yields:

$$\int \frac{d^2y}{y} = \int -P(x).dx \tag{22}$$

Or

$$y.dy = \exp(-\int P(x).dx) + C) \tag{23}$$

To find solution of Gamow function, therefore the integral below must be evaluated:

$$\gamma = \sqrt{\frac{2m}{\hbar^2}[V(x) - E]} \tag{24}$$

The general expression of Gamow function then is defined by:

$$\Gamma \approx \frac{1}{\eta^2} = \exp(-2\int_a^b \gamma(x)dx) \tag{25}$$

Therefore it should be clear that we can find different solutions for any given form of potential. In the present paper we will only consider a few potential, namely Takahashi's block-type potential (he called it STTBA model), and our PT-symmetric periodic potential. Rosen-Morse potential will be compared for the results only.

c.1. Takahashi's STTBA-block-type potential

For the case of Takahashi experiment [3][4][5], we can use b=5.6fm, and r_0=5fm, where the Gamow function is given by:

$$\Gamma = 0.218\sqrt{\mu}.\int_{r0}^{b}(V_b - E_d)^{1/2}.dr \tag{26}$$

Where he obtained V_b=0.256 MeV.

c.2. PT-symmetric periodic potential (14)

Here we assume that E=V_b=0.257MeV. Therefore the integral becomes:

$$\Gamma = 0.218\sqrt{\mu}.\int_{r0}^{b}(k_1 \sin(\beta r) - 0.257)^{1/2}.dr \tag{27}$$

By setting boundary conditions:

(a) at r=0 then Vo=-Vb—0.257 MeV
(b) at r=5.6fm then V1= $k_1 \sin(br) - 0.257$ =0.257Mev,therefore one can find estimate of m.
(c) Using this procedure solution of the equation (11) can be found.

The interpretation of this Gamow function is the tunneling rate of the fusion reaction of cluster of deuterium (with the given data) corresponding to Takahashi data, with the difference that here we consider a PT-symmetric periodic potential.

c.3. Rosen-Morse potential [8]

Another type of potential which may be considered here is known as Rosen-Morse potential [9][10]:

$$v = -2b.\cot|z| + a(a+a).\csc^2|z|, \tag{28}$$

Where z=r/d. Therefore the Gamow function can be written, respectively:

$$\Gamma = 0.218\sqrt{\mu}.\int_{r0}^{b}((-2b.\cot|z| + a(a+a).\csc^2|z|) - 0.257)^{1/2}.dr \tag{29}$$

(This section is not complete yet).

Some new findings indicating Condensed matter nuclear science and Mesofusion

In this section, we can mention that the most obvious objection against cold fusion is that the Coulomb wall between two nuclei makes the mentioned processes extremely unlikely to happen at low temperature. We can also mention here that there are three known reaction types in thermo fusion:

a. D+D → ^{4}He+γ (23.8 MeV)
b. D+D → ^{3}He+n
c. D+D → ^{3}He+p

In this regards we would like to mention here some clear reasons why cold fusion cannot be analyzed in the classical framework of fission or 'thermo' fusion:

a. No gamma rays are seen;
b. The flux of energetic neutron is much lower than expected on basis of the heat production rate;
c. Lack of signature of D-D reaction;
d. Isotopes of Helium and also tritium accumulate to the Pd samples;
e. Cold fusion appears to occur more effective in Pd nano-particles [6][7];
f. The ratio of x to D atoms to Pd atoms in Pd particle must be in the critical range [0.85,0.90] for the process to occur.

Other strict experimental conditions may also be considered before we can expect repeatability of this process. In this regards, a recent experiment in Arata Hall, Osaka University, on May 22 2008 by Arata has clearly demonstrated that this process did happen. Because the experiment took place at Arata-Zhang laboratory, it then was referred to as Arata-Zhang experiment [6]. Other teams also produced excellent results, for example Prof. Takahashi and his Kobe University team [7].

The basic element of Takahashi's series of experiments is that a periodic potential of the Bloch wave type, as shown in the Figure 1 below.

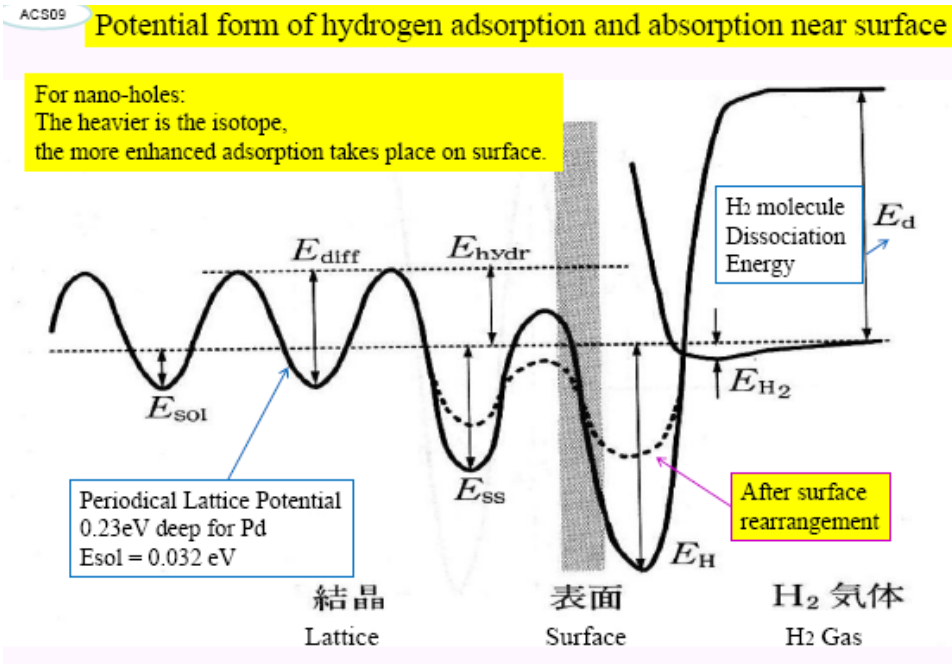

Figure 1. Lattice periodic potential used by Takahashi et al. [7]

From another line of reasoning, one can also consider this possibility of low-temperature fusion. Consider the heat production in our Earth, that some researchers consider it produced by nuclear fission or fusion. But considering that the Earth is lacking uranium (by statistical distribution), chance is that fission is unlikely, but the temperature inside the Earth is clearly much lower than

the Sun, therefore the hotfusion is also unlikely to happen. Therefore apparently we can infer that inside the Earth, the heat is produced either as Condensate Nuclear transmutation (CMNS), or other types of low-energy nuclear reaction (LENR).

In other words, if we would like to keep ourselves a bit open-minded, then there other questions too which we don't find quick answer even in the natural processes surrounding us. This would mean as an indication that new types of transmutation processes should be taken into consideration as a possibility.

In this regards perhaps it would be useful to discuss a possible categorization of these new possibilities beyond standard (thermo) fusion process:

a. CANR: or chemically aided nuclear reaction, which essentially uses special types of chemical substance or enzymes [8]. For instance, see hydrino experiments (hydrino.org). Other chemists may prefer to use isoprenoids to create this new effect.
b. LENR: low-energy nuclear reaction [8], or some researchers may prefer to call it 'Lattice fusion Reaction', that is perhaps a more proper name for cold-fusion and other types of deuterium reaction which happens far below the Gamow energy. The name 'lattice fusion' also implies that the process includes neutron in some kind of solid-state physics. An indication that the fusion associated to LENR is outside the domain of standard fusion processes is lack of signature of D-D reaction, which would mean that perhaps the process is much more complicated (for instance Takahashi considered tetra-deuterium model). There is also indication of lacking of neutron emission during this process [7]. We will discuss more on these issues in subsequent section.
c. Mesofusion (or mesoscopic fusion): this belongs to experiments which can be associated to nano-Pd samples for instance by Takahashi and his team in Japan [6]. While this term is not well accepted yet, in our opinion this type of reactions will be much more common in particular for industrial applications, since nanometer devices are much more manageable rather than materials at the order of lepton or hadron scale.

Concluding remarks: Next steps

We would like to conclude this note with a number of some kinds of wish-list.

First of all, a rigorous theoretical framework is clearly on demand. This for instance, will include both to clarify the distinction between Mesofusion and Chromodynamics fusion, and also to consider new type of potentials.

And then, in terms of experiments it appears to be more interesting to introduce new types of tools in order to enhance the performance of these Mesofusion or Chromodynamics fusions. For instance, perhaps it would be interesting to see whether the performance can be improved by introducing either laser or external electromagnetic pulse, just like what has been done in the conventional thermo fusion.

All of these remarks are written here to emphasize that based on recent publication [5]-[8], we are clearly in the beginning of observing new types of fusion technologies, by harnessing our knowledge of hadron and chromodynamics theory.

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Generalized Quaternion Quantum Electrodynamics from Ginzburg-Landau-Schrödinger type Equation

(Proposed Research Abstract)

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Despite incomparable achievement of Quantum Electrodynamics and its subsequent theories, there are some known limitations and unsolved theoretical problems until this time, including 'renormalization' condition [1][2] and its generalization to larger systems. While renormalization problem has been declared as 'settled', yet it is known for their own founding fathers (Feynman & Dirac, for instance) this question remains unsolved satisfactorily. Other known problems include limitation to explain anti-hydrogen phenomena [5][39], and confinement problem in quantum chromodynamics theory.

In the meantime, electrodynamics theories have advanced beyond established stage and it has become possible to extend these theories to include self-similarity (scale-invariance). There are also some recent interests to re-consider quaternion and biquaternion numbers in describing electrodynamics phenomena in original form as conceived by Maxwell.

For generalised case, it could also be expected that by using quaternion numbers we could also achieve scale-invariant quantum electrodynamics, which could yield explanation for quantization of celestial systems [3], which have been observed in recent years. For these known reasons, the sought-after theory will be called here: Generalized Quaternion Quantum Electrodynamics from Ginzburg-Landau-Schrödinger type equation, or for simple term (GQQED). It is expected that by the end of tenureship, the basic principles of this sought-after new theory could have been formulated and presented in understandable way.

1 Research description

Despite incomparable achievement of Quantum Electrodynamics and its subsequent theories, there are some known limitations and unsolved theoretical problems until this time, including 'renormalization' condition [1][2] and QED generalization to larger systems. Other known problems include limitation to explain anti-hydrogen phenomena, and confinement problem in quantum chromodynamics theory.

In the meantime, there are some recent interests to re-consider quaternion and biquaternion numbers [2][6][7][9] for describing electrodynamics phenomena in original form as conceived by Maxwell. Therefore, it seems possible to generalize this new approach to use quaternion/biquaternion number towards a new Quaternionic Quantum Electrodynamics theory, which is free from renormalization problem. It appears that the new theory should be consistent with topological

electronic interpretation of QED [10][23][27][28][30], which could also arrive at the same Bohr-type quantization condition for large-scale systems [28].

The research will be conducted in a few steps as follows:

a. literature survey: examine historical development on the use of quaternion/biquaternion numbers in QED;
b. theoretical development: algebraic structures of quaternion/biquaternion numbers & interpretation;
c. derive implications of the theory: derive implications of the proposed theory and to find physical phenomena corresponding to the theory. This step includes making quantitative prediction;
d. data collection: collect quantitative data from astrophysical observation etc.
e. comparison: compare observed data and theory;
f. experiment: develop method to verify theory for practical purposes, for instance using scale-invariant

quantum electrodynamics theory to build better antenna systems.

Methodology to be used in this proposed research is 50% pure theoretical investigation, 30% data collection and analysis, and 20% experimental work.

2 Significance of the proposed research

In recent years, there are numerous exoplanet observations [11][12][13], which could be predicted via Bohr-type quantization condition with a remarkable precision [14][15].

An alternative method to describe this quantization of celestial system is by generalizing quantum electrodynamics. It is known that quantum electrodynamics (QED) is one of the most profound discoveries in the past century, but it has not been used to describe classical-celestial systems.

By generalizing quantum electrodynamics, the proposed research could open a new way of thinking the nature of astroparticle physics field.

3 How an Institution could contribute

The Institution should be well known for its high reputation in frontier research in various fields, including astrophysics. Therefore the applicants believe that there are numerous previous observation data which could be collected and re-organized in much more meaningful way, provided the new hypothesis is available (including perhaps exoplanets data, planetary migration, planetary precession, etc.).

In the meantime, there should be senior fellows in the Institution who also work in areas related to planetary formation and migration, which perhaps could contribute to the research to be conducted herein.

4 Possible Research Advisors

From the list of Smithsonian Institution scholars, there are some scholars, who perhaps would like to be research advisor for this proposed research:

- K. Kirby (Bose-Einstein condensate and astroparticle)
- Rudolph E Schild (Navier-Stokes and cosmology)
- Charles J. Lada (star & planet formation)

For co-advisor / consultant, the applicant has identified a few scholars:

- Robert P. Kirshner (accelerating universe hypothesis)
- Willie Soon (Earth and planetary studies)
- L. Hartmann (senior astrophysicist / lecturer)

Nonetheless, along the way of this research, the authors would like to consider numerous discussions with other research fellows within or outside the Smithsonian Institute, in particular those who have conducted previous experimental/theoretical works in the similar line of research (i.e. new advancement of QED theories).

5 Estimate of time period: 12 (twelve) months (max.).

6 Estimated budget (research allowance)

While the majority of activities only include mathematical/theoretical development, by the end of tenureship we expect to build example of practical tool, which could serve as 'model' where the proposed theory could play a role. For instance, the applicants expect to develop a new method of antennae design for electronic transmitter or wireless communication.

To build such a practical tool, it is required to purchase raw material and toolkit. We expect to build four or five antenna designs as an alternative of present design (with various scales from small-scale to full-scale antenna). List of tool expected is described in section #7.

The estimated budget is around $4,000 (for four – five antenna designs), unless these tools could be found in lab without necessarily purchasing them.

January 8, 2006

7. List of toolkit expected:

The present estimate to conduct experiment includes:

- one (1) Weller soldering iron by Cooper Tools;
- one (1) mini tubing cutter;
- one (1) mini drilling tool;
- one (1) handheld dril;
- electric rod;
- electric Copper wire;
- stainless steel plate (2 mm);
- N Connector;
- RF Connector;
- Multitester.

Other tools/materials as per need.

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Unleashing the Quark within:
LENR, Klein-Gordon Equation, and Elementary Particle Physics

(Preliminary report)

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Introduction.
Recently we've read that there is an excellent Cold Fusion experiment performed by Prof Arata, showing that the promise of CF/LENR (Low Energy Nuclear Reaction) is rekindled.

With regards to this experiment, in our opinion part of the problem is to explain how the intraatomic interactions happen in low temperature. A hint on this issue is that perhaps what we know about QM is flawed under the fact of antihydrogen, see Van Hoydoonk [1]-[5]. And considering topological quantization, then can we expect to observe Bohr-Sommerfeld quantization inside the quarks too?

Of course, we don't mean to say that focusing on CF/LENR is because we're inclined to this kind of fusion, but because of our conviction to the idea that deep inside the nuclei, the structure resembles condensed matter physics (or superconductor), either using Ervin Goldfain's CGLE model, 'compressed hydrogen' (Rutherford), or Wilczek's theory [6]. Furthermore, one can find another hint by studying the Klein-Gordon equation for elementary particles, which suggest that deep inside the hadronic interaction is governed by boson. Similar conjecture can be found from Interactive Boson Model.

Here are comments from some fellows physicists on how this elementary particle can be understood via Klein/Gordon or condensed matter physics. They address these simple questions:

- Do you think that we can further extend your KGE to become quaternion Klein-Gordon equation? (see Nottale et al. [8]).

- Is it possible to replace Higgs field with boson field (reminiscent to Schwinger model)? See for instance the paper by Fujita et al. [7]

Comments by:

(a) Takehisha Fujita [7]

You may try to think of any possibilities of constructing fundamental scalar fields (complex fields, of course) in some way or the other. But I believe that this should not be a proper starting point for the scalar field. Simply there exists no fundamental scalar field which can couple to the electromagnetic field. I believe you may find a good explanation of these theoretical points in the textbook "Symmetry and breaking in quantum field theory." The Higgs mechanism itself is physically not acceptable. Unfortunately, people have been pretending that they understood the symmetry breaking theory, without examining its physics in depth. But in reality they did not understand the basic point of the vacuum structure in the symmetry breaking physics. The success of the Glashow-Weinberg-Salem model is entirely due to the final version of their Lagrangian density which has nothing to do with the gauge theory.

(b) M. Apostol

Dirac equation can be derived from Klein-Gordon equation by using quaternions. However, in curved spaces, this may raise problems, and fractal geometry seems to be needed in addition. A convenient covariance seems to be a prerequisite with quaternions, and this is not known to me. The difficulties reside in noncommutativity. As regards the Higgs, I incline to think that it should be a real scalar, after breaking, not a boson. After all, Schwinger model is essentially one-dimensional.

(c) Ervin Goldfain

My explanation is that Wilczek and other theorists from his generation belong to a school of thought that is no longer effective in explaining many experimental observations and "anomalies". This fact is one of the reasons progress in particle theory has been so slow. This generation has been trained primarily in perturbative Quantum Field Theory (QFT), Feynman diagrams and Path Integrals. These methods are mainly applicable in equilibrium QFT but fail almost completely when used in critical phenomena, nonlinear dynamics and chaos, complex behavior, phase transitions in extended systems, self-organized criticality, non-extensive statistical physics and so on. The problem is that many of such "old school" theorists are not ready to acknowledge that these traditional techniques simply do not work when studying nonlinear, open and irreversible systems and processes. I am not the only one that says that: there are studies that have reported this "unwillingness" or lack of training in modern analytic tools.

There are indeed many opportunities for developing condensed matter theory to a point where certain cooperative phenomena (such as cold-fusion) become better understood. Quantum phase transitions (phase transitions at low temperatures) and the physics of strongly correlated quantum systems in different dimensions are two prime examples of topics that are under active investigation.

A similar type of issues are present when talking about phase transitions in Quantum Chromodynamics, a theory with an unexpectedly rich spectrum of behaviors. Understanding quark-gluon plasma,

restoration of chiral symmetry, formation of strange bound states of quarks (quarkonia) and gluons (glueballs) and so on, may also help explaining room-temperature collective phenomena such as cold-fusion.

The physics (and the spectroscopy) of macroscopic states involving anti-matter (anti-hydrogen and the like) are also far from being completely understood. It is my view that, until one is able to explain the mechanism of CP symmetry breaking in field theory, one is not in a position to comprehend the underlying physics of anti-matter. It is here where complexity theory (so called emergent physics) and approaches using CGLE may be of practical value.

With regards to CMNS/LENR experiments, it would be fascinating to come up with a sound theoretical model explaining these CF experiments. There are at least three avenues to such a model:

1) quantum phase transitions at sufficiently low temperatures (above 0 K). See work by Subir Sachdev and others.

2) mesoscopic non-equilibrium thermodynamics for quantum systems. See work by D. Bedeaux and P. Mazur et al.

3) non-equilibrium phase transitions (by analogy with reaction-diffusion processes). See work by Lubeck, Hinrichsen and others.

Hope this discussion will be found a bit useful.

FS, VC

{First version: 14th June 2008}

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Appendix A: Cold Fusion experiment performed by Prof Arata (from [9],[10],[11])

June 11th, 2008

On 23 March 1989 Martin Fleischmann of the University of Southampton, UK, and Stanley Pons of the University of Utah, US, announced that they had observed controlled nuclear fusion in a glass jar at room temperature, and — for around a month — the world was under the impression that the world's energy woes had been remedied. But, even as other groups claimed to repeat the pair's results, skeptical reports began trickle in. An editorial in *Nature* predicted cold fusion to be unfounded. And a US Department of Energy report judged that the experiments did "not provide convincing evidence that useful sources of energy will result from cold fusion."

This hasn't prevented a handful of scientists persevering with cold-fusion research. They stand on the sidelines, diligently getting on with their experiments and, every so often, they wave their arms frantically when they think have made some progress.

There is a reasonable chance that the naysayers are (to some extent) right and that cold fusion experiments in their present form will not amount to anything. But it's too easy to be drawn in by the crowd and overlook a genuine breakthrough, which is why I'd like to let you know that one of the handfuls of diligent cold-fusion practitioners has started waving his arms again. His name is Yoshiaki Arata, a retired (now emeritus) physics professor at Osaka University, Japan. Yesterday, Arata performed a demonstration at Osaka of one his cold-fusion experiments.

Although I couldn't attend the demonstration (it was in Japanese, anyway), I know that it was based on reports published here and here. Essentially Arata, together with his co-researcher Yue-Chang Zhang, uses pressure to force deuterium (D) gas into an evacuated cell containing a sample of palladium dispersed in zirconium oxide (ZrO2–Pd). He claims the deuterium is absorbed by the sample in large amounts — producing what he

calls dense or “pynco” deuterium — so that the deuterium nuclei become close enough together to fuse.

So, did this method work yesterday? Here’s an email I received from Akito Takahashi, a colleague of Arata’s, this morning:

> “Arata’s demonstration…was successfully done. There came about 60 people from universities and companies in Japan and few foreign people. Six major newspapers and two TV [stations] (Asahi, Nikkei, Mainichi, NHK, et al.) were there…Demonstrated live data looked just similar to the data they reported in [the] papers…This showed the method highly reproducible. Arata’s lecture and Q & A were also attractive and active.”

I also received a detailed account from Jed Rothwell, who is editor of the US site LENR (Low Energy Nuclear Reactions) and who has long thought that cold-fusion research shows promise. He said that, after Arata had started the injection of gas, the temperature rose to about 70 °C, which according to Arata was due to both chemical and nuclear reactions. When the gas was shut off, the temperature in the centre of the cell remained significantly warmer than the cell wall for 50 hours. This, according to Arata, was due solely to nuclear fusion.

Is There Iso-PT Symmetric Potential in Nature?

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In recent years there are new interests on special symmetry in physical systems, called PT-symmetry with various ramifications. Along with the isodual symmetry popularized by RM Santilli, these ideas form one of cornerstone in hadron physics. In the present article, we argue that it is plausible to generalise both ideas to become iso-PT symmetry which indicate there should be new potential obeying this symmetry. We also discuss some possible interpretation of the imaginary solution of the solution of biquaternionic KGE (BQKGE); which indicate the plausible existence of the propose iso-PT symmetry. Further observation is of course recommended in order to refute or verify this proposition.

Introduction

There were some attempts in literature to generalise the notion of symmetries in Quantum Mechanics, for instance by introducing CPT symmetry, chiral symmetry, etc.

In recent years there are new interests on special symmetry in physical systems, called PT-symmetry with various ramifications. Along with the isodual symmetry popularized by RM Santilli, these ideas form one of cornerstone in hadron physics. In the present article, we argue that it is plausible to generalise both ideas to become iso-PT symmetry which indicate there should be new potential obeying this symmetry. We also discuss some possible interpretation of the imaginary solution of the solution of biquaternionic KGE (BQKGE); which indicate the plausible existence of the propose iso-PT symmetry.

This biquaternion effect may be useful in particular to explore new effects in the context of low-energy reaction (LENR) [14]. Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

Basic ideas: PT-Symmetric Potential and Isoselfdual Symmetry

It has been argued elsewhere that it is plausible to derive a new PT-symmetric Quantum Mechanics (PT-QM) which is characterized by a PT-symmetric potential [1]:

$$V(x) = V(-x)\,. \tag{1}$$

One particular example of such PT-symmetric potential can be found in sinusoidal-form potential:

$$V = \sin\alpha\,. \tag{2}$$

PT-symmetric harmonic oscillator can be written accordingly [2]. Znojil has argued too [1] that condition (1) will yield Hulthen potential:

$$V(\xi) = \frac{A}{\left(1-e^{2i\xi}\right)^2} + \frac{B}{\left(1-e^{2i\xi}\right)}. \tag{3}$$

Interestingly, the similar Hulthen potential has often been cited with respect to the isodual symmetric proposed by RM. Santilli in a number of published works [3][4]. Therefore it appears quite interesting to find out generalization of these types of symmetries to become (iso-PT symmetry). In other words, we would like to ask in this paper, whether there is isoselfdual-PT symmetric potential in nature, which is the subject of the present paper.

Now we're going to discuss some remarkable result from isoselddual theory popularized by RM. Santilli under the flagship of Hadronic Mechanics (HM theory). With regards to isodual symmetry we note some basic relations in according with [3][4]:

*The imaginary unit is isoselfdual because [3, p.8]:

$$i^d = -\bar{i} = i \tag{4}$$

*The correct left and right multiplicative unit [3, p.6]:

$$A \times^d I^d = I^d \times^{d^d} A = A \tag{5}$$

*The isodual functional analysis also includes [3, p.8]:

$$\sin^d \theta^d = -\sin(-\theta) \tag{6}$$

Therefore, with respect to the aforementioned basic relations of isoselfdual theory, then a new generalization can be sug-

gested, i.e. an isoselfdual-PT symmetry is such that the potential follows this relation:

$$` V^{d}{}_{isoselfdual}(x) = V^{d}(-x). \tag{7}$$

In other words, a possible solution of equation (7), with respect to the isodual functional analysis (6) and (2), can be given by:

$$V^{d} = -\sin(-\alpha). \tag{8}$$

The next section will discuss solution of biquaternion Klein-Gordon equation [5][7] and how it will yield a sinusoidal form potential with appears to be related either to (2) or to (8). See also [8].

Review of solution of biquaternionic Klein-Gordon equation

In our preceding paper [5], we argue that it is possible to write biquaternionic extension of Klein-Gordon equation as follows:

$$\left[\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right) + i\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\right]\varphi(x,t) = -m^2\varphi(x,t), \tag{9}$$

Or this equation can be rewritten as:

$$\left(\Diamond\overline{\Diamond} + m^2\right)\varphi(x,t) = 0, \tag{10}$$

Provided we use this definition:

$$\begin{aligned}\Diamond = \nabla^q + i\nabla^q &= \left(-i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right) \\ &+ i\left(-i\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z}\right)\end{aligned} \tag{11}$$

Where e_1, e_2, e_3 are *quaternion imaginary units* obeying (with ordinary quaternion symbols: e_1=i, e_2=j , e_3 =k):

$$\begin{aligned}&i^2 = j^2 = k^2 = -1,\ ij = -ji = k, \\ &jk = -kj = i, ki = -ik = j.\end{aligned} \tag{12}$$

And quaternion *Nabla operator* is defined as [5]:

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} \tag{13}$$

Note that equation (11) already included partial time-differentiation.

It is worth nothing here that equation (10) yields solution containing imaginary part, which differs appreciably from known solution of KGE [5]:

$$y(x,t) = \left(\frac{1}{4} - \frac{i}{4}\right)m^2t^2 + cons\tan t \tag{14}$$

Solution of radial biquaternion Klein-Gordon equation and a new sinusoidal form potential

One can expect to use the same method described above to find solution of radial biquaternion KGE [7][8].

First, the standard Klien-Gordon equation reads:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi(x,t) = -m^2\varphi(x,t)\,. \tag{15}$$

At this point we can introduce polar coordinate by using the following transformation:

$$\nabla = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\ell^2}{r^2} \tag{15a}$$

Therefore by introducing this transformation (15a) into (15) one gets (setting $\ell = 0$):

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + m^2\right)\varphi(x,t) = 0\,. \tag{16}$$

Using similar method (15)-(16) applied to equation (10), then one gets radial solution of BQKGE for 1-dimensional condition [7][8]:

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) + m^2\right)\varphi(x,t) = 0\,, \tag{17}$$

Using Maxima computer package we find solution of (18) as a new potential taking the form of sinusoidal potential:

$$y = k_1 \sin\left(\frac{|m|r}{\sqrt{-i-1}}\right) + k_2 \cos\left(\frac{|m|r}{\sqrt{-i-1}}\right), \qquad (18)$$

Where k1 and k2 are parameters to be determined.

In a recent paper [8], we interpret and compare this result from the viewpoint of EQPEt/TSC model which has been suggested by Prof. Takahashi in order to explain some phenomena related to Condensed matter nuclear Science (CMNS).

Nonetheless what appears to us as more interesting question is whether it is possible to find out proper generalisation of PT-symmetric potential (1) to become isoselfdual-PT symmetric potential (7). Further theoretical and experiments are therefore recommended to verify or refute the proposed new isoselfdual-PT symmetric potential in Nature.

Acknowledgment

Special thanks to Prof. C Castro for bringing this idea of PT-QM to our attention.

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On the Meaning of Imaginary Part of Solution of Biquaternion Klein-Gordon Equation

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In the preceding article we argue that biquaternionic extension of Klein-Gordon equation has solution containing imaginary part, which differs appreciably from known solution of KGE. In the present article we discuss some possible interpretation of this imaginary part of the solution of biquaternionic KGE (BQKGE). Further observation is of course recommended in order to refute or verify this proposition.

Some interpretations of preceding result of biquaternionic KGE

In our preceding paper [1], we argue that it is possible to write biquaternionic extension of Klein-Gordon equation as follows:

$$\left[\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)+i\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\right]\varphi(x,t)=-m^2\varphi(x,t)\,, \tag{1}$$

Or this equation can be rewritten as:

$$\left(\Diamond\bar{\Diamond}+m^2\right)\varphi(x,t)=0\,, \tag{2}$$

Provided we use this definition:

$$\begin{aligned}\Diamond=\nabla^q+i\nabla^q&=\left(-i\frac{\partial}{\partial t}+e_1\frac{\partial}{\partial x}+e_2\frac{\partial}{\partial y}+e_3\frac{\partial}{\partial z}\right)\\&+i\left(-i\frac{\partial}{\partial T}+e_1\frac{\partial}{\partial X}+e_2\frac{\partial}{\partial Y}+e_3\frac{\partial}{\partial Z}\right)\end{aligned} \tag{3}$$

Where e_1, e_2, e_3 are *quaternion imaginary units* obeying (with ordinary quaternion symbols: e_1=i, e_2=j , e_3 =k):

$$\begin{aligned}&i^2=j^2=k^2=-1\,,\ ij=-ji=k\,,\\&jk=-kj=i\,,ki=-ik=j\,.\end{aligned} \tag{4}$$

And quaternion *Nabla operator* is defined as [5]:

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} \tag{5}$$

Note that equation (3) and (5) included partial time-differentiation.

It is worth nothing here that equation (2) yields solution containing imaginary part, which differs appreciably from known solution of KGE:

$$y(x,t) = \left(\frac{1}{4} - \frac{i}{4}\right)m^2t^2 + cons\tan t \tag{6}$$

Some possible alternative interpretations of this *imaginary part* of the solution of biquaternionic KGE (BQKGE) are:

(a) The imaginary part implies that there is exponential term of the wave solution, which is quite similar to the Ginzburg-Landau extension of London phenomenology [3]:

$$\psi(r) = \left|\psi(r)\right|e^{i\varphi(r)}, \tag{7}$$

because equation (6) can be rewritten (approximately) as:

$$y(x,t) = \frac{e^i}{4}m^2t^2 \tag{8}$$

(b) The aforementioned exponential term of the solution (8) can be interpreted as signature of vortices solution. Interestingly Navier-Stokes equation which implies vorticity equation can also be rewritten in terms of Yukawa equation [8].

(c) The imaginary part implies that there is a spiral wave, which suggests spiralling motion of meson or other particles. Interestingly it has been argued that one can explain electron phenomena by assuming spiralling electrons [5]. Alternatively this spiralling wave may already be known in the form of Bierkeland flow. For meson observation, this could be interpreted as another form of meson, which may be called 'supersymmetric-meson' [1].

(d) The imaginary part of solution of BQKGE also implies that it consists of standard solution of KGE [1], and its alteration because of imaginary differential operator. That would mean the resulting wave is composed of two complementary waves.

(e) Considering some recent proposals suggesting that neutrino can have *imaginary mass* [6], the aforementioned imaginary part of solution of BQKGE can also imply that the (supersymmetric-) meson may be composed of neutrino(s). This new proposition may require new thinking both on the nature of neutrino and also supersymmetric-meson. [7]

While some of these propositions remain to be seen, in deriving the preceding BQKGE we follow Dirac's phrase that '*One can always generalize his physics by generalizing his mathematics.*' More specifically, we focus on using a 'theorem' from this principle, i.e.: '*One can generalize his mathematics by generalizing his (differential) operator.*'

Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

Acknowledgment

Thanks to Prof. D. Rapoport who mentioned Sprossig's interesting paper [8]. VC would like to dedicate this work for RFF.

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Introduction to the Mu-bit

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1. **Definition:**
   Mu-bit is defined here as 'multi-space bit'. It is different from standard meaning of bit in conventional computation, because in Smarandache's multi-space theory, the bit is created simultaneously at multi-spaces.
   This new 'bit' term is different from multi-valued-bit already known in computer technology, for example as MVLong [8][9].
   This new concept is also different from qu-bit from quantum computation terminology [10]. We know that using quantum mechanics logic we could introduce new way of computation with 'qubit' (quantum bit), but the logic remains Neumann. Now from the viewpoint of m-valued multi-space logic, we introduce a new term: 'mu-bit' (from ' multi-space bit').
2. **Conceptual foundation:**
   It is known that multi-valued logic is required to understand the general theory of process [6][7]. Multi-valued atomic variable is also known in theory [12].
   Similarly, multi-space concept could be viewed as an attempt to comprehend various phenomena altogether beyond multi-valuedness..
   In a (finite or infinite) multi-space M, which is a space formed by union of n (where n can be finite or infinitely large), spaces S_1, S_2, ..., S_n, we can consider these spaces S_1, S_2, ..., S_n overlapping and a particle P can have a degree of truth t_1 and can be in position p_1 in space S_1, and (t_2, p_2) in space S_2, and so on. Hence the same particle is logically infinite-valued and in infinitely many positions in the same time.

3. **Reasoning:**
   The reason for submitting a different term for this multi-space-bit is as follows. We all know that computer scientists always look for computation beyond Turing machine, and we believe that multi-space-bit from multi-space theory could offer significant theoretical leap beyond Turing machine paradigm.

4. **Theoretical implications**
   The idea was to abandon the notion of 'quantum computation', i.e. to use Neumann logic. We know that scientists began to use Bose condensate to model “quantum bit” but how to extend it to large systems remains unknown. Therefore, the best way is to forgo this “quantum bit” term,

and start from scratch, i.e. using Cantor sets instead.[3] In this context, multi-space hypothesis includes 'infinity' position of bits, which corresponds to Cantor sets, albeit from entirely different viewpoint.

Using the multi-space theory, we could expect to explain wave-particle duality problem in Young interference experiment [4]. It is known that there is a paradox that light could resemble particle (photon) yet it could behave like wave (Fresnel). It becomes apparent that this paradox could be reconciled once we introduce multi-space-bit, that creation of 'bit' will multiply itself into multi-space, which triggers wave pattern in Young slit experiment.

5. Possible practical use

(a) parallel quantum computation [1][2]. We could also think about 'parallel quantum computing' [somehow similar to parallel computer programming]. Since we work in a multi-space $S= S_1 \vee S_2 \vee \ldots \vee S_n$, we may consider a quantum computing in the same time in each space $S_1, S_2, ..., S_n$, – connecting this to mu-bit. This is different from standard logic used in quantum computation [10].

(b) theoretical biology: going beyond DNA model;

(c) quantum electrodynamics of wireless communication: advance radio frequency etc.;

(d) advanced brain modeling & human-consciousness theory [5];

(e) new chapter on artificial intelligence, pattern recognition, robotics [7a];

(f) multi-space CMOS [9].

(g) advanced coding/decoding symbolic language beyond multi-valued coder [11].

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8. Other information

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Introduction to Smarandache-Christianto (SC) Potential

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a. **Definition:**
A new type of potential for nucleus, which is different from Coulomb potential or Yukawa potential. This new potential may have effect for radius range within r = 5-10 fm.

b. **Reasoning:**
It is known that Yukawa potential has been derived from radial Klein-Gordon equation. Yukawa was able to predict new type of particle, which then it was coined as 'meson'.[1] Of course, in history the 'meson' associated to Yukawa was not observed with high-precision. [2][12]

But recently there are critics that Yukawa potential has problems because it uses Klein-Gordon with Lagrangeian over real. [3]

Alternatively, one can extend Klein-Gordon using biquaternion number, and it will lead to a new type of potential having sinusoidal form [4][5]. It is coined as 'SC-potential'. [6]

Interestingly, a quite similar form of potential has been derived by M. Geilhaupt. Using modified Klein-Gordon equation he comes up with sinusoidal wave representation of electron, which can be used to predict electron mass and charge. He called this equation: unified force equation. [7]

c. **Implications:**
For experimental verification of this new potential, we find possible application in the context of Condensed Matter Nuclear reaction [5][6]. According to Takahashi's research, it is more likely to get condensed matter nuclear reaction using cluster of deuterium (4D) rather than using D+D reaction (as in hot-fusion, in this process Coulomb barrier is very high). The probable reaction according to Takahashi is [8]:

4D --> 8Be

Then because Be is unstable, it will yield:

8Be --> 4He + 4He + 47.6 MeV

In recent work, Takahashi shows that in the TSC framework it is also possible to do CMNS reaction not only with DDDD, but also with DDDH, DDHH, DHHH, or HHHH [8], where the reaction can be different from above:

DDDH --> 7Be --> 3He + 4He + 29.3 MeV

or

DHDH --> 6Be --> 3He + 3He

In other words, TSC can be A mixture of heavy and light water. [8]

More interestingly, his EQPET/TSC (*tetrahedra symmetric condensate*) model, Takahashi can predict a new potential called STTBA (*sudden-tall thin barrier approximate*) which includes negative potential (reverse potential) and differs from Coulomb potential [8].

Therefore the SC-potential which has sinusoidal form can be viewed as a generalization of Takahashi's TSC/STTBA potential.[9]

Prof Akito Takahashi is chairman of ISCMNS (International Society of Condensed Matter Nuclear Science) [10].

Further experiments are recommended in order to verify this proposition.

(May 12th, 2008)

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Fractal links

Some useful links for drawing fractal from quaternion numbers:

Name	Description	URL link
QuaSZ QuaSZ Mac	Primarily for exploring quaternions, hypernions, octonions, cubics, and complexified quats.	www.mysticfractal.com
Hydra	This program graphs 3-D slices of formulas based on 4-D complex number planes, currently supporting quaternion, hypernion, and user-customized quad types of the Mandelbrot set and Julia sets.	www.mysticfractal.com
Fractal Agent Fractal Commander	Freeware programs orignally written to draw escape-type fractals using every conceivable complex math function. And now convergent and orbit-trap types, and extended basic complex math to hypercomplex and quaternion math	http://www.geocities.com/SoHo/Lofts/5601
Quaternion Julia Set VRML Server	**A CGI engine used to generate VRML quaternion Julia sets.**	http://www.ecs.wsu.edu/~hart

Source: http://home.att.net/~Paul.N.Lee/Fractal_Software.html

SCIENTIFIC RESEARCH METHODS

A SELF-RECURRENCE METHOD FOR GENERALIZING KNOWN SCIENTIFIC RESULTS

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A great number of articles widen known scientific results (theorems, inequalities, math/physics/chemical etc. propositions, formulas), and this is due to a simple procedure, of which it is good to say a few words:

Let suppose that we want to generalizes a known mathematical proposition $P(a)$, where a is a constant, to the proposition $P(n)$, where n is a variable which belongs to subset of N.

To prove that P is true for n by recurrence means the following: the first step is trivial, since it is about the known result $P(a)$ (and thus it was already verified before by other mathematicians!). To pass from $P(n)$ to $P(n+1)$, one uses too $P(a)$: therefore one widens a proposition by using the proposition itself, in other words the found generalization will be paradoxically proved with the help of the particular case from which one started!

We present below the generalizations of Hölder, Minkovski, and respectively Tchebychev inequalities.

1. A GENERALIZATION OF THE INEQUALITY OF HÖLDER

One generalizes the inequality of Hödler thanks to a reasoning by recurrence. As particular cases, one obtains a generalization of the inequality of Cauchy-Buniakovski-Scwartz, and some interesting applications.

Theorem: If $a_i^{(k)} \in \mathrm{R}_+$ and $p_k \in]1,+\infty[$, $i \in \{1,2,...,n\}$, $k \in \{1,2,...,m\}$, such that:, $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$, then:

$$\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)} \le \prod_{k=1}^{m}\left(\sum_{i=1}^{n}\left(a_i^{(k)}\right)^{p_k}\right)^{\frac{1}{p_k}} \text{ with } m \ge 2.$$

Proof:

For $m=2$ one obtains exactly the inequality of Hödler, which is true. One supposes that the inequality is true for the values which are strictly smaller than a certain m.

Then:,

$$\sum_{i=1}^{n}\prod_{k=1}^{m}a_i^{(k)}=\sum_{i=1}^{n}\left(\left(\prod_{k=1}^{m-2}a_i^{k}\right)\cdot\left(a_i^{(m-1)}\cdot a_i^{(m)}\right)\right)\le\left(\prod_{k=1}^{m-2}\left(\sum_{i=1}^{n}\left(a_i^{(k)}\right)^{p_k}\right)^{\frac{1}{p_k}}\right)\cdot\left(\sum_{i=1}^{n}\left(a_i^{(m-1)}\cdot a_i^{(m)}\right)^{p}\right)^{\frac{1}{p}}$$

where $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_{m-2}}+\frac{1}{p}=1$ and $p_h>1,\ 1\le h\le m-2,\ p>1;$

but

$$\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^{p}\cdot\left(a_i^{(m)}\right)^{p}\le\left(\sum_{i=1}^{n}\left(\left(a_i^{(m-1)}\right)^{p}\right)^{t_1}\right)^{\frac{1}{t_1}}\cdot\left(\sum_{i=1}^{n}\left(\left(a_i^{(m)}\right)^{p}\right)^{t_2}\right)^{\frac{1}{t_2}}$$

where $\frac{1}{t_1}+\frac{1}{t_2}=1$ and $t_1>1,\ t_2>2$.

It results from it:

$$\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^{p}\cdot\left(a_i^{(m)}\right)^{p}\le\left(\sum_{i=1}^{n}\left(a_i^{(m-1)}\right)^{pt_1}\right)^{\frac{1}{pt_1}}\cdot\left(\sum_{i=1}^{n}\left(a_i^{(m)}\right)^{pt_2}\right)^{\frac{1}{pt_2}}$$

with $\frac{1}{pt_1}+\frac{1}{pt_2}=\frac{1}{p}$.

Let us note $pt_1=p_{m-1}$ and $pt_2=p_m$. Then $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$ is true and one has $p_j>1$ for $1\le j\le m$ and it results the inequality from the theorem.

Note: If one poses $p_j=m$ for $1\le j\le m$ and if one raises to the power m this inequality, one obtains a generalization of the inequality of Cauchy-Buniakovski-Scwartz:

$$\left(\sum_{i=1}^{n}\prod_{k=1}^{m}a_i^{(k)}\right)^{m}\le\prod_{k=1}^{m}\sum_{i=1}^{n}\left(a_i^{(k)}\right)^{m}.$$

Application:

Let a_1,a_2,b_1,b_2,c_1,c_2 be positive real numbers.

Show that:

$$(a_1b_1c_1+a_2b_2c_2)^6\le 8(a_1^6+a_2^6)(b_1^6+b_2^6)(c_1^6+c_2^6)$$

Solution:

We will use the previous theorem. Let us choose $p_1=2,\ p_2=3,\ p_3=6$; we will obtain the following:

$$a_1b_1c_1+a_2b_2c_2\le(a_1^2+a_2^2)^{\frac{1}{2}}(b_1^3+b_2^3)^{\frac{1}{3}}(c_1^6+c_2^6)^{\frac{1}{6}},$$

or more:

$$(a_1b_1c_1 + a_2b_2c_2)^6 \le (a_1^2 + a_2^2)^3 (b_1^3 + b_2^3)^2 (c_1^6 + c_2^6),$$

and knowing that

$$(b_1^3 + b_2^3)^2 \le 2(b_1^6 + b_2^6)$$

and that

$$(a_1^2 + a_2^2)^3 = a_1^6 + a_2^6 + 3(a_1^4 a_2^2 + a_1^2 a_2^4) \le 4(a_1^6 + a_2^6)$$

since

$$a_1^4 a_2^2 + a_1^2 a_2^4 \le a_1^6 + a_2^6 \text{ (because: } -\left(a_2^2 - a_1^2\right)^2 \left(a_1^2 + a_2^2\right) \le 0)$$

it results the exercise which was proposed.

2. A GENERALIZATION OF THE INEQUALITY OF MINKOWSKI

Theorem : If p is a real number ≥ 1 and $a_i^{(k)} \in \mathbf{R}^+$ with $i \in \{1,2,...,n\}$ and $k \in \{1,2,...,m\}$, then:

$$\left(\sum_{i=1}^{n} \left(\sum_{k=1}^{m} a_i^{(k)} \right)^p \right)^{1/p} \le \left(\sum_{k=1}^{m} \left(\sum_{i=1}^{n} a_i^{(k)} \right)^p \right)^{1/p}$$

Demonstration by recurrence on $m \in \mathbf{N}^*$.
First of all one shows that:

$\left(\sum_{i=1}^{n} \left(a_i^{(1)}\right)^p \right)^{1/p} \le \left(\sum_{i=1}^{n} \left(a_i^{(1)}\right)^p \right)^{1/p}$, which is obvious, and proves that the inequality

is true for $m = 1$.

(The case $m = 2$ precisely constitutes the inequality of Minkowski, which is naturally true!).

Let us suppose that the inequality is true for all the values less or equal to m

$$\left(\sum_{i=1}^{n} \left(\sum_{k=1}^{m+1} a_i^{(k)} \right)^p \right)^{1/p} \le \left(\sum_{i=1}^{n} a_i^{(1)^p} \right)^{1/p} + \left(\sum_{i=1}^{n} \left(\sum_{k=2}^{m+1} a_i^{(k)} \right)^p \right)^{1/p} \le$$

$$\le \left(\sum_{i=1}^{n} \left(a_i^{(1)}\right)^p \right)^{1/p} + \left(\sum_{k=2}^{m+1} \left(\sum_{i=1}^{n} a_i^{(k)} \right)^p \right)^{1/p}$$

and this last sum is $\left(\sum_{k=1}^{m+1}\left(\sum_{i=1}^{n} a_i^{(k)}\right)^p\right)^{1/p}$ therefore the inequality is true for the level $m+1$.

3. A GENERALIZATION OF AN INEQUALITY OF TCHEBYCHEV

Statement: If $a_i^{(k)} \geq a_{i+1}^{(k)}$, $i \in \{1,2,...,n-1\}$, $k \in \{1,2,...,m\}$, then:

$$\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)} \geq \frac{1}{n^m}\prod_{k=1}^{m}\sum_{i=1}^{n} a_i^{(k)}.$$

Demonstration by recurrence on m.

Case $m=1$ is obvious: $\frac{1}{n}\sum_{i=1}^{n} a_i^{(1)} \geq \frac{1}{n}\sum_{i=1}^{n} a_i^{(1)}$.

In the case $m=2$, this is the inequality of Tchebychev itself:

If $a_1^{(1)} \geq a_2^{(1)} \geq ... \geq a_n^{(1)}$ and $a_1^{(2)} \geq a_2^{(2)} \geq ... \geq a_n^{(2)}$, then:

$$\frac{a_1^{(1)}a_1^{(2)} + a_2^{(1)}a_2^{(2)} + ... + a_n^{(1)}a_n^{(2)}}{n} \geq \frac{a_1^{(1)} + a_2^{(1)} + ... + a_n^{(1)}}{n} \times \frac{a_1^{(2)} + ... + a_n^{(2)}}{n}$$

One supposes that the inequality is true for all the values smaller or equal to m. It is necessary to prove for the rang $m+1$:

$$\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m+1} a_i^{(k)} = \frac{1}{n}\sum_{i=1}^{n}\left(\prod_{k=1}^{m} a_i^{(k)}\right)\cdot a_i^{(m+1)}.$$

This is $\geq \left(\frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{m} a_i^{(k)}\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{(m+1)}\right) \geq \left(\frac{1}{n^m}\prod_{k=1}^{m}\sum_{i=1}^{n} a_i^{(k)}\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{(m+1)}\right)$

and this is exactly $\frac{1}{n^{m+1}}\prod_{k=1}^{m+1}\sum_{i=1}^{n} a_i^{(k)}$ (*Quod Erat Demonstrandum*).

{Translated from French by the Author.}

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The Neutrosophic Research Method in Scientific and Humanistic Fields

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The Neutrosophic Research Method is a generalization of Hegel's dialectic, and suggests that scientific and humanistic research will progress via studying not only the *opposite ideas* but the *neutral ideas* related to them as well in order to have a bigger picture of the whole problem to solve.

We have qualitative research methods and quantitative research methods.

In a way we do research in social science, in another way in biology and anatomy, or in physics and mathematics, or in psychology, etc.

Two types of scientific research methods are: descriptive (analysis) and experimental (testing).

1) **Analysis:**
   Make a plan of research: What to find out? What tools are needed? What procedure to follow? Observation and formulation of the problem to solve. What other people did before? Analyze the data and draw conclusions. What is your hypothesis? What evidence supports it? How many variables (unknowns) are in your research? What is the interaction between them (the correlational coefficient can be positive, negative, or no relation between variables)? There are dependent and independent variable – be able to distinguish these categories. Measure your variable. Get help from others (your future co-authors) if parts of the topic is outside of your deep knowledge. Do a survey. Design a guide. Split the big problem into small problems in order to check each of them. Don't be bias, or at least reduce it as much as possible. Try to be more objective than subjective. Don't be guided by interest, but by the scientific or humanistic truth. Inquire yourself and others. Use modern logics (fuzzy logic, neutrosophic logic, paraconsistent logic) for prediction. Avoid misconceptions.
2) **Testing:**
   How to test your results? How to interpret the results? How to connect them with other researches? Collect data from your experiment and control. Communicate it to other experts in your field and ask their opinions. Your experiment has to be repeatable, i.e. if somebody else reproduces it he or she should get the same result as yours.
3) **Re-Testing:**

After experiment and control, you check again your hypothesis, theory. Analyze the resulted data and repeat the experiment. Do statistics on your repeated trials. Look for patterns. What is the reliability of your test? Do you get a valid result (i.e. is your result in contradiction with classically confirmed results)? Educate others about your method of research and your experimental result. When sharing your new idea, expect that some people may be opposed (because of common intellectual… inertia) to new concepts, so don't get discouraged (see, for example, how Quantum Physics is so… strange). If your hypothesis is right (valid), this might lead you to develop it into a law or new theory. Or, you might try to disprove a hypothesis (called null-hypothesis).

4) **Study the opposite ideas**.
   Why those ideas are in contradiction with yours? What conditions apply for your ideas and for those opposed to yours? Can you hypothesis be true in some conditions and the opposite ideas be true in other conditions? What is the explanation for this contradiction?
5) **Study the neutral theories.**
   This point makes the difference between dialectics and neutrosophy. A neutral idea (which neither opposes not asserts your hypothesis) could influence you in generalizing your hypothesis in a larger scientific space. Or, can give a new impulse to interconnect your hypothesis with others that apparently have no connections.

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STATISTICS

A GENERAL FAMILY OF ESTIMATORS FOR ESTIMATING POPULATION MEAN USING KNOWN VALUE OF SOME POPULATION PARAMETER(S)

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Abstract

A general family of estimators for estimating the population mean of the variable under study, which make use of known value of certain population parameter(s), is proposed. Under Simple Random Sampling Without Replacement (SRSWOR) scheme, the expressions of bias and mean-squared error (MSE) up to first order of approximation are derived. Some well known estimators have been shown as particular member of this family. An empirical study is carried out to illustrate the performance of the constructed estimator over others.

Keywords: Auxiliary information, general family of estimators, bias, mean-squared error, population parameter(s).

1. Introduction

Let y and x be the real valued functions defined on a finite population $U = (U_1, U_2, ..., U_N)$ and $\overline{Y}$ and $\overline{X}$ be the population means of the study character y and auxiliary character x respectively. Consider a simple random sample of size n drawn without replacement from population U. In order to have a survey estimate of the population mean $\overline{Y}$ of the study character y, assuming the knowledge of population mean $\overline{X}$ of the auxiliary character x, the well-known ratio estimator is

$$t_1 = \overline{y}\frac{\overline{X}}{\overline{x}} \qquad (1.1)$$

Product method of estimation is well-known technique for estimating the populations mean of a study character when population mean of an auxiliary character is known and it is negatively correlated with study character. The conventional product estimator for $\overline{Y}$ is defined as

$$t_2 = \overline{y}\frac{\overline{x}}{\overline{X}} \qquad (1.2)$$

Several authors have used prior value of certain population parameters (s) to find more precise estimates. Searls (1964) used Coefficient of Variation (CV) of study character at estimation stage. In practice this CV is seldom known. Motivated by Searls (1964) work, Sisodiya and Dwivedi (1981) used the known CV of the auxiliary character for estimating population mean of a study character in ratio method of estimation. The use of prior value of Coefficient of Kurtosis in estimating the population variance of study character y was first made by Singh et.al.(1973). Later, used by Sen (1978), Upadhyaya and Singh (1984) and Searls and Interpanich (1990) in the estimation of population mean of study character. Recently Singh and Tailor (2003) proposed a modified ratio estimator by using the known value of correlation coefficient.

In this paper, under SRSWOR, we have suggested a general family of estimators for estimating the population mean $\overline{Y}$. The expressions of bias and MSE, up to the first order of approximation, have been obtained, which will enable us to obtain the said expressions for any member of this family. Some well known estimators have been shown as particular member of this family.

2. The suggested family of estimators

Following Walsh (1970), Reddy (1973) and Srivastava (1967), we define a family of estimators $\overline{Y}$ as

$$t = \overline{y}\left[\frac{a\overline{X}+b}{\alpha(a\overline{x}+b)+(1-\alpha)(a\overline{X}+b)}\right]^{g} \qquad (2.1)$$

where a(≠0), b are either real numbers or the functions of the known parameters of the auxiliary variable x such as standard deviation (σ_x), Coefficients of Variation (C_X), Skewness ($\beta_1(x)$), Kurtosis ($\beta_2(x)$) and correlation coefficient (ρ).

To obtain the bias and MSE of t, we write

$$\overline{y} = \overline{Y}(1+e_0), \overline{x} = \overline{X}(1+e_1)$$

such that

$$E(e_0)=E(e_1)=0,$$

and

$$E(e_0^2) = f_1C_y^2, E(e_1^2) = f_1C_x^2, E(e_0e_1) = f_1\rho C_yC_x,$$

where

$$f_1 = \frac{N-n}{nN},\ C_y^2 = \frac{S_y^2}{\overline{Y}^2},\ C_x^2 = \frac{S_x^2}{\overline{X}^2}.$$

Expressing t in terms of e's, we have

$$t = \overline{Y}(1+e_0)(1+\alpha\lambda e_1)^{-g} \tag{2.2}$$

where $\lambda = \dfrac{a\overline{X}}{a\overline{X}+b}$. (2.3)

We assume that $|\alpha\lambda e_1| < 1$ so that $(1+\alpha\lambda e_1)^{-g}$ is expandable.

Expanding the right hand side of (2.2) and retaining terms up to the second powers of e's, we have

$$t \approx \overline{Y}\left[1+e_0-\alpha\lambda g e_1+\frac{g(g+1)}{2}\alpha^2\lambda^2 e_1^2-\alpha\lambda g e_0 e_1\right] \tag{2.4}$$

Taking expectation of both sides in (2.4) and then subtracting $\overline{Y}$ from both sides, we get the bias of the estimator t, up to the first order of approximation, as

$$B(t) \approx \left(\frac{1}{n}-\frac{1}{N}\right)\overline{Y}\left[\frac{g(g+1)}{2}\alpha^2\lambda^2 C_x^2-\alpha\lambda g\rho C_y C_x\right] \tag{2.5}$$

From (2.4), we have

$$(t-\overline{Y}) \cong \overline{Y}[e_0-\alpha\lambda g e_1] \tag{2.6}$$

Squaring both sides of (2.6) and then taking expectations, we get the MSE of the estimator t, up to the first order of approximation, as

$$MSE(t) \approx \left(\frac{1}{n}-\frac{1}{N}\right)\overline{Y}^2\left[C_y^2+\alpha^2\lambda^2 g^2 C_x^2-2\alpha\lambda g\rho C_y C_x\right] \tag{2.7}$$

Minimization of (2.7) with respect to α yields its optimum value as

$$\alpha = \frac{K}{\lambda g} = \alpha_{opt} \text{ (say)} \tag{2.8}$$

where

$$K = \rho \frac{C_y}{C_x}.$$

Substitution of (2.8) in (2.7) yields the minimum value of MSE (t) as

$$\min.MSE(t) = f_1 \overline{Y}^2 C_y^2 (1 - \rho^2) = MSE(t)_0 \qquad (2.9)$$

The min. MSE (t) at (2.9) is same as that of the approximate variance of the usual linear regression estimator.

3. Some members of the proposed family of the estimators' t

The following scheme presents some of the important known estimators of the population mean which can be obtained by suitable choice of constants α, a and b:

Estimator	**Values of**			
	α	**a**	**b**	**g**
1. $t_0 = \overline{y}$ The mean per unit estimator	0	0	0	0
2. $t_1 = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)$ The usual ratio estimator	1	1	0	1
3. $t_2 = \overline{y}\left(\frac{\overline{x}}{\overline{X}}\right)$ The usual product estimator	1	1	0	-1
4. $t_3 = \overline{y}\left(\frac{\overline{X} + C_x}{\overline{x} + C_x}\right)$ Sisodia and Dwivedi	1	1	C_x	1

(1981) estimator				
5. $t_4 = \bar{y}\left(\frac{\bar{x} + C_x}{\bar{X} + C_x}\right)$ Pandey and Dubey (1988) estimator	1	1	C_x	-1
6. $t_5 = \bar{y}\left[\frac{\beta_2(x)\bar{x} + C_x}{\beta_2(x)\bar{X} + C_x}\right]$ Upadhyaya and Singh (1999) estimator	1	$\beta_2(x)$	C_x	-1
7. $t_6 = \bar{y}\left[\frac{C_x\bar{x} + \beta_2(x)}{C_x\bar{X} + \beta_2(x)}\right]$ Upadhyaya, Singh (1999) estimator	1	C_x	$\beta_2(x)$	-1
8. $t_7 = \bar{y}\left[\frac{\bar{x} + \sigma_x}{\bar{X} + \sigma_x}\right]$ G.N.Singh (2003) estimator	1	1	σ_x	-1
9. $t_8 = \bar{y}\left[\frac{\beta_1(x)\bar{x} + \sigma_x}{\beta_1(x)\bar{X} + \sigma_x}\right]$ G.N.Singh (2003) estimator	1	$\beta_1(x)$	σ_x	-1
10. $t_9 = \bar{y}\left[\frac{\beta_2(x)\bar{x} + \sigma_x}{\beta_2(x)\bar{X} + \sigma_x}\right]$ G.N.Singh (2003) estimator	1	$\beta_2(x)$	σ_x	-1
11. $t_{10} = \bar{y}\left[\frac{\bar{X} + \rho}{\bar{x} + \rho}\right]$ Singh, Tailor (2003)	1	1	ρ	1

estimator				
12. $t_{11} = \bar{y}\left[\frac{\bar{x}+\rho}{\bar{X}+\rho}\right]$ Singh, Tailor (2003) estimator	1	1	ρ	-1
13. $t_{12} = \bar{y}\left[\frac{\bar{X}+\beta_2(x)}{\bar{x}+\beta_2(x)}\right]$ Singh et.al. (2004) estimator	1	1	$\beta_2(x)$	1
14. $t_{13} = \bar{y}\left[\frac{\bar{x}+\beta_2(x)}{\bar{X}+\beta_2(x)}\right]$ Singh et.al. (2004) estimator	1	1	$\beta_2(x)$	-1

In addition to these estimators a large number of estimators can also be generated from the proposed family of estimators t at (2.1) just by putting values of α ,g, a, and b.

It is observed that the expression of the first order approximation of bias and MSE/Variance of the given member of the family can be obtained by mere substituting the values of α ,g, a and b in (2.5) and (2.7) respectively.

4. Efficiency Comparisons

Up to the first order of approximation, the variance/MSE expressions of various estimators are:

$$V(t_0) = f_1\bar{Y}^2C_y^2 \tag{4.1}$$

$$MSE(t_1) = f_1\bar{Y}^2\left[C_y^2 + C_x^2 - 2\rho C_yC_x\right] \tag{4.2}$$

$$MSE(t_2) = f_1\bar{Y}^2\left[C_y^2 + C_x^2 + 2\rho C_yC_x\right] \tag{4.3}$$

$$MSE(t_3) = f_1\bar{Y}^2\left[C_y^2 + \theta_1^2C_x^2 - 2\theta_1\rho C_yC_x\right] \tag{4.4}$$

$$MSE(t_4) = f_1\overline{Y}^2\left[C_y^2 + \theta_1^2 C_x^2 + 2\theta_1 \rho C_y C_x\right] \quad (4.5)$$

$$MSE(t_5) = f_1\overline{Y}^2\left[C_y^2 + \theta_2^2 C_x^2 + 2\theta_2 \rho C_y C_x\right] \quad (4.6)$$

$$MSE(t_6) = f_1\overline{Y}^2\left[C_y^2 + \theta_3^2 C_x^2 + 2\theta_3 \rho C_y C_x\right] \quad (4.7)$$

$$MSE(t_7) = f_1\overline{Y}^2\left[C_y^2 + \theta_4^2 C_x^2 + 2\theta_4 \rho C_y C_x\right] \quad (4.8)$$

$$MSE(t_8) = f_1\overline{Y}^2\left[C_y^2 + \theta_5^2 C_x^2 + 2\theta_5 \rho C_y C_x\right] \quad (4.9)$$

$$MSE(t_9) = f_1\overline{Y}^2\left[C_y^2 + \theta_6^2 C_x^2 + 2\theta_6 \rho C_y C_x\right] \quad (4.10)$$

$$MSE(t_{10}) = f_1\overline{Y}^2\left[C_y^2 + \theta_7^2 C_x^2 - 2\theta_7 \rho C_y C_x\right] \quad (4.11)$$

$$MSE(t_{11}) = f_1\overline{Y}^2\left[C_y^2 + \theta_7^2 C_x^2 + 2\theta_7 \rho C_y C_x\right] \quad (4.12)$$

$$MSE(t_{12}) = f_1\overline{Y}^2\left[C_y^2 + \theta_8^2 C_x^2 - 2\theta_8 \rho C_y C_x\right] \quad (4.13)$$

$$MSE(t_{13}) = f_1\overline{Y}^2\left[C_y^2 + \theta_8^2 C_x^2 + 2\theta_8 \rho C_y C_x\right] \quad (4.14)$$

where

$$\theta_1 = \frac{\overline{X}}{\overline{X} + C_x}, \theta_2 = \frac{\beta_2(x)\overline{X}}{\beta_2(x) + C_x}, \theta_3 = \frac{C_x\overline{X}}{C_x\overline{X} + C_x},$$

$$\theta_4 = \frac{\overline{X}}{\overline{X} + \sigma_x}, \theta_5 = \frac{\beta_1(x)\overline{X}}{\beta_1(x) + \sigma_x}, \theta_6 = \frac{\beta_2(x)\overline{X}}{\beta_2(x)\overline{X} + \sigma_x},$$

$$\theta_7 = \frac{\overline{X}}{\overline{X} + \rho}, \theta_8 = \frac{\overline{X}}{\overline{X} + \beta_2(x)}.$$

To compare the efficiency of the proposed estimator t with the existing estimators t_0-t_{13}, using (2.9) and (4.1)-(4.14), we can, after some algebra, obtain

$$V(t_0) - MSE(t)_0 = C_y^2\rho^2 > 0 \quad (4.15)$$

$$MSE(t_1) - MSE(t)_0 = (C_x - \rho C_y)^2 > 0 \quad (4.16)$$

$$MSE(t_2) - MSE(t)_0 = (C_x + \rho C_y)^2 > 0 \quad (4.17)$$

$$MSE(t_3) - MSE(t)_0 = (\theta_1 C_x - \rho C_y)^2 > 0 \quad (4.18)$$

$$MSE(t_4) - MSE(t)_0 = (\theta_1 C_x + \rho C_y)^2 > 0 \quad (4.19)$$

$$MSE(t_5) - MSE(t)_0 = (\theta_2 C_x + \rho C_y)^2 > 0 \quad (4.20)$$

$$MSE(t_6) - MSE(t)_0 = (\theta_3 C_x + \rho C_y)^2 > 0 \quad (4.21)$$

$$MSE(t_7) - MSE(t)_0 = (\theta_4 C_x + \rho C_y)^2 > 0 \quad (4.22)$$

$$MSE(t_8) - MSE(t)_0 = (\theta_5 C_x + \rho C_y)^2 > 0 \quad (4.23)$$

$$MSE(t_9) - MSE(t)_0 = (\theta_6 C_x + \rho C_y)^2 > 0 \quad (4.24)$$

$$MSE(t_{10}) - MSE(t)_0 = (\theta_7 C_x - \rho C_y)^2 > 0 \quad (4.25)$$

$$MSE(t_{11}) - MSE(t)_0 = (\theta_7 C_x + \rho C_y)^2 > 0 \quad (4.26)$$

$$MSE(t_{12}) - MSE(t)_0 = (\theta_8 C_x - \rho C_y)^2 > 0 \quad (4.27)$$

$$MSE(t_{13}) - MSE(t)_0 = (\theta_8 C_x + \rho C_y)^2 > 0 \quad (4.28)$$

Thus from (4.15) to (4.28), it follows that the proposed family of estimators ‘t’ is more efficient than other existing estimators t_0 to t_{13}. Hence, we conclude that the proposed family of estimators ‘t’ is the best (in the sense of having minimum MSE).

5. Numerical illustrations

We consider the data used by Pandey and Dubey (1988) to demonstrate what we have discussed earlier. The population constants are as follows:

N=20,n=8, $\overline{Y} = 19.55$, $\overline{X} = 18.8$, $C_x^2 = 0.1555$, $C_y^2 = 0.1262$, $\rho_{yx} = -0.9199$, $\beta_1(x) = 0.5473$,

$\beta_2(x) = 3.0613$, $\theta_4 = 0.7172$.

We have computed the percent relative efficiency (PRE) of different estimators of $\overline{Y}$ with respect to usual unbiased estimator $\bar{y}$ and compiled in table 5.1.

Table 5.1: Percent relative efficiency of different estimators of $\overline{Y}$ with respect to $\bar{y}$

Estimator	PRE
$\bar{y}$	100
t_1	23.39
t_2	526.45
t_3	23.91
t_4	550.05
t_5	534.49
t_6	582.17
t_7	591.37
t_8	436.19
t_9	633.64
t_{10}	22.17
t_{11}	465.25
t_{12}	27.21
t_{13}	644.17
$t_{(opt)}$	650.26

where $t_{(opt)}$ is the value of t at (2.1) and replacing α by $\alpha_{(opt)}$ given in (2.8) and the resulting MSE given by (2.9).

Conclusion

From table 5.1, we observe that the proposed general family of estimators is preferable over all the considered estimators under optimum condition. The choice of the estimator mainly depends upon the availability of information about the known parameters of the auxiliary variable x such as standard deviation (σ_x), Coefficients of Variation (C_X), Skewness ($\beta_1(x)$), Kurtosis ($\beta_2(x)$) and correlation coefficient (ρ).

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DISTRICT LEVEL ANALYSIS OF URBANIZATION FROM RURAL-TO-URBAN MIGRATION IN THE RAJASTHAN STATE

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Abstract

Migration has various dimensions; urbanization due to migration is one of them. In Rajasthan State, District level analysis of urbanization due to migrants shows trend invariably for all the districts of the state though the contribution in urbanization by the migrants varies from district to district. In some districts the share of migrants moving to urban areas is very impressive though in others it is not that much high. The migrants' contribution in urbanization is on the rising over the decades. In this paper district level migration in the Rajasthan state is examined in relation to total urbanization and urbanization due to migration.

Broadly speaking rural to urban migration is due to diverse economic opportunities across space. Throughout history migration has played substantial role in the urbanization process of several countries and still continues to play similar role. In many cases it is witnessed that more the migration higher the urbanization rate. In general, it is perceived that migration has a fairly large share in urbanization and migrants constitute a significant portion in urbanization.

At all India level rural-urban migration seems to be modest as 2001 census discloses that net rural to urban migration in 1961-71 had been 18.7 percent, in 1971-81 it was 19.6 percent, in 1981-91 migration was 21.7 percent and in 1991-01 it was 21.0 percent. So the figures reveal that there

has been continuous rise in the contribution of net migration to total urban growth since the sixties though between 1991 and 2001 there has been slight decline in the rate compared to previous decade.

Migration is defined on the basis on the last residence concept hence migration in the 2001 census refers to those who migrated in ten years (1991-2001) preceding the year of survey 2001. The gross decadal inflow of rural to urban migrants as a percentage of total urban population in 2001 turns out to be a little above 7 per cent at the all-India level (Table on next page). However, it varies considerably across states. Both industrialized states like Gujarat and Maharashtra and the backward states like Orissa and Madhya Pradesh show high rates of migration. Similarly examples can be found from both the types of states which have recorded sluggish migration rate, e.g. industrialized states such as Tamilnadu and West Bengal and backward states such as Uttar Pradesh, Bihar and Rajasthan. This reveals that share of the rural to urban migrants in urbanization differs from state to state. A table giving rural to urban migrants for the period 1991-2001 as a % of urban population relation is given in table on ensuing page.

Table 1

Rural-Urban migration for 1991-2001 as a % of urban population

States	Rural-to-Urban Migrants (1991-2001) as a % of Urban Population
Andhra Pradesh	6.72
Assam	7.12
Bihar	6.28
Gujarat	10.63
Haryana	11.45
Karnataka	7.03
Kerala	6.99
Madhya Pradesh	9.50
Maharashtra	10.41
Orissa	10.97
Punjab	7.63
Rajasthan	6.18
Tamil Nadu	3.34
Utter Pradesh	4.44

West Bengal	4.83
All India	7.32

Source: Census of India 2001, Migration Tables.

Nevertheless, rural-urban migration rates at intrastate level have been a phenomenal in India as this flow dominates the interstate flows. Since the intrastate migration rates are much higher in magnitude than the interstate migration rates therefore it makes an interesting subject area to comprehend various economic, social and cultural factors connected closely with it. A district level analysis for Rajasthan state is thus attempted to perceive urbanization due to migration their interlinkages and affiliations.

Urbanization Trend in the State of Rajasthan

According to the census report of 2001 the share of urban population in Rajasthan has inched up to 23.38% as compared to 15.06% mentioned in the census report 1901. Number of towns in the Rajasthan increased to 216 in the census 2001 against 133 in the 1901 census that depicts 62.4% of growth in this period of time whereas at national level this growth has been 169.36% in same time span. Share of Rajasthan's urban population in the country dropped to 4.6% from 5.98% over a period of century whereas in terms of growth of number of towns, state share also slipped down to 4.18% from 6.94% in this same period of time. Therefore, it can be clearly claimed that Rajasthan has to go a long way to match with national figures as regards the characteristics of urbanization is concerned whether it is growth in urban population or towns. However, there has been a meager improvement in the percentage share of state's urban population in the national urban population as it has grown to 4.1% to 4.52%, 4.52% to 4.62% and then to 4.64% in last three successive censuses.

District Level Analysis for Rajasthan

The migrants contribution in urbanization is on the rising over the decades as 16.4% of the total migrants in the Rajasthan settled in urban areas during the period 1971-80 and the figure which went up to 22.4%

during the duration 1981-1990 and further advanced to 25.4% in the duration 1991-2000. This trend is evident invariably in all the districts of the state though the contribution in urbanization by the migrants varies from district to district. In some districts the share of migrants moving to urban areas is very impressive though in others it is not that much high.

The census analysis of Barmer district of Rajasthan reveals that 7.7%, 7.1% & 4.0% of total migrants moved to urban areas in last three decades i.e. 1991-2000, 1981-90 & 1971-1980. This percentage share for Jalore was 9.6, 8.1 & 4.7%, and for Banswara it was 9.1, 7.9 & 4.7%. The figures disclose that these districts had poor share of migrants to urban areas. On the other hand there are districts like Jaipur, Ajmer, Kota and Bhilwara where the percentage share of migrants settling in urban areas with context to the total migrants is comparatively much higher. This percentage share of rural migrants in three last successive decades for these districts is given in table placed below.

Table 2

Share of Rural Migrants in selected Districts during last three decadal period

District / period	1991-2000	1981-90	1971-1980
Kota	56.8	54.3	50.7
Jaipur	53.2	48.5	35
Ajmer	41.4	35.6	28.7
Bhilwara	31.1	25.0	14.8
Jodhpur	26.8	18.7	12.4

To apprehend the trends in the migration of population to the urban areas in different districts of Rajasthan, based on the share of urbanization due to migration can be categorized as follows:

Category 1: Higher During all the three decades

Category 2: Higher during 1991-2000 & 1981-91 but lower in 1971-80

Category 3: Higher during 1991-2001 but lower in last two decades

Category 4: Lower During all the three decades

Category 5: Lower during 1991-2000 & 1981-90 but higher in 1971-80

Category 6: Lower during 1991-2000 but higher in 1981-90 & 1971-80

Districts falling in Category 1 are those, which observed higher urbanization due to migration in comparison with state level figures during three consecutive decadal periods. In these Districts, the proportion of migrants coming to urban areas is higher than the state proportion of such migration.

District falling in Category 2 performed better as far urbanization due to migration in last two decades is concerned. Districts in this category observed higher urbanization share due to migration than what was seen in the state in last two decadal times whereas three decades back share of migrants to urban areas was lower in these districts from that of state in overall.

Similarly, Category 3 is featuring districts that have observed higher urbanization share due to migration than to state in recent decade though that particular district was falling below than state share in two previous consecutive decades.

Category 4 to 6 are counterpart of category 1 to 3 where share of migrants moving to urban areas in total migrants for a district is lower than state share of migrants moving to urban areas as regards total migrants of the state.

Table 3

Classification of District according to Urbanization Trends in last three decades

Category	Districts
Category 1	Ganganagar, Bharatpur, Swaimadhopur, Jaipur, Pali, Ajmer, Kota

Category 2	Bhilwara
Category 3	Jodhpur
Category 4	Alwar, Dholpur, Karauli, Dausa, Sikar, Nagaur, Barmer, Jalore, Sirohi, Tonk, Bundi, Rajsamand, Udaipur, Dungarpur, Banswara, Baran, Chittorgarh, Jhalawar
Category 5	Bikaner, Jhunjhunu
Category 6	Hanumangarh,Churu

Classification elucidated above undoubtedly depicts that there are only seven districts where there is larger urbanization due to rural migrants in context with the overall state level migration and urbanization figures over three consecutive decades. Notwithstanding there are 18 districts having lower urbanization due to migration than to state level migrant urbanization.

2001 census report explains that Jodhpur is the only district where urbanization due to migration has improved with regard to the figure of state in total. Similarly, district Bhilwara has witnessed this edge in two recent decades. In two recent decades there is improvement in the data of Bhilwara in relation to urbanization due to migration is concerned decades otherwise three decades back the urbanization due to migration for Bhilwara was lower than state figures. Jodhpur showed this improvement in last decade even though it was lagging behind in two previous decades.

Bikaner and Jhunjunu are way behind in showing any improvement in urban migration to state share in last two decades while Hanumagarh & Churu showed no improvement only in last decade. Jaiselmer is the district that doesn't observe any clear-cut pattern on account of migrants share in relation to state.

Urbanization and Migration:

It is well evident that number of rural migrants as regards total migrants is considered as an extent of urbanization by migration in a particular category. Districts are classified in the groups where percentage of migrants attributing to urbanization is <20%, 20-50 and >50% in all the three durations 1971-80, 1981-90 and 1991-2000 and the result is summarized as below:

Table 4

Number of Districts according to range of Urbanization in last three Census

Range of urbanization (in%)	2001	1991	1981
	Number of Districts		
<20	10	16	28
20-50	20	14	3
>50	2	2	1

Its is evident from above classification that there is stark variation in the urbanization by migrants in various census barring the category of the districts that are having >50% of urban migrants in total migrants as there are only district since last two census against one three decade back where as considerably shift in the other two categories of 20-50% and <20% urbanization due to migration is there in this three decadal period. There are more districts classified in the category 20-50% during the recent decades whereas the number of districts in the category <20% has gone down in the recent decades.

Comparative Analysis of Total Urbanization & Urbanization due to Migration:

Migration is an important part of the urbanization and in many cases it is attributing predominately in the urbanization. Indicator of rate of Urbanization can be defined as below:

1. Total Urbanization rate: is the percentage of population living in urban areas to the total population

2. Urbanization rate due to migration: is the percentage share of rural migrants to the total migrants.

The result of the comparative investigation made on the basis of above mentioned two indicators for the last decadal period i.e. 1991-2001 is examined in coming paragraphs.

State urbanization rate is the share of urban population to the total population at state level and similarly it is counted on districts level. Consequently these two rates are compared at state and districts level to analyze the urbanization trend and to establish its association with the migration. At state level 23.4% of the total population is urbanized and 22.9% of migrants are coming to urban areas thus at state level the urbanization rate through migrants is compatible to the total urbanization rate. Barmer and Jalore are two districts in which urbanization through migrants' rate is below 20% as the urbanization rate of the migrants to these districts is mere 15 & 19%.

Rate of urbanization through migrants in Jaipur is (73.6%), Kota (68.2%), Ajmer (53.8%) and Udaipur (50%) and thus these districts have more than 50% of rural migrants and this can be summed up as more than half of the migrants to these districts are settling in urban areas. Bikaner and Churu are the only districts observed where urbanization through migrants rate is lower than total urbanization rate of the state. This difference was more than 32% for the Udaipur and Banswara districts and for seven districts it was more than 20%. The classification of number of districts based on the range of these two urbanization indicators is classified in coming table.

Table 5

Total Urbanization Rate vis-à-vis Urbanization Rate due to Migration

Range of Urbanization rate		>50%	40-50%	30-40%	20-30%	<20%
Combined (Male & female)	**Total Urbanization rate**	1	2	2	8	19
Male		1	1	2	9	19
Female		1	1	3	7	20
Combined (Male & female)	**Urbanization rate due to migration**	4	5	8	13	2
Male		12	8	4	9	12
Female		2	2	11	10	7

Clearly the migration witnesses a better urbanization rate and there are more districts classified in higher range of urbanization rates than the number of district classified in lower range in accordance with the total urbanization rate of the districts.

Technique of non-parametric test is used for district level analysis of the urbanization to examine the migration to different districts having same size of population. District are ranked on the basis of the total urban population and urban population due to migration and these formed two groups of Non-parametric test and Wilcoxon - Mann/Whitney Non parametric Test is employed for equality of K universes for total population and Male & Female population and results of the analysis done in Megastat is as below:

TOTAL		
n	sum of ranks	
32.00	698.00	Group 1
32.00	1382.00	Group 2
64.00	2080.00	Total
	1040.00	expected value
	74.48	standard deviation
	-4.59	Z

	0.00	p-value (two-tailed)
MALE		
n	sum of ranks	
32.00	612.00	Group 1
32.00	1468.00	Group 2
64.00	2080.00	Total
	1040.00	expected value
	74.48	standard deviation
	-5.74	Z
	0.00	p-value (two-tailed)
FEMALE		
n	sum of ranks	
32.00	775.00	Group 1
32.00	1305.00	Group 2
64.00	2080.00	Total
	1040.00	expected value
	74.48	standard deviation
	-3.55	Z
	.0004	p-value (two-tailed)
GROUP 1 URBANIZATION IN TOTAL POPULATION		
GROUP 2 URBANIZATION BY MIGRATION		

Clearly, the above examined district level analysis reveals that total urbanization and urbanization due to migration differs significantly. Male and female population and districts have significant impact on total urbanization & urbanization due to migration. Thus the relative magnitude of total urbanization and urbanization due to migration differs significantly for the districts for both genders and combined.

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Estimation of Mean in Presence of Non Response Using Exponential Estimator

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Abstract

This paper considers the problem of estimating the population mean using information on auxiliary variable in presence of non response. Exponential ratio and exponential product type estimators have been suggested and their properties are studied. An empirical study is carried out to support the theoretical results.

Keywords: population mean, study variable, auxiliary variable, exponential ratio, exponential, product estimator, Bias, MSE

1. Introduction

In surveys covering human populations, information is in most cases not obtained from all the units in the survey even after some call-backs. Hansen and Hurwitz(1946) considered the problem of non response while estimating the population mean by taking a sub sample from the non respondent group with the help of some extra efforts and an estimator was proposed by combining the information available from response and non response groups. In estimating population parameters like the mean, total or ratio, sample survey experts sometimes use auxiliary information to improve precision of the estimates. When the population mean $\overline{X}$ of the auxiliary variable x is known and in presence of non response, the problem of estimation of population mean $\overline{Y}$ of the study variable y has been discussed by

Cochran (1977), Rao (1986), Khare and Srivastava(1997) and Singh and Kumar (2008). In Hansen and Hurwitz(1946) method, questionnaires are mailed to all the respondents included in a sample and a list of non respondents is prepared after the deadline is over. Then a sub sample is drawn from the set of non respondents and a direct interview is conducted with the selected respondents and the necessary information is collected.

Assume that the population is divided into two groups, those who will not respond called non–response class. Let N_1 and N_2 be the number of units in the population that belong to the response class and the non-response class respectively ($N_1+N_2=N$). Let n_1 be the number of units responding in a simple random sample of size n drawn from the population and n_2 the number of units not responding in the sample. We may regard the sample of n_1 respondents as a simple random sample from the response class and the sample of n_2 as a simple random sample from the non-response class. Let h_2 denote the size of the sub sample from n_2 non-respondents to be interviewed and $f = \frac{n_2}{h_2}$. Let $\bar{y}_1$ and $\bar{y}_{h2}$ denote the sample means of y character based on n_1 and h_2 units respectively. The estimator proposed by Hansen and Hurwitz (1946) is given by-

$$\bar{y}^* = \frac{n_1\bar{y}_1 + n_2\bar{y}_{h2}}{n} \tag{1.1}$$

The estimator $\bar{y}^*$ is unbiased and has variance

$$V(\bar{y}^*) = (\frac{1}{n} - \frac{1}{N})S_y^2 + (f-1)\frac{N2}{N}.\frac{S_{y2}^2}{n} \tag{1.2}$$

The population mean square of the character y is denoted by S_y^2 and the population mean square of y for N_2 non-response units of the population is denoted by S_{y2}^2.

Bahl and Tuteja (1991) introduced an exponential ratio –type estimator for population mean as given by

$$\bar{y}_{er} = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \tag{1.3}$$

and exponential product- type estimator as

$$\bar{y}_{ep} = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \tag{1.4}$$

The objective of this paper is to study Bahl and Tuteja (1991) exponential ratio- type and product- type estimators in presence of non-response.

2. Suggested estimator

First we assume that the non response is only on study variable. The estimator $\bar{y}_{er}$ under non response will take the form

$$\bar{y}^{*}_{er} = \bar{y}^{*}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \tag{2.1}$$

To obtain the bias and MSE of the estimator $\bar{y}^{*}_{er}$ we write

$\bar{y}^{*} = \bar{Y}(1+e^{*}_{0}), \quad \bar{x} = \bar{X}(1+e_1)$

Such that

$E(e_0^{*})=E(e_1)=0,$

and

$$\left.\begin{aligned} E(e_0^2) &= \frac{V(\bar{y}^*)}{\bar{Y}^2} \\ E(e_1^2) &= \frac{V(\bar{x})}{\bar{X}^2} = (\frac{1}{n} - \frac{1}{N})\frac{S_x^2}{\bar{X}^2} \\ E(e_o^* e_1) &= \frac{\mathrm{cov}(\bar{y}^*, \bar{x})}{\bar{Y}\bar{X}} = (\frac{1}{n} - \frac{1}{N})\frac{S_{XY}}{\bar{X}\bar{Y}} \end{aligned}\right\} \quad (2.2)$$

where $S_{xy} = \frac{1}{(N-1)}\sum_{j=1}^{N}(x_j - \bar{X})(y_j - \bar{Y})$.

Now expressing $\bar{y}_{er}^*$ in terms of e's we have

$$\bar{y}_{er}^* = \bar{Y}(1 + e_0)\exp\left\{\frac{-e_1}{2(1+\frac{e_1}{2})}\right\} \quad (2.3)$$

Expanding the right hand side of (2.3) and neglecting the terms involving powers of e's greater than two we have

$$\bar{y}_{er}^* = \bar{Y}(1 + e_0 - \frac{e_1}{2} - \frac{e_0 e_1}{2} + \frac{3e_1^2}{8}) \quad (2.4)$$

Taking expectations of both sides of (2.4), we get the bias of $\bar{y}_{er}^*$ to the first degree of approximation, as

$$B(\bar{y}_{er}^*) = (\frac{1}{n} - \frac{1}{N})\bar{Y}\left[\frac{3C_x^2}{8} - \frac{\rho C_y C_x}{2}\right] \quad (2.5)$$

Squaring both sides of (2.4) and neglecting terms of e's involving powers greater than two we have

$$(\bar{y}_{er}^{*} - \bar{Y})^2 = \bar{Y}^2\{e_0^2 + \frac{e_1^2}{4} - e_0 e_1\} \quad (2.6)$$

Taking expectations of both sides of (2.6) we get the MSE (to the first degree of approximation) as

$$MSE(\bar{y}_{er}^{*}) = (\frac{1}{n} - \frac{1}{N})\bar{Y}^2[C_y^2 + \frac{C_x^2}{4} - \rho C_y C_x] + \frac{(f-1)N_2}{nN} S_{y2}^2 \quad (2.7)$$

3. Exponential product type estimator

The estimator $\bar{y}_{ep}$ under non response (only on study variable) will take the form

$$\bar{y}_{ep}^{*} = \bar{y}^{*} \exp\left[\frac{\bar{x} - \bar{X}}{\bar{x} + \bar{X}}\right] \quad (3.1)$$

Following the procedure of section 2, we get the bias and MSE of $\bar{y}_{ep}^{*}$ as

$$Bias(\bar{y}_{ep}^{*}) = (\frac{1}{n} - \frac{1}{N})\bar{Y}\left[\frac{-C_x^2}{8} + \frac{\rho C_x C_y}{2}\right] \quad (3.2)$$

and

$$MSE(\bar{y}_{er}^{*}) = (\frac{1}{n} - \frac{1}{N})\bar{Y}^2[C_y^2 + \frac{C_x^2}{4} + \rho C_y C_x] + \frac{(f-1)N_2}{nN} S_{y2}^2 \quad (3.3)$$

4. Non response on both y and x

We assume that the non response is both on study and auxiliary variable. The estimator $\bar{y}_{er}$ and $\bar{y}_{ep}$ under non response on both the variables takes the following form respectively-

$$\bar{y}_{er}^{**} = \bar{y}^{*} \exp\left[\frac{\bar{X} - \bar{x}^{*}}{\bar{X} + \bar{x}^{*}}\right] \quad (4.1)$$

$$\bar{y}_{ep}^{**} = \bar{y}^{*} \exp\left[\frac{\bar{x}^{*} + \bar{X}}{\bar{x}^{*} + \bar{X}}\right] \tag{4.2}$$

To obtain the bias and MSE of the estimator $\bar{y}_{er}^{**}$ and $\bar{y}_{ep}^{**}$ we write

$$\bar{y}^{*} = \bar{Y}(1+e_0)\ ,\ \bar{x}^{*} = \bar{X}(1+e_1^{*}),\ \bar{x}\ = \bar{X}(1+e_1)$$

Such that

$E(e_0)=E(e_1^{*})=0$

$$\left.\begin{aligned} E(e_1^{*2}) &= \frac{v(\bar{x}^{*})}{\bar{X}^2} = (\frac{1}{n}-\frac{1}{N})S_x^2 + (f-1)\frac{N_2}{N}.\frac{S_{x2}^2}{n} \\ E(e_0e_2) &= \frac{cov(\bar{y}^{*},\bar{x}^{*})}{\bar{Y}\bar{X}} \\ &= \frac{1}{\bar{Y}\bar{X}}\left[(\frac{1}{n}-\frac{1}{N})S_{xy} + (f-1)\frac{N_2}{N}.\frac{S_{xy}}{n}\right] \end{aligned}\right\} \tag{4.3}$$

The population mean square of the character x is denoted by S_x^2 and the population mean square of x for N_2 non response units of the population is denoted by S_{x2}^2.

The biases and MSE of the estimators $\bar{y}_{er}^{**}$ and $\bar{y}_{ep}^{**}$ are given by $\bar{y}_{er}^{**}$ and $\bar{y}_{ep}^{**}$

$$B(\bar{y}_{er}^{**}) = (\frac{1}{n}-\frac{1}{N})\bar{Y}\left[\frac{3}{8}.\frac{S_x^2}{\bar{X}^2} - \frac{S_{xy}}{2\bar{Y}\bar{X}}\right] + \bar{Y}\frac{(f-1)}{n}.\frac{N_2}{N}\left[\frac{3}{8}.\frac{S_{x2}^2}{\bar{X}^2} - \frac{S_{xy2}}{2\bar{Y}\bar{X}}\right]$$

$$= (\frac{1}{n}-\frac{1}{N})\bar{Y}\left[\frac{3}{8}.C_x^2 - \frac{\rho C_x C_y}{2}\right] + \bar{Y}\frac{(f-1)}{n}.\frac{N_2}{N}\left[\frac{3}{8}.C_x^{'2} - \frac{\rho_2 C_x' C_y'}{2}\right] \tag{4.4}$$

where $C_x^{'} = \frac{S_{x2}}{\overline{X}}$, $C_y^{'} = \frac{S_{y2}}{\overline{Y}}$.

$$B(\overline{y}_{ep}^{**}) = (\frac{1}{n} - \frac{1}{N})\overline{Y}\left[\frac{-1}{8}.\frac{S_x^2}{\overline{X}^2} + \frac{S_{xy}}{2\overline{Y}\overline{X}}\right] + \overline{Y}\frac{(f-1)}{n}.\frac{N_2}{N}\left[\frac{-1}{8}.\frac{S_{x2}^2}{\overline{X}^2} + \frac{S_{xy2}}{2\overline{Y}\overline{X}}\right]$$

$$= (\frac{1}{n} - \frac{1}{N})\overline{Y}\left[\frac{-1}{8}.C_x^2 + \frac{\rho C_x C_y}{2}\right] + \overline{Y}\frac{(f-1)}{n}.\frac{N_2}{N}\left[\frac{-1}{8}.C_x^{'2} + \frac{\rho_2 C_x^{'} C_y^{'}}{2}\right] \qquad (4.5)$$

$$MSE(\overline{y}_{er}^{**}) = (\frac{1}{n} - \frac{1}{N})\overline{Y}^2\left[C_y^2 + \frac{C_x^2}{4} - \rho C_x C_y\right]$$

$$+ \quad \overline{Y}^2\frac{(f-1)}{n}.\frac{N_2}{N}\left[C_y^{'2} + \frac{C_x^{'}}{4} - \rho_2 C_x^{'} C_y^{'}\right] \qquad (4.6)$$

$$MSE(\overline{y}_{ep}^{**}) = (\frac{1}{n} - \frac{1}{N})\overline{Y}^2\left[C_y^2 + \frac{C_x^2}{4} + \rho C_x C_y\right]$$

$$+ \quad \overline{Y}^2\frac{(f-1)}{n}.\frac{N_2}{N}\left[C_y^{'2} + \frac{C_x^{'}}{4} + \rho_2 C_x^{'} C_y^{'}\right] \qquad (4.7)$$

From expressions (2.7),(3.3),(4.6),(4.7), we observe that the MSE expressions of suggested estimators have an additional term (which depends on non-response)as compaired to the estimator proposed by Bahl and Tuteja ((1991)(without non response).

5. Efficiency comparisons:

From (1.2), (2.7), (3.3), (4.6) and (4.7), we have

First we compare the efficiencies of $\overline{y}_{er}^{*}$ and $\overline{y}^{*}$

$MSE(\overline{y}_{er}^{*}) - V(\overline{y}^{*}) \leq 0$

$$\frac{C_x^2}{4} - \rho C_x C_y \le 0$$

$$\rho \ge \frac{C_x}{4C_y} \qquad (5.1)$$

When this condition is satisfied $\bar{y}_{er}^{*}$ will be better estimator than $\bar{y}^{*}$.

Next we compare the efficiencies of $\bar{y}_{ep}^{*}$ and $\bar{y}^{*}$

$$MSE(\bar{y}_{ep}^{*}) - V(\bar{y}^{*}) \le 0$$

$$\frac{C_x^2}{4} + \rho C_x C_y \le 0$$

$$\rho \le -\frac{C_x}{4C_y} \qquad (5.2)$$

When this condition is satisfied $\bar{y}_{ep}^{*}$ will be better estimator than $\bar{y}^{*}$.

Next, we compare the efficiencies of $\bar{y}_{er}^{**}$ and $\bar{y}^{*}$

$$MSE(\bar{y}_{er}^{**}) - V(\bar{y}^{*}) \le 0$$

$$(N-n)\alpha \le (1-f)N_2\alpha'$$

$$\frac{\alpha}{\alpha'} \le \frac{(1-f)N_2}{N-n} \qquad (5.3)$$

where $\alpha = \frac{C_x^2}{4} - \rho C_y C_x$, $\alpha' = \frac{C_x^2}{4} - \rho C_y' C_x'$

When this condition is satisfied $\bar{y}_{er}^{**}$ will be better estimator than $\bar{y}^{*}$.

Finally we compare the efficiencies of $\bar{y}_{ep}^{**}$ and $\bar{y}^{*}$

$$MSE(\bar{y}_{ep}^{**}) - V(\bar{y}^{*}) \le 0$$

$$(N-n)\lambda \le (1-f)N_2\lambda'$$

$$\frac{\lambda}{\lambda'} \le \frac{(1-f)N_2}{N-n} \qquad (5.4)$$

where $\lambda = \frac{C_x^2}{4} + \rho C_y C_x$, $\lambda' = \frac{C_x^2}{4} + \rho C_y' C_x'$

When this condition is satisfied $\bar{y}_{ep}^{**}$ will be better estimator than $\bar{y}^*$.

6. Empirical study

For numerical illustration we consider the data used by Khare and Sinha (2004,p.53). The values of the parameters related to the study variate y (the weight in kg) and the auxiliary variate x (the chest circumference in cm) have been given below.

$\overline{Y} = 19.50$ $\overline{X} = 55.86$ $S_y = 3.04$

$S_x = 3.2735$ $S_{y2} = 2.3552$ $S_{x2} = 2.51$

$\rho = 0.85$ $\rho_2 = 0.7290$

$N_1 = 71$ $N_2 = 24$ $N = 95$ $n = 35$

Here, we have computed the percent relative efficiencies (PRE) of different suggested estimators with respect to usual unbiased estimator $\bar{y}^*$ for different values of f.

Table 5.1: PRE of different proposed estimators

Values of w	f values	$\bar{y}^{*}$	$\bar{y}^{*}_{er}$	$\bar{y}^{*}_{ep}$	$\bar{y}^{**}_{er}$	$\bar{y}^{**}_{ep}$
0.10	1.50	100	263.64	45.47	263.65	45.47
	2.00	100	263.62	45.47	263.65	45.47
	2.50	100	263.61	45.48	263.65	45.47
	3.00	100	263.59	45.48	263.64	45.47
0.20	1.50	100	263.62	45.47	263.65	45.47
	2.00	100	263.59	45.48	263.64	45.47
	2.50	100	263.56	45.48	263.64	45.47
	3.00	100	263.52	45.48	263.63	45.47
0.30	1.50	100	263.61	45.48	263.65	45.47
	2.00	100	263.56	45.48	263.64	45.47
	2.50	100	263.51	45.48	263.63	45.47
	3.00	100	263.46	45.48	263.62	45.47

From table 5.1, we conclude that the estimators which use auxiliary information performs better than Hansen and Hurwitz(1946) estimators $\bar{y}^{*}$. Also when non response rate increases, the efficiencies of suggested estimators decreases.

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A Class Of Separate-Type Estimators For Population Mean In Stratified Sampling Using Known Parameters Under Non-Response

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ABSTRACT

The objective of the present paper is to propose a family of separate-type estimators of population mean in stratified random sampling in presence of non-response based on the family of estimators proposed by Khoshnevisan et al. (2007). Under simple random sampling without replacement (SRSWOR) the expressions of bias and mean square error (MSE) up to the first order of approximation are derived. The comparative study of the family with respect to usual estimator has been discussed. The expressions for optimum sample sizes of the strata in respect to cost of the survey have also been derived. An empirical study is carried out to shoe the properties of the estimators.

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1. INTRODUCTION

Upadhyaya and Singh (1999) have suggested the class of estimators in simple random sampling using some known population parameter(s) of an auxiliary variable. These estimators have been extended by Kadilar and Cingi (2003) for stratified random sampling. In an attempt to improve the estimators, Kadilar and Cingi (2005), Shabbir and Gupta (2005, 2006) and Singh and Vishwakarma (2008) have suggested new ratio estimators in stratified random sampling. Using power transformation Singh et al. (2008) have suggested a class of estimators adapting the estimators developed by Kadilar and Cingi (2003). Koyuncu and Kadilar (2008, 2009) have proposed a family of combined-type estimators in stratified random sampling based on the family of estimators proposed by Khoshnevisan et al. (2007). Singh et al. (2008) suggested some exponential ratio type estimators in stratified random sampling. Recently Koyuncu and Kadilar (2010) have suggested a family of estimators in stratified random sampling following Diana (1993) and Kadilar and Cingi (2003).

Let Y and X be the study and auxiliary variables respectively, with respective population means $\overline{Y}$ and $\overline{X}$. Khoshnevisan et al. (2007) have proposed a family of estimators for population mean using known values of some population parameters in simple random sampling (SRS) given by

$$t = \bar{y}\left[\frac{a\overline{X}+b}{\alpha(a\bar{x}+b)+(1-\alpha)(a\overline{X}+b)}\right]^{g} \qquad (1.1)$$

where $a \neq 0$, b are either real numbers or functions of known parameters of the auxiliary variable X.

In this paper, we have proposed a family of separate-type estimators of population mean in stratified random sampling in presence of non-response on study variable adapting the above family of estimators. The properties of the proposed family of estimators in comparison with usual estimators have been discussed. The expressions for optimum sample sizes of the strata with respect to cost of the survey have been obtained.

2. SAMPLING STRATEGY AND ESTIMATION PROCEDURES

Let us consider a finite population of size, N, is divided into k strata. Let N_i be the size of i^{th} stratum $(i = 1,2,...,k)$ and a sample of size n_i is drawn from the i^{th} stratum using SRSWOR scheme such that $\sum_{i=1}^{k} n_i = n$. It is assumed that the non-response is detected on study variable Y only and auxiliary variable X is free from non-response.

Let $\bar{y}_i^*$ and $\bar{x}_i$ are the unbiased estimators of population means $\bar{Y}_i$ and $\bar{X}_i$ respectively, for the i^{th} stratum, given as

$$\bar{y}_i^* = \frac{n_{i1}\bar{y}_{ni1} + n_{i2}\bar{y}_{ui2}}{n_i} \tag{2.1}$$

where $\bar{y}_{ni1}$ and $\bar{y}_{ui2}$ are the means based on n_{i1} response units and u_{i2} non-response units of sub sample selected from n_{i2} non-response units respectively. $\bar{x}_i$ be the sample mean based on n_i units.

Therefore an unbiased estimator of population mean $\bar{Y}$ is given by

$$\bar{y}_{st}^* = \sum_{i=1}^{k} p_i \bar{y}_i^* \tag{2.2}$$

and variance of the estimator is expressed as

$$V(\bar{y}_{st}^*) = \sum_{i=1}^{k}\left(\frac{1}{n} - \frac{1}{N}\right)p_i^2 S_{yi}^2 + \sum_{i=1}^{k}\frac{(k_i - 1)}{n_i} W_{i2} p_i^2 S_{yi2}^2 \tag{2.3}$$

where S_{yi}^2 and S_{yi2}^2 are respectively the mean-squares of entire group and non-response group of study variable in the population for the i^{th} stratum. $k_i = \frac{n_{i2}}{u_{i2}}$, $p_i = \frac{N_i}{N}$ and $W_{i2} =$ non-response rate of the i^{th} stratum in the population $= \frac{N_{i2}}{N_i}$.

2.1 SUGGESTED FAMILY OF ESTIMATORS

Adapting the idea of Khoshnevisan et al. (2007), we propose a family of separate-type estimators of population mean $\overline{Y}$, given by

$$T_S = \sum_{i=1}^{k} p_i T_i^* \tag{2.4}$$

where $$T_i^* = \bar{y}_i^* \left[\frac{a\overline{X}_i + b}{\alpha(a\bar{x}_i + b) + (1-\alpha)(a\overline{X}_i + b)} \right]^g \tag{2.5}$$

Obviously, T_S is biased for $\overline{Y}$. Therefore, bias and MSE of T_S can be obtained on using large sample approximations. Let

$$\bar{y}_i^* = \overline{Y}_i(1+e_0) \; ; \; \bar{x}_i = \overline{X}_i(1+e_1)$$

such that $E(e_0) = E(e_1) = 0$ and

$$E(e_0^2) = \frac{V(\bar{y}_i^*)}{\overline{Y}_i^2} = f_i C_{Yi}^2 + \frac{(k_i - 1)W_{i2}S_{Yi2}^2}{n_i \overline{Y}_i^2},$$

$$E(e_1^2) = \frac{V(\bar{x}_i)}{\overline{X}_i^2} = f_i C_{Xi}^2,$$

$$E(e_0 e_1) = \frac{Cov(\bar{y}_i^*, \bar{x}_i)}{\overline{Y}_i \overline{X}_i^2} = f_i \rho_i C_{Yi} C_{Xi},$$

where $f_i = \frac{N_i - n_i}{N_i n_i}$, $C_{Yi}^2 = \frac{S_{Yi}^2}{\overline{Y}_i^2}$, $C_{Xi}^2 = \frac{S_{Xi}^2}{\overline{X}_i^2}$, S_{Xi}^2 be the mean-square of entire group of auxiliary variable in the population for the i^{th} stratum and ρ_i is the correlation coefficient between Y and X in the i^{th} stratum.

Expressing the estimator T_S in terms of e_0 and e_1, we get

$$T_S = \sum_{i=1}^{k} p_i \overline{Y}_i (1+e_0)[1+\alpha\lambda_i e_1]^{-g} \tag{2.6}$$

where $\lambda_i = \frac{a\overline{X}_i}{a\overline{X}_i + b}$.

Suppose $|\alpha\lambda_i e_1| < 1$ so that $[1+\alpha\lambda_i e_1]^{-g}$ is expandable. Expanding the right hand side of the equation (2.6) up to the first order of approximation, we obtain

$$\left(T_S - \overline{Y}\right) = \sum_{i=1}^{k} \overline{Y}_i \left[e_0 - g\alpha\lambda_i e_1 + \frac{g(g+1)}{2}\alpha^2\lambda_i^2 e_1^2 - g\alpha\lambda_i e_0 e_1 \right] \quad (2.7)$$

Taking expectations of both sides of (2.7), we get the bias of T_S up to the first order of approximation, as

$$B(T_S) = \sum_{i=1}^{k} p_i f_i \overline{Y}_i \left[\frac{g(g+1)}{2}\alpha^2\lambda_i^2 C_{Xi}^2 - \alpha\lambda_i g\rho_i C_{Yi} C_{Xi} \right] \quad (2.8)$$

Squaring both side of equation (2.7) and taking expectations on both sides of this equation, we get the MSE(T_S) to the first order of approximation as given below:

$$MSE(T_S) = \sum_{i=1}^{k} p_i^2 \left[f_i \overline{Y}_i^2 \left(C_{Yi}^2 + \alpha^2\lambda_i^2 g^2 C_{Xi}^2 - 2\alpha\lambda_i g\rho_i C_{Yi} C_{Xi} \right) + \frac{(k_i - 1)}{n_i} W_{i2} S_{Yi2}^2 \right] \quad (2.9)$$

2.2 SOME SPECIAL CASES

Case 1: If we put $\alpha = 1$, $a = 1, b = 0$ and $g = 1$ in equation (2.4), we get

$$T_S = \sum_{i=1}^{k} p_i \overline{y}_i^{*} \frac{\overline{X}_i}{\overline{x}_i} \quad (2.10)$$

which is separate ratio estimator of population mean $\overline{Y}$ under non-response.

Case 2: If $\alpha = 1$, $a = 1, b = 0$ and $g = -1$, the equation (2.4) gives

$$T_S = \sum_{i=1}^{k} p_i \overline{y}_i^{*} \frac{\overline{x}_i}{\overline{X}_i} \quad (2.11)$$

which is separate product estimator of population mean $\overline{Y}$ under non-response.

Case 3: If we take $\alpha = 0$, $a = 0, b = 0$ and $g = 0$, the equation (2.4) provides

$$T_S = \sum_{i=1}^{k} p_i \overline{y}_i^{*} \quad (2.12)$$

which is the usual estimator of population mean $\overline{Y}$ under non-response.

Similarly, we can obtain the various existing estimators of the family under non-response on different choices of α, a, b and g [See Khoshnevisan et al. (2007)].

2.3 OPTIMUM CHOICE OF α

In order to obtain the optimum α we minimize $MSE(T_S)$ with respect to α . Differentiating $MSE(T_S)$ with respect to α and equating the derivative to zero, we get the normal equation

$$\frac{\partial MSE(T_S)}{\partial \alpha} = \sum_{i=1}^{k} p_i^2 f_i \overline{Y}_i^2 \left[2\alpha\lambda_i^2 C_{Xi}^2 - 2\alpha\lambda_i g\rho_i C_{Yi} C_{Xi}\right] = 0 \qquad (2.13)$$

$$\Rightarrow \alpha_{(opt)} = \frac{\sum_{i=1}^{k} p_i^2 f_i \overline{Y}_i^2 \rho_i C_{Yi} C_{Xi}}{\lambda_i g \sum_{i=1}^{k} p_i^2 f_i \overline{Y}_i^2 C_{Xi}^2} \qquad (2.14)$$

Thus the equation (2.14) provides the value of α at which $MSE(T_S)$ would be minimum.

2.4 OPTIMUM n_i WITH RESPECT TO COST OF THE SURVEY

Let C_{i0} be the cost per unit of selecting n_i units, C_{i1} be the cost per unit in enumerating n_{i1} units and C_{i2} be the cost per unit of enumerating u_{i2} units. Then the total cost for the i^{th} stratum is given by

$$C_i = C_{i0} n_i + C_{i1} n_{i1} + C_{i2} u_{i2} \qquad \forall \ i = 1,2,...,k \qquad (2.15)$$

Now, we consider the average cost per stratum

$$E(C_i) = n_i \left[C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{k_i} \right] \qquad (2.16)$$

Thus the total cost over all the strata is given by

$$C_0 = \sum_{i=1}^{k} E(C_i)$$

$$= \sum_{i=1}^{k} n_i \left[C_{i0} + C_{i1} W_{i1} + C_{i2} \frac{W_{i2}}{k_i} \right] \qquad (2.17)$$

Let us consider the function

$$\phi = MSE(T_S) + \mu C_0 \qquad (2.18)$$

where μ is Lagrangian multiplier. Differentiating the equation (2.18) with respect to n_i and k_i respectively and equating the derivatives to zero, we get the following normal equations:

$$\frac{\partial \phi}{\partial n_i} = -\frac{p_i^2}{n_i^2}\left[\overline{Y}_i^2\left(C_{Yi}^2 + \alpha^2\lambda_i^2 g^2 C_{Xi}^2 - 2\alpha\lambda_i g\rho_i C_{Yi}C_{Xi}\right) + (k_i - 1)W_{i2}S_{Yi2}^2\right] +$$

$$\mu\left(C_{i0} + C_{i1}W_{i1} + C_{i2}\frac{W_{i2}}{k_i}\right) = 0 \qquad (2.19)$$

$$\frac{\partial \phi}{\partial k_i} = \frac{p_i^2 W_{i2}S_{Yi2}^2}{n_i} - \mu n_i C_{i2}\frac{W_{i2}}{k_i^2} = 0 \qquad (2.20)$$

From the equations (2.19) and (2.20), we have

$$n_i = \frac{p_i\sqrt{\overline{Y}_i^2\left(C_{Yi}^2 + \alpha^2\lambda_i^2 g^2 C_{Xi}^2 - 2\alpha\lambda_i g\rho_i C_{Yi}C_{Xi}\right) + (k_i - 1)W_{i2}S_{Yi2}^2}}{\sqrt{\mu}\sqrt{C_{i0} + C_{i1}W_{i1} + C_{i2}\frac{W_{i2}}{k_i}}} \qquad (2.21)$$

and $\sqrt{\mu} = \frac{k_i p_i S_{Yi2}}{n_i\sqrt{C_{i2}}}$ (2.22)

Putting the value of the $\sqrt{\mu}$ from equation (2.22) into the equation (2.21), we get

$$k_{i(opt)} = \frac{\sqrt{C_{i2}}B_i}{S_{Yi2}A_i} \qquad (2.23)$$

where $A_i = \sqrt{C_{i0} + C_{i1}W_{i1}}$

and $B_i = \sqrt{\overline{Y}_i^2\left(C_{Yi}^2 + \alpha^2\lambda_i^2 g^2 C_{Xi}^2 - 2\alpha\lambda_i g\rho_i C_{Yi}C_{Xi}\right) - W_{i2}S_{Yi2}^2}$

On substituting $k_{i(opt)}$ into equation (2.21), n_i can be expressed as

$$n_i = \frac{p_i \sqrt{B_i^2 + \frac{\left(\sqrt{C_{i2}} B_i W_{i2} S_{Yi2}\right)}{A_i}}}{\sqrt{\mu}\sqrt{A_i^2 + \frac{\sqrt{C_{i2}} A_i W_{i2} S_{Yi2}}{B_i}}} \tag{2.24}$$

The $\sqrt{\mu}$ in terms of total cost C_0 can be obtained by putting the values of $k_{i(opt)}$ and n_i from equations (2.23) and (2.24) respectively into equation (2.17) as

$$\sqrt{\mu} = \frac{1}{C_0} \sum_{i=1}^{k} p_i \left(A_i B_i + \sqrt{C_{i2}} W_{i2} S_{Yi2}\right) \tag{2.25}$$

Thus the n_i can be expressed in terms of the total cost C_0 as

$$n_{i(opt)} = \frac{C_0}{\sum_{i=1}^{k} p_i \left(A_i B_i + \sqrt{C_{i2}} W_{i2} S_{Yi2}\right)} \frac{p_i \sqrt{B_i^2 + \frac{\left(\sqrt{C_{i2}} B_i W_{i2} S_{Yi2}\right)}{A_i}}}{\sqrt{A_i^2 + \frac{\sqrt{C_{i2}} A_i W_{i2} S_{Yi2}}{B_i}}} \tag{2.26}$$

The optimum values of n_i and k_i can be obtained by the expressions (2.26) and (2.23) respectively.

3. EMPIRICAL STUDY

In this section, we use the data set in Koyuncu and Kadilar (2009). The data concerning the number of teachers as study variable and the number of students as auxiliary variable in both primary and secondary school for 923 districts at 6 regions (as 1: Marmara, 2: Agean, 3: Mediterranean, 4: Central Anatolia, 5: Black Sea, 6: East and Southeast Anatolia) in Turkey in 2007 (Source: Ministry of Education Republic of Turkey).

Table 1: Stratum means, Mean Squares and Correlation Coefficients

Stratum No.	N_i	n_i	$\overline{Y}_i$	$\overline{X}_i$	S_{Yi}	S_{Xi}	S_{XYi}	ρ_i	S_{Yi2}
1	127	31	703.74	20804.59	883.835	30486.751	25237153.52	.936	440
2	117	21	413.00	9211.79	644.922	15180.769	9747942.85	.996	200
3	103	29	573.17	14309.30	1033.467	27549.697	28294397.04	.994	400
4	170	38	424.66	9478.85	810.585	18218.931	14523885.53	.983	405
5	205	22	527.03	5569.95	403.654	8497.776	3393591.75	.989	180
6	201	39	393.84	12997.59	711.723	23094.141	15864573.97	.965	300

Table 2: Percent Relative Efficiency (P.R.E.) of T_S with respect to $\overline{y}^*_{st}$ at $\alpha_{(opt)}$ = 0.9317, $a=1$ and $b=1$

W_{i2}	k_i	$P.R.E.(T_S)$
0.1	2.0	1319.17
	2.5	1153.15
	3.0	1026.92
	3.5	927.72
0.2	2.0	1026.92
	2.5	847.70
	3.0	726.55
	3.5	639.18
0.3	2.0	847.70
	2.5	679.59
	3.0	573.20
	3.5	499.81
0.4	2.0	726.55
	2.5	573.20
	3.0	480.16
	3.5	417.69

4. CONCLUSION

In this paper, a class of separate-type estimators for estimating the population mean in stratified random sampling under non-response has been proposed and method of finding the optimum estimator of the family has also been discussed. We have derived the expressions for optimum sample sizes in respect to cost of the survey. From the Table 2, it is easily observed that the optimum estimator of the proposed class T_S provides better estimate than usual estimator $\bar{y}_{st}^{*}$ under non-response. It is also observed that the relative efficiency of T_S decreases with increase in the non-response rate W_{i2} and k_i.

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Alternatives To Pearson's and Spearman's Correlation Coefficients

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Abstract. This article presents several alternatives to Pearson's correlation coefficient and many examples. In the samples where the rank in a discrete variable counts more than the variable values, the mixture of Pearson's and Spearman's gives a better result.

Introduction

Let's consider a bivariate sample, which consists of $n \geq 2$ pairs (x,y). We denote these pairs by:

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

where x_i = the value of x for the i-th observation,
and y_i = the value of y for the i-th observation,
for any $1 \leq i \leq n$.

We can construct a scatter plot in order to detect any relationship between variables x and y, drawing a horizontal x-axis and a vertical y-axis, and plotting points of coordinates (x_i, y_i) for all $i \in \{1, 2, \ldots, n\}$.

We use the standard statistics notations, mostly used in regression analysis:

$$\sum x = \sum_{i=1}^{n} x_i, \quad \sum y = \sum_{i=1}^{n} y_i, \quad \sum xy = \sum_{i=1}^{n} (x_i y_i),$$
$$\sum x^2 = \sum_{i=1}^{n} x_i^2, \quad \sum y^2 = \sum_{i=1}^{n} y_i^2, \tag{1}$$

$\overline{X} = \dfrac{\sum_{i=1}^{n} x_i}{n}$ = the mean of sample variable x,

$\overline{Y} = \dfrac{\sum_{i=1}^{n} y_i}{n}$ = the mean of sample variable y.

Let's introduce a notation for the median:

X_M = the median of sample variable x, (2)

Y_M = the median of sample variable y.

Correlation Coefficients.

Correlation coefficient of variables x and y shows how strongly the values of these variables are related to one another. It is denoted by r and $r \in [-1, 1]$.

If the correlation coefficient is positive, then both variables are simultaneously increasing (or simultaneously decreasing).

If the correlation coefficient is negative, then when one variable increases while the other decreases, and reciprocally.

Therefore, the correlation coefficient measures the degree of line association between two variables.

We have strong relationship if $r \in [0.8, 1]$ or $r \in [-1, -0.8]$;
moderate relationship if $r \in (0.5, 0.8)$ or $r \in (-0.8, -0.5)$; (3)
And weak relationship if $r \in [-0.5, 0.5]$.

Correlation coefficient does not depend on the measurement unit, neither on the order of variables: (x, y) or (y, x).

If r = 1 or -1, then there is a perfectly linear relationship between x and y. If r = 0, or close to zero, then there is not a strong linear relationship, but there might be a strong non-linear relationship that can be checked on the scatter plot.

The coefficient of determination, denoted by r^2, represents the proportion of variation in ***y*** due to a linear relationship between x and y in the sample:

$$r^2 = \frac{SSTo - SS\,\mathrm{Re}\,sid}{SSTo} = 1 - \frac{SS\mathrm{Re}\,sid}{SSTo} \quad (4)$$

where SSTo = total sum of squares = $\sum (y - \bar{y})^2 = \sum_{i=1}^{n} (y_i - \bar{y})^2$ (5)

and SSResid = residual sum of squares = $\sum (y - \hat{y})^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)$ (6)

with $\hat{y}_i$ = the i-th predicted value = $a + bx_i$ for $i \in \{1,2,\ldots,n\}$

resulting from substituting each sample x value into the equation for the least-squares line

$\hat{y} = a + bx$

$$\text{where } b = \frac{\sum xy - [(\sum x)(\sum y)/n]}{\sum x \wedge 2 - [(\sum x) \wedge 2/n]} \tag{7}$$

$$\text{and } a = \bar{Y} - b\,\bar{X}\,. \tag{8}$$

Obviously: coefficient of determination = (correlation coefficient)^2.

Two sample correlation coefficients are well-known:

1) Pearson's sample correlation coefficient, let's denote it by r_p

$$r_p = \frac{\sum xy - [(\sum x)(\sum y)/n]}{\sqrt{\sum x \wedge 2 - [(\sum x) \wedge 2/n]} \bullet \sqrt{\sum y \wedge 2 - [(\sum y) \wedge 2/n]}} \tag{9}$$

which is the most popular;

and 2) Spearman's rank correlation coefficient, let's denote it by r_5, which is obtained from the previous one by replacing, for each $i \in \{1, 2, \ldots, n\}$, x_i by its rank in the variable x, and similarly for y_i.

*

We propose more alternative sample correlation coefficients in the following ways, replacing in Pearson's formula (9):

3.1. Each x_i by its deviation from the x mean: $x_i - \bar{x}$,
and each y_i by its deviation from the y mean: $y_i - \bar{y}$.

3.2. Each x_i by its deviation from the x minimum: $x_i - x_{min}$, and each y_i by its deviation from the y minimum: $y_i - y_{min}$.

3.3. Each x_i by its deviation from the x maximum: $x_{max} - x_i$, and each y_i by its deviation from the y maximum: $y_{max} - y_i$

3.4. Each x_i by its deviation from a given x_k (for $k \in \{1, 2, \ldots, n\}$):

$$x_i - x_k$$

and each y_i by its deviation from the corresponding given y_k:

$$y_i - y_k$$

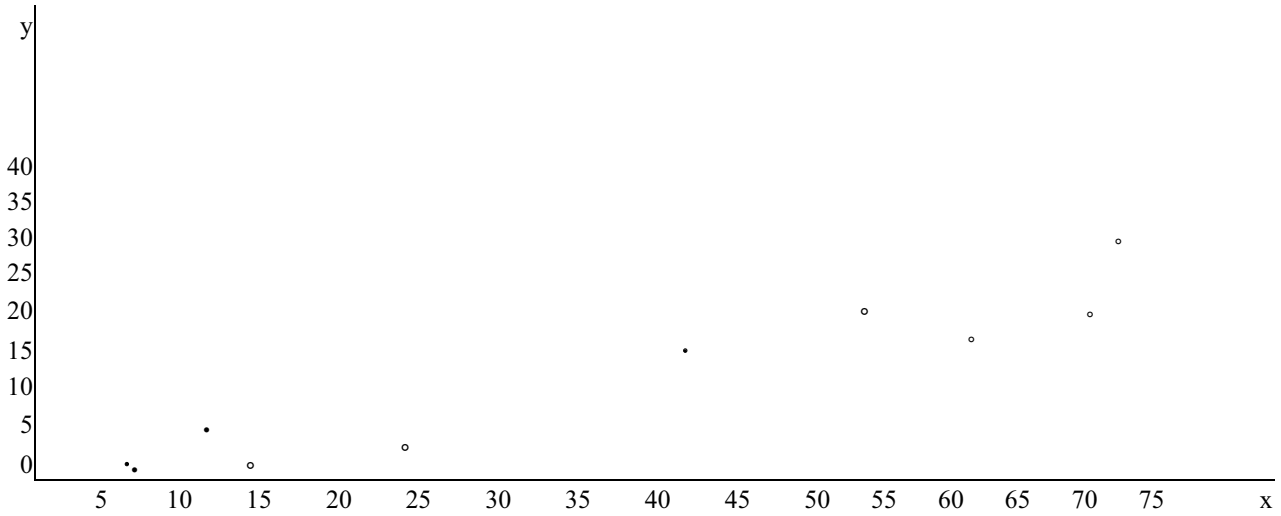
Not surprisingly, all these four alternative sample correlation coefficients are equal to Pearson's since they are simply related to translations of Cartesian axes, whose origin (0,0) is moved to $(\overline{x}, \overline{y})$, (x_{min}, y_{min}), (x_{max}, y_{max}), or (x_k, y_k) respectively.

Example: Let the variables x, y be given below:

x	6	7	12	14	23	41	53	60	69	72
y	2.5	1.1	6.3	2.1	2.9	15.3	20.7	18.4	22	33

Table 1

and their scatter plot:



Graph 1

1) Calculating Pearson's correlation coefficient:

$$\sum x = 357; \qquad \overline{x} = 35.7;$$

$$\sum y = 124.3; \qquad \overline{y} = 12.43;$$

$$\sum x^2 = 18{,}989;$$

$$\sum y^2 = 2{,}634.11;$$

$$\sum xy = 6{,}916.8;$$

$r_p = 0.95075.$

2) Calculating Spearman's rank correlation coefficient:

x	1	2	3	4	5	6	7	8	9	10
y	3	1	5	2	4	6	8	7	9	10

Table 2

$$\sum x = \frac{(1+10)\bullet 10}{2} = 11.5 = 5.5;$$

$$\sum y = 55;$$

$$\sum x^2 = 385;$$

$$\sum y^2 = 385;$$

$$\sum xy = 377;$$

$r_s = 0.90303.$

3.1) Replacing x_i by $x_i - \overline{x}$ and y_i by $y_i - \overline{y}$ for all i (deviations from the mean):

x	-29.7	-28.7	-23.7	-21.7	-12.7	5.3	17.3	24.3	33.3	36.3
y	-9.93	-11.33	-6.13	-10.33	-9.53	2.87	8.27	5.97	9.57	20.57

Table 3

Similarly: $\sum x = 0,$

because $\sum x = \sum_{i=1}^{10}(x_i - \overline{x}) = x_1 - \overline{x} + x_2 - \overline{x} + \ldots + x_{10} - \overline{x} = (x_1 + x_2 + \ldots + x_{10}) - 10\,\overline{x}$

$$= (x_1 + x_2 + \ldots + x_{10}) - 10\bullet\frac{x_1 + x_2 + \ldots + x_n}{10} = 0;$$

$$\sum y = 0;$$

$$\sum x^2 = 6{,}244.10;$$

$$\sum y^2 = 1{,}089.06;$$

$$\sum xy = 2{,}479.29;$$

$$r_{mean} = 0.95075.$$

3.2) Replacing x_i, y_i by their deviations from the smaller x: = x-x_{small} and y: = y-y_{small} we have a translation of axes again.

x	0	1	6	8	17	35	47	54	63	66
y	1.4	0	5.2	1	1.8	14.2	19.6	17.3	20.9	31.9

Table 4

$$\sum x = 297;$$
$$\sum y = 113.3;$$
$$\sum x^2 = 15{,}065;$$
$$\sum y^2 = 2{,}372.75;$$
$$\sum xy = 5{,}844.30;$$
$$r_{(small)} = 0.95075.$$

3.3) Replacing x_i, y_i by their deviations from the maximum:

x	66	65	60	58	49	31	19	12	3	0
y	30.5	31.9	26.7	30.9	30.1	17.7	12.3	14.6	11	0

Table 5

$$\sum x = 363;$$
$$\sum y = 205.7;$$
$$\sum x^2 = 19{,}421;$$
$$\sum y^2 = 5{,}320.31;$$
$$\sum xy = 9{,}946.20;$$
$$r_{(max)} = 0.95075.$$

3.4) Replacing x_i by $x_i - x_4$ and y_i by $y_i - y_4$ (in this case k = 4), $(x_4, y_4) = (14, 2.1)$:

x	-8	-7	-2	0	9	27	39	46	55	58
y	0.4	-1	4.2	0	0.8	13.2	18.6	16.3	19.9	30.9

Table 6

$$\sum x = 217;$$
$$\sum y = 103.3;$$

$$\sum x^2 = 10{,}953;$$
$$\sum y^2 = 2{,}156.15;$$
$$\sum xy = 4{,}720.9;$$

$r_4 = r_i = 0.95075$ for any $i \in \{1, 2, \ldots, 10\}$.

Similarly if we replace in Pearson's formula (9) and also getting the same result equals to r_p:

3.5) Each x_i by its deviation from x's median, and each y_i by its deviation from y's median.

3.6) Each x_i by its deviation from x's standard deviation, and each y_i by its deviation from y's standard deviation.

3.7) Each x_i by $x_i \pm a$ (where a is any number), and each y_i by $y_i \pm b$ (where b is any number).

3.8) Each x_i by x_i * a (where a is any non-zero number and "*" is either division or multiplication), and each y_i by y_i * b (similarly for b and "*").

Since the cases 3.5 – 3.7 are similar to 3.1 - 3.4, let's consider two examples for the case 3.8:

3.8.1) Suppose each x_i in the original example, Table 1, is divided by 5, while each y_i is divided by 2.

Then:
$$\sum x = 71.4;$$
$$\sum y = 62.15;$$
$$\sum x^2 = 759.56;$$
$$\sum y^2 = 658.528;$$
$$\sum xy = 691.68;$$
$$r_{(\text{division, division})} = 0.95075.$$

3.8.2) Now, let's still divide each x_i in Table 1 by 5, but this time multiply each y_i with 2.

Then:
$$\sum x = 71.4;$$
$$\sum y = 248.6;$$

$$\sum x^2 = 759.56;$$
$$\sum y^2 = 10{,}536.4;$$
$$\sum xy = 2{,}766.72;$$
$$r_{(\text{division, multiplication})} = 0.95075.$$

So, again these results coincide with Pearson's.

More interesting alternative correlation coefficients [and given different results from Pearson's and Spearman's] are obtained by doing:

A mixture of Pearson's and Spearman's correlation coefficients.

4.1 We only replace x_i by its rank among x's, while y_i remains unchanged:

x rank	1	2	3	4	5	6	7	8	9	10
y	2.5	1.1	6.3	2.1	2.9	15.3	20.7	18.4	22	33

Table 7

$$\sum x = 55;$$
$$\sum y = 124.3;$$
$$\sum x^2 = 385;$$
$$\sum y^2 = 2{,}634.11;$$
$$\sum xy = 958.4;$$
$$r_{s,p} = 0.91661 \in [0.90303, 0.95075].$$

4.2. Similarly, as above, let's only replace y_i by its rank among y's, while x_i remains unchanged.

x	6	7	12	14	23	41	53	60	69	72
y rank	3	1	5	2	4	6	8	7	9	10

Table 8

$$\sum x = 357;$$
$$\sum y = 55;$$
$$\sum x^2 = 18{,}989;$$
$$\sum y^2 = 385;$$
$$\sum xy = 2{,}636;$$
$$r_{p,s} = 0.93698 \in [0.90303, 0.95075].$$

Both mixture correlation coefficients give different results from Pearson's and Spearman's, actually they are in between.

Conclusion:
In the samples where the rank in a discrete variable counts more than the variable values, this mixture of correlation coefficients brings better results than Pearson's or Spearman's.

Reference:

Jay Devore, Roxy Peck, "Introductory Statistics", second edition, West Publ. Co., 1994.

LETTERS TO THE EDITORS

Forward to Military Research

Dr. Florentin Smarandache
Associate Professor
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University of New Mexico
Gallup Campus, USA

It is with great honor that I write a forward for this issue of *Review of the Air Force Academy*, Romanian journal dedicated to military research, and I'd like to thank the editorial board: Lt. Col. Associate Prof. Marian Pearsică, Ph. D., Capt. Adrian Lesenciuc, Associate Prof. Doru Luculescu, Ph. D., Lecturer Diana Ilişoi, Ph. D., and Capt. Laurian Gherman for a such opportunity and for their initiative of launching an important journal in the world.

Unfortunately, Science History was not very friendly with the Romanian scientists, and important results by some of our prominent men were simply ignored by world scientific community.
I cite a few examples where the World Science History was unfair to Romanians:

- Recently Gheorghe Benga and his group have discovered the presence and location of the first water channel protein in the human red blood cell membrane in 1986, before Peter Agre. Benga and his group proposed in the same year further studies consisting in purification of protein and reconstitution in lipid vesicles.
- Ştefan Odobleja (1902-1978) created the psychocybernetics and the generalized cybernetics in 1938-1939, ten years before Norbert Wiener! Odobleja, based on observations, intuition, and rationality, set up a cybernetic model that he called "The Consonantist Psychology", which was presented in <Psychological Abstracts> in 1941.
- The savant Nicolae Paulescu (1969-1931) discovered the insulin used in the treatment of the diabetes in 1922 (that he called "pancreina"), before Frederick Banting and his team. Not his work was recognized, but he was virulently denigrated.
- Gabriel Sudan is the first mathematician to have constructed a recursive function that is not a primitive recursive function, before Wilhelm Ackermann. Sudan worked with functions of transfinite ordinals, while Ackermann with functions of natural numbers. Yet, David Hilbert cited Ackermann, whence today's *Ackermann Function*, but never Sudan, although both were Hilbert's students.

Now, in the third millennium, we hope that scientific research of small countries, such as ours, should not be ignored or denigrated any longer. It has to be incorporated in the human thesaurus that it belongs to.

For the future we would like to see more scientific names, ideas and theories from our country recognized in the universal science and culture.
And we believe the *Review of the Air Force Academy* Romanian journal should fully contribute to this imperative.

Request for Support Letters

I am a member of the Mathematical Association of America. I write to you because I wish to relate to you the following facts:

- Last year I received an invitation, from Professor Richard K. Guy, to participate in the Eugene Strenus Memorial Conference on Intuitive and Recreational Mathematics and its History (July 17-August 2, 1986; University of Calgary, Alberta, Canada) to present a 15-20 minute talk: *Mathematical Fancies & Paradoxes*, and another one, from Professor Andrew M. Gleason, to participate in the International Congress of Mathematics (August 2-11, 1986, University of California, Berkley, California, USA) to present a 10 minute short communication, An *infinity of unsolved problems concerning a function in number theory* in the Section 3 (Number Theory). I had already paid $125 as my preregistration fee.

But the Romanian authorities did not grant me a visa, although I had obtained the Canadian and American visas. Moreover, they have caused me to be unemployed since September 1986 and office in Craiova will give me a job.

- This year I received another invitation from Dr. B. Stankovič (Novi Sad), to participate in the International Conference on Generalized Functions, Convergence Structures and Their Applications (June 23-27, 1987; University of Dubrovnik, Yugoslavia) and to present a paper. But the Romanian authorities have told me that I may not participate in the Conference because I am unemployed!

In the future I should desire to obtain the Romanian visa so I may participate in international conferences. Hence, as the mathematician Radu Rosu did (see Notices, November 1985, page 795), "I appeal to the mathematical community, especially to former members of the IAS to express concern for such cases by support letters to Nicolae Ceauşescu, Honorary President of Romanian Academy, Victoriei 125, Bucharest 1, Romania and Elena Ceauşescu, Head of the Romanian Council for Scienes and Technology, Victoriei Str. 1, Bucharest 1, Romania".

Thank you very much for your support.

Florentin Smarandache
Craiova, Romania
(Received May, 26, 1987)

{Letter to the Editor, in "*Notices* of the American Mathematical Society", Providence, RI, USA, Vol.34, ISS 6, 924-925, October 1987.}

ADDENDA:
The next three pages reproduce (1) a Note from the Securitate [Secret Police] about two letters sent to me by the ICM-86 Organizers and by the University of Calgary in 1986 that I never received (I was nicknamed "Savu" by the Securitate), (2) a copy of a Letter from Dr. Andrew M. Gleason that my case was turned over to the Committee of Concerned Scientists in New York - letter that I never received either, and (3) a Translation done by Secret Agents of another letter sent to me from ICM-86 that I never received too. Copies of the pages were recovered by the Author after the 1989 Romanian Revolution through the CNSAS [National Council of Studying the Archives of the Secret Police].

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23 -07- 2002
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- Notă -

- „Sorin" a primit două materiale de la:

- International Congress of Mathematicians 1986
Post Office Box 6887
Providence, Rhode. USA 02940.

- The University of Calgary
Department of Mathematics and Statistics
2500 University Drive N.W. Calgary, Alberta,
Canada.

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PROFESSOR FLORENTIN SMARANDACHE

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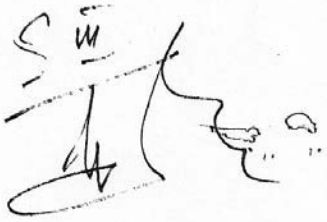

ICM-86 HAS TURNED YOUR CASE OVER TO THE COMMITTEE OF CONCERNED SCIENTISTS IN NEW YORK. WE HOPE THIS WILL DO SOME GOOD.

SINCERELY.

ANDREW M. GLEASON

CHAIRMAN, ICM-86 STEERING COMMITTEE

COL B1 36 19 ICM-86 ICM-86

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0065667

23 -07- 2002

C.N.S.A.S.
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„Dragă profesore STARANDACHE,

... În comunicarea dvs. din 22 mai ne-ați solicitat de asemenea ajutorul pentru o viză pentru Conferința 1979-86. Din păcate nu credem că avem nici o influență asupra guvernului dvs. Sperăm că veți putea participa la conferință.. SALE M. ITUGLER"

N.N. Totodată obiectivul dvs. a primit numeroase din partea secretarului prof. GUY RICHARD din Canada, Calgary, University of Calgary (ne lipsă invitația și programul Conferinței memoriale Eugene Strens, între 27 iulie - 2 august).

Sînt sigur că prof. GUY va putea să vă răspundă la dificilele dvs. întrebări cînd veți veni în Calgary. El este plecat până la începerea lunii iulie. Sperăm că eforturile noastre depuse în împrejurări v-am permis să vă aranjați vizita la conferința noastră..."

De asemenea președintele Asociației americane de matematică, STEEN LYNN-ARTHUR din SUA, Washington, 1529 18th Street îl anunță pe obiectivul dvs. că, în anul acest an se percepe o taxă suplimentară (dar voluntară) de 10 dolari în plus fiecărui membru al asociației.

Biography of Scientist, Writer, and Artist
Florentin Smarandache
at 55

Updated and extended by Prof. Mihály Bencze
Department of Mathematics
Áprily Lajos College
Braşov, Romania

Dr. Florentin Smarandache is a *polymath*: as author, co-author, translator, co-translator, editor, or co-editor of **143 books** and **183 scientific papers and notes**.
On December 10th, 2009, he was 55 years old.

Florentin Smarandache in Alaska (August 2009)

Actually he is a *Renaissance man* since he published in many fields, such as: **mathematics** (number theory, statistics, non-Euclidean geometry), **computer science** (artificial intelligence, information fusion), **physics** (quantum physics, particle physics), **economics** (cultural economics, poly-emporium theory), **philosophy** (neutrosophy – a generalization of dialectics, neutrosophic logic – a generalization of intuitionistic fuzzy

logic), **social sciences** (political essays), **literature** (poetry, prose, essays, novel, dramas, children plays, translations), **arts** (avant-garde/experimental drawings, collages, paintings).
He works as a Professor of Mathematics at the University of New Mexico, Gallup Campus, USA.

His books are to be found in Amazon.com, Amazon Kindle, Google Book Search, Library of Congress (Washington D. C.), and in many libraries around the world.
In arXiv.org international scientific database, sustained by Cornell University, he together with co-authors has about 150 scientific papers.

Dr. Smarandache is the creator of *Dezert-Smarandache Theory* in Information Fusion (applied mathematics), together with Dr. J. Dezert from France. This theory is internationally known since it is used in robotics, medicine, military, cybernetics, and every year since 2003 he is invited to present tutorials and papers about it at Fusion International Conferences in Australia (2003), Sweden (2004), USA (2005), Italy (2006), Canada (2007), Germany (2008), or at Marcus Evans's Defense Seminars in Spain (2006), Belgium (2007), or at other universities (in Indonesia in 2006).
He was invited speaker and sponsored by NASA in 2004 and by NATO in 2005.
His papers are published by the Proceedings of these Conferences.
Many Ph D Theses have been sustained at universities in Canada, France, Italy, and a M. Sc. Thesis at Tehran University in Iran.
See the site of DSmT that he designed and maintained himself at: http://fs.gallup.unm.edu//DSmT.htm.

In *Smarandache algebraic structures*, such as monoid, semigroup, vector space, linear algebra, etc., students from IIT (Indian Institute of Technology) in Chennai, Tamil Nadu, India, did and still do Ph D theses under the direction of Dr. W. B. Vasantha Kandasamy, who is one of his contributors to many such algebraic structures' studies (see http://fs.gallup.unm.edu//algebra.htm).

He set up and developed the *Neutrosophic Logic/Set/Probability*, which are generalizations of fuzzy logic (especially intuitionistic fuzzy logic), fuzzy set (especially intuitionistic fuzzy set), and respectively imprecise probability.
He was an invited speaker at University of Berkeley in 2003 at a conference organized by the famous L. Zadeh, the father of fuzzy sets; also invited speaker in India (2004), Indonesia (2006), Egypt (2007).
There were two Ph D theses on them at Georgia State University in Atlanta, and at Queensland University in Australia (see: http://fs.gallup.unm.edu//neutrosophy.htm).

Smarandache notions in Number Theory, also internationally known, such as *Smarandache sequences, Smarandache functions*, *Smarandache constants* (which are included even in the prestigious "CRC Encyclopedia of Mathematics", by E. Weinstein, CRC Press, Florida, 1998; see http://mathworld.wolfram.com/).
Several *Smarandache functions* are included in the "Handbook of Number Theory", by Jozsef Sandor, Springer-Verlag, 2006.

Smarandache-Wellin Numbers and Primes are also treated in a book by the well-known number theorists R. Crandall and C. Pomerance, called "Prime Numbers: A Computational Perspective", 2nd ed., New York: Springer-Verlag, 2005.
Other work by Dr. Florentin Smarandache in Number Theory and Combinatorics, such as open problems and conjectures, are subject to many research papers published by Xi'an University from China in the "Scientia Magna" international journal (see its last issue at: http://fs.gallup.unm.edu//ScientiaMagna4no3.pdf), and by Chinese Academy of Sciences from Beijing in "International Journal of Mathematical Combinatorics" (see its last issue at: http://fs.gallup.unm.edu//IJMC-3-2008.pdf).
In Number Theory there has been organized an International Conference on Smarandache Notions in Number Theory in 1997 at the University of Craiova, Romania (where he graduated as first of his graduates in 1979), organized by Dr. C. Dumitrescu & Dr. V. Seleacu (see http://fs.gallup.unm.edu/ProgramConf1SmNot.pdf).
This conference is listed in the prestigious "Notices of the American Mathematical Society" journal, Providence, NJ, USA, Vol. 48, No. 8, p. 903, 2001.
In China there have been organized four "International Conferences on Number Theory and Smarandache Problems" in 2005, 2006, 2007, and 2008. Dr. Zhang Wenpeng together with his students from Northwest University in Xi'an, China, edited an international journal called "Scientia Magna" where there are many papers on Smarandache notions in number theory. Several of them are listed in the prestigious "Notices of the American Mathematical Society" journal.
See for example the proceedings of the 2008 international conference:
http://fs.gallup.unm.edu//ScientiaMagna4no1.pdf.

He is the editor of the international journal "Progress in Physics", printed and edited at UNM-Gallup, with international contributors and sponsors, and subscriptions from various research nuclear institutes from around the world.
See one of its issues at: http://fs.gallup.unm.edu//PP-03-2008.pdf.

In Physics he coined the notion of "unmatter", revealed some "Sorites quantum paradoxes", used the neutrosophic logic (which is a multi-valued logic) to extend physics spaces, and extended together with V. Christianto physical differential equations from quaternion form to biquaternion form; see also the Smarandache-Christianto potential (http://fs.gallup.unm.edu//physics.htm).

In Economics he wrote together with V. Christianto about cultural economics as an alternative for underdeveloped countries, and proposed a poly-emporium theory (http://fs.gallup.unm.edu//economics.htm).

In Philosophy he did a synthesis of multiple contradictory philosophical ideas and schools of thought, extending Hegel's dialectics to neutrosophy, which means analysis of not only the opposites but also the neutralities in between that interact with them (http://fs.gallup.unm.edu//neutrosophy.htm).

In humanistic fields, he is the father of "*paradoxism*" in literature, which is an avant-garde movement based on excessive use of antitheses, oxymorons, contradictions,

paradoxes in creations set up by him in 1980's in Romania. He published five International Anthologies on Paradoxism, where have contributed hundreds of writers from over the world (http://fs.gallup.unm.edu//a/Paradoxism.htm).

In the frame of or related to the paradoxism he introduced:

- *New types of poetry with fixed form*, such as: the Paradoxist Distich, Tautological Distich, Dualistic Distich, Paradoxist Tertian, Tautological Tertian, Paradoxist Quatrain, Tautological Quatrain, Fractal Poem, Non-Poems (1990), and more poetical avant-garde experiments behind the outer limits of poetry in "Encyclopoetria (Everything is Poetry and Nothing is Poetry)" (2006);
- *New types of short story*, such as: Syllogistic Short Story, Circular Short Story ["Infinite Tale", 1997];
- *New types of drama*, such as: Neutrosophic Drama, Sophistic Drama, Combinatory Drama (a drama whose scenes are permuted and combined in so many ways producing over a billion of billions of different dramas! ["Upside-Down World", 1993];
- and *New types of science fiction genres in prose*, such as:

military science fiction ["The Art of antiWAR / paradoxistINSTRUCTION Notebooks of Captain Gook (or Kook)", 2008];
information technology science fiction ["Inform Technology", 2008];
political science fiction ["International Fonfoism (Manual of Therrory)", 2008];
business and finance science fiction ["Reproduction's disOrganization", 2009];
psychological science fiction ["Textbook of Psychunlogy (MASTER DECREE Thesis)", 2009];
and educational science fiction ["Treatise of Parapedagogy (Ph D Dissentation)", 2009].
These books can be downloaded from the site:
http://fs.gallup.unm.edu//eBooks-otherformats.htm.

And *linguistic literary experiments* in the volumes: "Florentin's Lexicon" (2008), interpreting in an opposite sense language clichés, homonyms, etc. ["If anything can go wrong, pass it on to someone else (Florenitn's Laws)"; "The dictator lift the state of emergency with a crane (Florenitn's Clichés)"; "Send me an e-male (Florentin's Homonyms)'; etc.].

Also, a combination of very short poetry, art, and science he did in the volumes "Lyriphoto(n)s / At Mind's Infinite Speed" (2009), and "Aph(l)orisms in Unistiches" (2008).

His anti-dictatorial drama "Country of the Animals", drama with no words!, was performed at the International Festival of Student Theaters, Casablanca (Morocco), September 1-21, 1995, it was staged three times by Thespis Theater (producer Diogene V. Bihoi) and it received The Jury Special Award;
it was also staged at Karlsruhe (Germany), September 29, 1995.

While a children play written by him "Pacala, Ursul si Balaurul" [Trickster, the Bead, and the Dragon], was staged by the National Dramatic Theater <I.D.Sîrbu>, director: Dumitru Velea, at Petrosani, Romania, in September 1997;
(http://fs.gallup.unm.edu//a/theatre.htm).

He also did *electronic art* (using computer programs), experimental art (*outer-art*), and pledged for the Unification of Art Theories
(http://fs.gallup.unm.edu//a/oUTER-aRT.htm).

At Arizona State University, Hayden Library, in Tempe, Arizona, there is a large special collection called "The Florentin Smarandache Papers" (which has more than 30 linear feet) with books, journals, manuscripts, documents, CDs, DVDs, video tapes by him or about his work.

Another special collection "The Florentin Smarandache Papers" is at The University of Texas at Austin, Archives of American Mathematics (within the Center for American History).

His professional web site: http://fs.gallup.unm.edu// has about ¼ million hits per month! It is the largest and most visited site at UNM Gallup campus.
Inside this, his sub-directory site **Digital Library of Science** (http://fs.gallup.unm.edu//eBooks-otherformats.htm), with many of his published scientific books but also with books and journals of others about his scientific creations, gets about 1,000 hits per day!
His **Digital Library of Arts & Letters** (http://fs.gallup.unm.edu//eBooksLiterature.htm), with many of his literary and art books or albums, or about his literary and artistic creations, gets about 100 hits per day.
He became very popular around the world since over **3,000,000 people** per year from about 110 countries read and download his e-books; many of his books have thousands of hits per month.

{And because the biography of a living person is continuously developing and improving I mention the date when I completely it:

March 15, 2010.}

This is an eclectic tome of 100 papers in various fields of sciences, alphabetically listed, such as: astronomy, biology, calculus, chemistry, computer programming codification, economics and business and politics, education and administration, game theory, geometry, graph theory, information fusion, neutrosophic logic and set, non-Euclidean geometry, number theory, paradoxes, philosophy of science, psychology, quantum physics, scientific research methods, and statistics – containing 800 pages.

It was my preoccupation and collaboration as author, co-author, translator, or co-translator, and editor with many scientists from around the world for long time. Many ideas from this book are to be developed and expanded in future explorations.

I dreamt with the engineer and friend Vic Christianto to build a Lunar Space Base and travel from there inside the Solar System and outside in order to discover new planetoids and respectively exoplanets and to quantize the Universe. Or use the multispace and multistructure together with the physicist and editor-in-chief Dmitri Rabounski to re-interpret and extend scientific theories and even to induce New Physics if possible. Generalize the qu-bit to a *mu-bit* (multi-bit in a multi-space) for a (multi-) parallel computing (*mu-computing* with a *mu-supercomputer*), and search for an SC Potential.

Go to the outer-limits of science, not in a fiction but in a realistic way, and apply a neutrosophic interdisciplinary method of study and research, not being ashamed to ask and seek even elementary or impossible questions.

The Author

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